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BLOCH-TYPE CONJECTURES AND AN EXAMPLE OF A THREE-FOLD OF GENERAL TYPE

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The hypothetical existence of a good theory of mixed motives predicts many deep phenomena related to algebraic cycles. One of these, a generalization of Bloch's conjecture says that "small Hodge diamonds" go with "small Chow groups". Voisin's method [19] (which produces examples with small Chow groups) is analyzed carefully to widen its applicability. A three-fold of general type without 1- and 2-forms is exhibited for which this extension yields Bloch's generalized conjecture.

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1. Introduction

Very little is known about the Chow groups of algebraic varieties. This is even true for 0-cycles on surfaces. Mumford [12] has shown that if Albanese equivalence and rational equivalence coincide on 0-cycles of degree 0, then there are no holomorphic 2-forms. In this case, the group of 0-cycles of degree 0 modulo rational equivalence is *representable*, i.e. isomorphic to an abelian variety (the Albanese variety in this case). Bloch conjectured the converse and in [2] this is shown for surfaces which are not of general type. Later it was shown for several classes of surfaces of general type without holomorphic 2-forms (see [8, 1, 19]). For higher dimensions an analogue of Mumford's result can be found in [3, 4].

An analogue of (a weaker form of) Bloch's conjecture can be formulated in terms of the *level* of a non-zero Hodge structure $H = \bigoplus H^{p,q}$, i.e. the largest difference $|p-q|$ for which $H^{p,q} \neq 0$. In particular, for a projective manifold X the cohomology group $H^k(X,\mathbb{C})$ has level $< (k-2p)$ if and only if $H^{k,0}(X) = \cdots = H^{k-p,k}(X) = 0$. For instance H^1 has level < 1 if it vanishes and H^2 has level < 1 if $H^2 = H^{1,1}$. For surfaces, the two combined mean exactly that $q = p_g = 0$. Bloch's conjecture in this case states that for 0-cycles rational and homological equivalence coincide. Laterveer [10, Corollary 1.10] and Schoen [17] have found a generalization of Mumford's theorem: if for all cycles of dimension $\leq s$ rational and homological equivalence coincide, then every cohomology group $H^k(X)$ has level $< k - 2s$. Bloch's generalized conjecture is the converse. See also [9], where such conjectures are deduced from the Beilinson's conjectures on the existence of a nice filtration on the Chow groups.

An obvious class to test these types of conjectures are hypersurfaces of degree d in projective space $\mathbb{P}^{n+1}$. By Lefschetz hyperplane theorem, only the middle cohomology group can have level > 0. In fact, it has level $< n - 2s$ if $(s+1)d < n+2$ and so one expects rational and homological equivalence to coincide for cycles of dimension $\leq s$.

For $s = 0$ this is true (Roitman's theorem [16]), but for $s > 0$ the Bloch conjecture has only been shown for degrees much below the optimum which is the largest integer $< (n+2)/(s+1)$. See [5] and Sec. 3 below. Note that d falls below the optimum if one lets n grow. The same is true for the bound from [5]. Note also that such hypersurfaces are Fano.

To get examples of projective varieties which on the one hand have small Hodge diamonds, but on the other hand have ample canonical bundle, one has to look for complete intersections in weighted projective spaces. The disadvantage is that one has to allow "quasi-smooth" complete intersections which have certain mild singularities. However, provided one works with rational coefficients the Chow groups behave as in the smooth case. Also, the cohomology groups carry pure Hodge structures as in the smooth case. See Sec. 6.2 and the references therein for details. These examples thus provide more interesting testing ground for Bloch-type conjectures since they are far from Fano.

Of crucial importance here is that the analogues of the bounds from [5] are valid for weighted projective spaces $P = \mathbb{P}(a_0, \ldots, a_{n+1})$ with n replaced by the sum of the weights $\sum a_i := N + 1$. See Sec. 3. In other words, a complete intersection in P whose multidegree is "small" with respect to N has "small" Chow groups.[a]

In [19], Voisin considers a variant of the Bloch conjecture for certain smooth hypersurfaces in $\mathbb{P}^3$ and $\mathbb{P}^4$ equipped with a finite group of automorphisms; instead of looking to the entire cohomology and the entire Chow group, one restricts to one or more character subspaces at a time. In loc. cit. this modified conjecture is proven for dimension 2 and the trivial character (and for the sum over the non-trivial characters), but also for smooth quintics of dimension 3 invariant under an involution.

Voisin's proof, apart from ingenious cycle theoretic constructions, basically exploits the fact, as indicated above, that Chow groups of complete intersections of the same degree but one dimension up become "smaller". The above remarks for complete intersections in weighted projective spaces motivate to look for suitable generalizations of her method. Although this can be done in any dimension, in this

[a]The bounds one gets in this way are again not the optimal ones in general.

note I only propose such a generalization for odd dimensional varieties (Theorem 15) since I only could find an interesting class of new examples for dimension 3. In this dimension the result reads as follows (see Corollary 16):

Theorem. *Let W be a smooth $(4+k)$-fold in $\mathbb{P}^{N+1+k}$, and let $Y \supset X \supset B$ smooth linear sections of dimensions $4, 3, 2$ respectively. Suppose that the following assumptions hold*:

(1) $\mathrm{CH}_1^{\mathrm{hom}}(W) = 0$
(2) $\mathrm{CH}_0^{\mathrm{hom}}(Y) = \mathrm{CH}_2^{\mathrm{hom}}(Y) = 0$;
(3) *there exists a family of* 1*-cycles on X with* 1*-dimensional parameter space*, *say* S, *such that the Abel–Jacobi map $S \to J^2(X)$ is onto* ($J^2(X)$ *is the intermediate Jacobian of* X);
(4) $h^{3,0}(X) = 0 = h^{1,0}(B)$ *and* $h^{1,1}(X) = 1$.

Then, *modulo some further technical conditions* (*Assumption* 17), *and provided X is sufficiently general*, $\mathrm{CH}_0^{\mathrm{hom}}(X) = 0$.

Let me discuss the assumptions in the statement.

The third condition is well known to imply the generalized Hodge conjecture on X (see [14, Example 12.11]) and the condition $h^{3,0}(X) = 0$ is a consequence of $\mathrm{CH}_0^{\mathrm{hom}}(X) = 0$ (by the generalized Mumford theorem). Since it is often easier to prove that $\mathrm{CH}_0^{\mathrm{hom}}(Y) = 0$ for the variety Y of which X is a linear section it is natural to make this assumption. It will be explained below (Corollary 4) that in that case the Abel–Jacobi map for 2-cycles cohomologous to zero is injective, so the vanishing of $\mathrm{CH}_2^{\mathrm{hom}}(Y)$ results if one knows for instance that $b_3(Y) = 0$. In this situation the Lefschetz theorem on linear sections then shows that $b_1(B) = 0$ for the surface B.

One might wonder what the use of the variety W is. For this one goes back to condition (3). It remains true for the entire family of hypersurface sections of Y with base B and then gives a correspondence, say β from Y to B which one needs to show to vanish on the level of Chow-groups. This is not immediately clear, but the functoriality of the construction shows that β can be extended as a correspondence from W to B where W is *any* variety of which Y is a linear section. In that case β factors over W and by the philosophy of the linear section method $\mathrm{CH}_1^{\mathrm{hom}}(W) = 0$ provided k is large enough.

For the moment, this method does not yet yield new examples of complete intersections, but there is one new example of a weighted complete intersection of general type. See Example 6.2. Nevertheless, the main theorem has potentially wider applicability, even for complete intersections and for that reason this case is explained in detail in Sec. 6.1. One can see this as a warming-up for the weighted complete intersection case in Sec. 6.2.

Finally a word about the technical conditions mentioned in the theorem. These have to do with general position with respect to linear subspaces; one of these is true for complete intersections whose dimension is larger than half the dimension of the linear space (Sec. 3.2) and — provided suitably reformulated — is valid also for weighted complete intersections; the second is true for all smooth projective varieties and this is the only statement which will be used for the example.

Notation and conventions. In this paper "variety" is a reduced projective scheme over the complex numbers so it does not need to be irreducible.

Let X be any variety. Then $\mathrm{CH}_k(X)$ denotes the Chow-group of k-cycles with $\mathbb{Q}$-coefficients modulo rational equivalence and $\mathrm{CH}_k^{\mathrm{hom}}(X)$ denotes the subgroup of $\mathrm{CH}_k(X)$ consisting of cycles homologous to 0. Occasionally, one puts $\mathrm{CH}(X) = \oplus\, \mathrm{CH}_k(X)$. One conventionally uses superscripts to denote codimension so that $\mathrm{CH}^k(X)$ denotes the group of codimension k cycles on X modulo rational equivalence.

If a variety is considered as embedded in some fixed projective space, say $X \hookrightarrow \mathbb{P}^N$ an important role is played by $\mathrm{CH}_k^{\mathrm{lin}}(X)$, the subgroup of k-cycles generated by k-dimensional linear sections and linear k-spaces contained in X. Of course $\mathrm{CH}_0^{\mathrm{lin}}(X) = \mathrm{CH}_0(X)$, but if $k > 0$ the two may differ.

For any variety X the (singular) mth cohomology group with *rational* coefficients is denoted $H^m(X)$. For a *smooth* variety X over $\mathbb{C}$ one denotes by

$$J^k(X) = H^{2k-1}(X;\mathbb{C})/\left[F^k H^{2k-1}(X;\mathbb{C}) + H^{2k-1}(X;\mathbb{Z})\right],$$
$$J^k(X)_{\mathbb{Q}} = H^{2k-1}(X;\mathbb{C})/\left[F^k H^{2k-1}(X;\mathbb{C}) + H^{2k-1}(X)\right]$$

the kth intermediate jacobian, respectively the kth intermediate jacobian *modulo torsion*. Here $F^\bullet$ stands for the usual Hodge filtration. Given any $z \in \mathrm{CH}^k_{\mathrm{hom}}(X)$, integration over a topological cycle whose boundary is z defines an element in the kth intermediate jacobian (see [7] for for details) and hence a homomorphism

$$u_X^k : \mathrm{CH}^k_{\mathrm{hom}}(X) \to J^k(X)_{\mathbb{Q}},$$

the *Abel–Jacobi map*.

2. Correspondences

2.1. *Generalities*

Let X and Y be smooth projective varieties. A *correspondence from X to Y* is a cycle, or equivalence class of cycles on $X \times Y$. Write $\mathrm{Corr}(X,Y)$ for the set of $\mathbb{Q}$-correspondences from X to Y up to rational equivalence. If X, Y, Z are smooth, $\alpha \in \mathrm{Corr}(X,Y)$, $\beta \in \mathrm{Corr}(Y,Z)$, the composition $\beta\circ\alpha \in \mathrm{Corr}(X,Z)$ is defined as $\beta\circ\alpha = (p_{13})_*[p_{12}^*\alpha \cdot p_{13}^*\beta]$ where p_{12}, p_{13}, and p_{23} are the projections of $X \times Y \times Z$ onto $X \times Y, X \times Z$, respectively $Y \times Z$.

Any correspondence α from X to Y induces homomorphisms between the Chow groups $\alpha_* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ by the formula $\alpha_*(u) = (p_2)_*[p_1^*u \cdot \alpha]$, where p_1, p_2 is the projection of $X \times Y$ to X respectively Y. This homomorphism does not necessarily preserve the degree. If the components of α have the same dimension, say

$\dim X + d$, one says that α has degree d since in this case α_* induces homomorphisms

$$\alpha_k : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+d}(Y).$$

Write $\mathrm{Corr}_d(X, Y)$ for the degree d correspondences from X to Y.

The *transpose* ${}^{\mathsf{T}}\alpha$ of α is obtained by interchanging X and Y. If α has degree d, ${}^{\mathsf{T}}\alpha$ has degree $\dim X - \dim Y + d$. In particular, a degree 0 correspondence between varieties of the same dimension induces $\alpha_k : \mathrm{CH}_k(X) \to \mathrm{CH}_k(Y)$ and ${}^{\mathsf{T}}\alpha_k : \mathrm{CH}_k(Y) \to \mathrm{CH}_k(X)$.

Any correspondence α between X and Y induces homorphisms in homology by the same formula as the formula for α_*. In this formula $(p_1)_*$ is the usual map induced in homology while p_2^* is the Gysin map obtained from the map in cohomology after applying Poincaré duality. If α has degree d the map is homogeneous of degree $2d$ and one writes $\alpha_k : H_k(X) \to H_{k+2d}(Y)$. This map is compatible with the cycle class map $\mathrm{cl}_k : \mathrm{CH}_k(X) \to H_{2k}(X)$. If one interchanges the roles of p_1 and p_2 and passes to cohomology one gets the induced map in cohomology $\alpha^k : H^k(Y) \to H^{k-2d}(X)$. Under Poincaré duality it coincides with ${}^{\mathsf{T}}\alpha_k$ which now is compatible with the cycle-class map $\mathrm{cl}^k : \mathrm{CH}^k(X) \to H^{2k}(X)$ for cohomology. It also induces a map between intermediate jacobians $\alpha^k : J^k(Y) \to J^{k-d}(X)$ compatible with the Abel–Jacobi maps, i.e. there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^k_{\mathrm{hom}}(Y) & \xrightarrow{\alpha^k} & \mathrm{CH}^{k-d}_{\mathrm{hom}}(X) \\ \downarrow{\scriptstyle u_Y^k} & & \downarrow{\scriptstyle u_X^{k-d}} \\ J^k(Y)_{\mathbb{Q}} & \xrightarrow{\alpha^k} & J^{k-d}(X)_{\mathbb{Q}}. \end{array} \tag{1}$$

A degree d correspondence from S to X is nothing but a family Z_s of d-cycles in X parametrized by points $s \in S$. Fix $o \in S$. The d-cycles $Z_s - Z_o$ are homologous to zero and the map $S \to \mathrm{CH}_d^{\mathrm{hom}}(X)$ given by $s \mapsto [Z_s - Z_o]$ is the composition of $S \to \mathrm{CH}_0^{\mathrm{hom}}(S)$ given by $s \mapsto [s - o]$ and the map $\mathrm{CH}_0^{\mathrm{hom}}(S) \to \mathrm{CH}_d^{\mathrm{hom}}(X)$ induced by the correspondence. Assembling the Abel–Jacobi images of $[Z_s - Z_o]$ then gives a morphism $S \to J^{n-d}(X)$, $d = \dim X$, which is also called the *Abel–Jacobi map* associated to the correspondence.

2.2. *Degree zero correspondences between varieties of the same dimension*

The following easy result is crucial for many of the results in this paper.

Lemma 1. *Let α be a degree zero correspondence from X to Y,* $\dim X = \dim Y = n$. *Suppose that α can be represented by a cycle having support on $V \times W$, V a subvariety of X, W a subvariety of Y,* $\dim V = v, \dim W = w$. *Then*

(1) $\alpha_k : \mathrm{CH}_k(X) \to \mathrm{CH}_k(Y)$ *is zero if $k < n - v$ or $k > w$,*
(2) ${}^{\mathsf{T}}\alpha_k$ *acts trivially on* $\mathrm{CH}_k(Y)$ *for $k < n - w$ or $k > v$.*
(3) α_w *acts trivially on* $\mathrm{CH}_w^{\mathrm{hom}}(X)$ *and* ${}^{\mathsf{T}}\alpha_v$ *acts trivially on* $\mathrm{CH}_v^{\mathrm{hom}}(Y)$.

(4) $\alpha_{w-1}|\,\mathrm{CH}^{\mathrm{hom}}_{w-1}(X)$ *and* ${}^{\mathsf{T}}\alpha_{v-1}|\,\mathrm{CH}^{\mathrm{hom}}_{v-1}(Y)$ *factor over an abelian variety* (*mod torsion*). *Moreover* α_w *as well as* ${}^{\mathsf{T}}\alpha_{v-1}$ *vanishes on the respective Abel–Jacobi kernels, i.e. on* $\ker(u_X^{n-(w-1)})$, $\ker(u_Y^{n-(v-1)})$ *respectively.*

Proof. Let $\tilde{V} \to V$, $\tilde{W} \to W$ be desingularizations and $i : \tilde{V} \to X$, $j : \tilde{W} \to Y$ be the desingularization composed with the inclusions. Choose $\tilde{\alpha} \in \mathrm{Corr}(\tilde{V} \times \tilde{W})$ mapping to α. With p_1 and p_2 the obvious projections, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_{k+v+w-n}(\tilde{V} \times \tilde{W}) & \xrightarrow{\tilde{\alpha}_{k+v+w-n}} & \mathrm{CH}_k(\tilde{V} \times \tilde{W}) \\ \uparrow p_1^* & & \downarrow (p_2)_* \\ \mathrm{CH}_{k+v-n}(\tilde{V}) & & \mathrm{CH}_k(\tilde{W}) \\ \uparrow i^k & & \downarrow j_k \\ \mathrm{CH}_k(X) & \xrightarrow{\alpha_k} & \mathrm{CH}_k(Y). \end{array} \tag{2}$$

Since $\mathrm{CH}_{k+v-n}(\tilde{V}) = 0$ for $k < n - v$ and $\mathrm{CH}_k(\tilde{W}) = 0$ for $k > w$ the first assertion follows. The second follows from the fact that ${}^{\mathsf{T}}\alpha_k$ factors over $\mathrm{CH}_{k+w-n}(\tilde{W})$ and $\mathrm{CH}_k(\tilde{V})$.

Since $\mathrm{CH}^{\mathrm{hom}}_v(\tilde{V}) = 0 = \mathrm{CH}^{\mathrm{hom}}_w(\tilde{W})$ the third assertion is clear.

Using the functoriality expressed by diagram (1) one deduces the fourth from the equalities $\mathrm{CH}^{\mathrm{hom}}_{w-1}(\tilde{W}) = \mathrm{Pic}^0(\tilde{W})_{\mathbb{Q}}$, $\mathrm{CH}^{\mathrm{hom}}_{v-1}(\tilde{V}) = \mathrm{Pic}^0(\tilde{V})_{\mathbb{Q}}$ and the fact that the Abel–Jacobi kernels are zero for the Picard-varieties. □

To apply this Lemma, one needs a very strong decomposition for the correspondences. If the varieties have small Chow groups this can in some cases be achieved using the Bloch–Srinivas method [3]. For this note the following variant is useful.

Proposition 2. *Let* X, Y *be varieties of the same dimension* n *and let* $\alpha \in \mathrm{Corr}_0(X, Y)$ *be a degree* 0 *correspondence. Suppose that the image of* $\alpha_0 : \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$ *is supported on a subvariety* $W \subset Y$. *Then in* $\mathrm{CH}_n(X \times Y)$ *one has a decomposition* $\alpha = \alpha^{(1)} + \alpha^{(2)}$ *with* $\alpha^{(1)}$ *supported on* $X \times W$ *and* $\alpha^{(2)}$ *supported on* $D \times Y$ *where* D *is some* (*possibly reducible*) *divisor on* X.

This variant can be proved by copying the proof of [3, Proposition 1] which is the case where α is the diagonal inside $X \times X$. See also [20, Corollary 10.20]. A first consequence of Proposition 2, but in fact of [3, Proposition 1] is the following result:

Corollary 3. *Let* X *be a smooth variety of dimension* $n \geq 2$ *with* $\mathrm{CH}^{\mathrm{hom}}_0(X) = 0$. *Then there exists a smooth variety* V *of dimension* $n - 2$ *and* $\beta \in \mathrm{CH}_{n-1}(X \times V)$ *such that* ${}^{\mathsf{T}}\beta_k \circ \beta_k = \mathrm{id}$ *for* $1 \leq k \leq n - 1$.

Proof. By Proposition 2 (or by [3, Proposition 1]) there is a decomposition of the diagonal inside $\mathrm{CH}(X \times X)$ as $\Delta = x \times X + \gamma$, $x \in X$ and such that $\mathrm{supp}(\gamma) \subset X \times D$, $D \subset X$ a divisor, which one may assume to be very ample.

If $k < n$ the first factor acts trivially on $\mathrm{CH}_k(X)$, and then γ acts as the identity. The support of γ cannot be contained in $X \times E$, with E a variety of dimension $\leq n-2$: the identity factors over $\mathrm{CH}_k(\tilde{E})$, with $\tilde{E}$ a desingularization of E and so is trivial for $k \leq n$, while $\mathrm{CH}_{n-1}(X) \neq 0$. In particular, taking a hyperplane section $D \cap H$ of D, the cycle

$$\beta = \gamma \cdot X \times H$$

has support of dimension $n-1$. Choose a desingularization $\tilde{D} \to D$ which contains a desingularization $V \to D \cap H$ of $D \cap H$. Let $i : \tilde{D} \to X$ and $j : V \to \tilde{D}$ be induced by the obvious inclusions and let $\tilde{\gamma} \in \mathrm{Corr}(X, \tilde{D})$ be a cycle class mapping to γ. By assumption, $i_* \circ \tilde{\gamma} = \gamma$ acts as the identity. The same argument for the transposed cycles proves that ${}^{\mathsf{T}}\tilde{\gamma} \circ i^* = {}^{\mathsf{T}}\gamma$ acts as the identity provided $k > 0$. One may assume that $H \in |D|$ so that $i_* \circ i^*$ and $j^* \circ j_*$ both equal cup product with the first Chern class of $c_1(D) = c_1(H)$, one arrives at a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_k(X) \arrow[r, "\tilde{\gamma}"'] \arrow[dr, "\beta_k"] & \mathrm{CH}_k(\tilde{D}) \arrow[r, "i_*"'] \arrow[d, "j^*"'] & \mathrm{CH}_k(X) \arrow[r, "i^*"'] & \mathrm{CH}_{k-1}(\tilde{D}) \arrow[r, "{}^{\mathsf{T}}\tilde{\gamma}"'] & \mathrm{CH}_k(X) \\
& \mathrm{CH}_{k-1}(V) \arrow[rr, equal] & & \mathrm{CH}_{k-1}(V) \arrow[u, "j_*"'] \arrow[ur, "{}^{\mathsf{T}}\beta_k"'] &
\end{tikzcd}$$

It shows that ${}^{\mathsf{T}}\beta_k \circ \beta_k$ is the identity. $\square$

We need another consequence of [3, Proposition 1]:

Corollary 4 ([3, Theorem 1, p. 1238]). *Suppose that* $\mathrm{CH}_0^{\mathrm{hom}}(X) = 0$. *Then the Abel–Jacobi map*

$$u_X^2 : \mathrm{CH}^2_{\mathrm{hom}}(X) \to J^2(X)_{\mathbb{Q}}$$

is injective; *in particular*, *if* $H^3(X) = 0$, *one has* $\mathrm{CH}^2_{\mathrm{hom}}(X) = 0$.

2.3. *Reflexive vanishing for correspondences*

Definition 5. Let X be a smooth variety. It has the mth *reflexive vanishing property with respect to correspondences* if the following property holds:

$$\mathrm{C}(X)_m \begin{cases} \text{For any smooth } V \text{ with } \dim V = \dim X \text{ and any degree } 0 \\ \text{correspondence } \alpha \text{ from } V \text{ to } X, \text{ if the induced map} \\ \alpha_m : \mathrm{CH}_m^{\mathrm{hom}}(V) \to \mathrm{CH}_m^{\mathrm{hom}}(X) \text{ vanishes then } {}^{\mathsf{T}}\alpha_m = 0. \end{cases}$$

This reflexive vanishing property is difficult to test but for $m = 0$ one can use the Bloch–Srinivas method from the previous section.

Proposition 6. *Let X be a surface with* $\mathrm{Alb}(X) = 0$. *Then X has the* 0*th reflexive vanishing property with respect to correspondences.*

Proof. Suppose that $\alpha \in \mathrm{Corr}_0(V, X)$ has the property that $\alpha_0 = 0$ on $\mathrm{CH}_0^{\mathrm{hom}}(V)$. This means that the image of α_0 is supported on points. By Proposition 2, one can write $\alpha = \alpha^{(1)} + \alpha^{(2)}$ with $\alpha^{(1)}$ supported on $V\times$ [points], while $\alpha^{(2)}$ is supported on [divisor]$\times X$. Now apply Lemma 1 to see that ${}^{\mathsf{T}}\alpha_0$ factors over an abelian variety and hence over $\mathrm{Alb}(X) = 0$. □

Remark. The referee remarks that the property $\mathrm{C}(X)_0$ is satisfied if and only if for any variety Y of dimension $< \dim X$, and any correspondence $\alpha \in \mathrm{Corr}^0(X, Y)$, $\alpha_0 : \mathrm{CH}_0^{\mathrm{hom}}(X) \to \mathrm{CH}_0^{\mathrm{hom}}(Y)$ is trivial. Indeed, one implication follows immediately from the preceding proof. Conversely, if $\mathrm{C}_0(X)$ holds, pick a smooth projective variety E such that $\dim E + \dim Y = \dim X = n$; Lemma 1 shows that if $\alpha \in \mathrm{Corr}_0(X, Y)$ is viewed as a correspondence β from X to $Y \times E$ (by choosing a point e of E and identifying Y with $Y \times e$), one has ${}^{\mathsf{T}}\beta_0 = 0$ and hence, by reflexivity, $\beta_0 = 0$, whence $\alpha_0 = 0$.

In terms of the "naïve filtration" on $\mathrm{CH}_0(X)$ introduced by Voisin in [21] this means by definition that

$$F^i_{\text{naïve}}\,\mathrm{CH}_0(X) = F^n_{\text{naïve}}\,\mathrm{CH}_0(X), \quad 0 < i \le n,\ \ n = \dim X.$$

3. Varieties with Linearly Generated Chow Groups

3.1. *Complete intersections in (weighted) projective space*

Definition 7. Let $X \subset \mathbb{P}^N$. Let me say that $\mathrm{CH}_k(X)$ is *linearly generated* if it is generated by linear sections and linear subvarieties of $\mathbb{P}^N$ contained in X.

Proposition 8 ([5]). *Let X be a complete intersection of multidegree $(d_1, \dots, d_r)$ in $\mathbb{P}^{n+r}$. Assume that $d_1 \ge d_2 \ge \cdots \ge d_r \ge 2$. For any s such that*

$$\sum \binom{d_j + s}{s+1} \le n + r - s, \tag{3}$$

the Chow groups $\mathrm{CH}_k^{\mathrm{hom}}(X)$ vanish for $k \le s$, while $\mathrm{CH}_{s+1}(X)$ is linearly generated: $\mathrm{CH}_{s+1}(X) = \mathrm{CH}^{\mathrm{lin}}_{s+1}(X)$.

If $d_1 \ge 3$ or $r \ge k+1$ the vanishing result remains true if one replaces the right-hand side of (3) by $(n + r)$; for an intersection of $r \le k$ quadrics one can replace the right-hand side by $n + 2r - k - 1$.

From these examples one finds new ones in weighted projective spaces as follows. Recall that a weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is the quotient of $\mathbb{C}^{n+1}\setminus\{0\}$ by the action of $\mathbb{C}^*$ given by $t \cdot (z_0, \dots, z_n) = (t^{a_0} z_0, \dots, t^{a_n} z_n)$. Equivalently, letting ζ_i be a primitive a_ith root of unity, let the generator of the ith factor of $G = \mathbb{Z}/\mathbb{Z}_{a_0} \times \cdots \times \mathbb{Z}/\mathbb{Z}_{a_n}$ act on $\mathbb{P}^n$ by multiplying the ith coordinate with ζ_i, the quotient is $\mathbb{P}(a_0, \dots, a_n)$.

One has two different concepts of a linear space inside a weighted projective space $P = \mathbb{P}(a_0, \dots, a_n)$. The first kind is the image of any linear space in $\mathbb{P}^n$ and is called *weighted linear subspace*. Note that a weighted codimension 1 subspace need not be given by an equation in homogeneous coordinates. Those for which an equation can be found will be called *weighted hyperplanes*.

The second kind, called *generalized linear spaces*, come from the following construction. Set $M+1=\sum a_i$ and introduce homogeneous coordinates X_i^j, $i=0,\dots,n$, $j=1,\dots,a_i$ on $\mathbb{P}^M$. For any choice $J=(j_0,\dots,j_n)$, $1\le j_k\le a_k$, define L_J as the linear codimension $n+1$ subvariety of $\mathbb{P}^M$ given by the equations $\xi_k^{j_k}=0, k=0,\dots,n$ and set $Q=\mathbb{P}^M\setminus\bigcup_J L_J$. Define $\varphi:Q\to P$ by $\varphi(\dots,X_i^j,\dots)=(\dots,X_i^1\cdots X_i^{a_i},\dots)$. If one embeds $\mathbb{P}^n$ with homogeneous coordinates $(x_0,\dots,x_n)$ in Q by setting $X_i^j=x_i$, the restriction of φ to $\mathbb{P}^n$ is just the natural quotient map $\mathbb{P}^n\to P$. It is clear that Q only contains complete linear spaces of dimension $\le n$. Any linear k-space inside Q meeting the general fiber of φ in at most points maps to a variety in P of dimension k and is called a *generalized k-plane* of P.

Since z_i has weight a_i, a homogeneous polynomial in the z_i of weighted degree d defines a hypersurface in $P=\mathbb{P}(a_0,\dots,a_n)$. It is called *quasi-smooth* if the corresponding hypersurface in $\mathbb{C}^{n+1}$ has 0 as its only singularity. A similar definition holds for complete intersections. The construction of generalized linear spaces can be used to prove:

Proposition 9 ([11]). *Let X be a complete intersection of multidegree $(d_1,\dots,d_r)$ in $P=\mathbb{P}(a_0,\dots,a_{n+r})$. Put $M+1=\sum_{j=0}^{n+r}a_j$. Assume that $d_1\ge d_2\le\cdots\ge d_r\ge 2$. For any s such that*

$$\sum\binom{d_j+s}{s+1}\le M-s,$$

the Chow groups $\mathrm{CH}_k^{\mathrm{hom}}(X)$ vanish for $k\le s$, while $\mathrm{CH}_{s+1}(X)$ is generated by generalized linear subspaces contained in X.

Example 10. I am particularly interested in the following weighted projective space $\mathbb{P}(2^a,3^b)$, where the notation means that one takes a weights to be equal to 2 and b weights equal to 3. Hence $M=2a+3b-1$. The complete intersections X given by homogeneous equations of degree 6 are the simplest ones. For these, $\mathrm{CH}_0^{\mathrm{hom}}=0$ whenever $r\le\frac{1}{6}(2a+3b-1)$ and $\mathrm{CH}_1^{\mathrm{hom}}=0$ whenever $r\le\frac{1}{21}(2a+3b-2)$. Rephrasing this in terms of the dimension n of the intersection, the first condition becomes

$$4a+3b\le 6(n+1).$$

One sees that for three-folds this gives $4a+3b\le 24$ and for four-folds $4a+3b\le 30$. The special case $a=4$, $b=3$ is treated later. For the three-fold $X=X_{6,6,6}$ this result does not give $\mathrm{CH}_0^{\mathrm{hom}}(X)=0$ but for the four-fold $Y=Y_{6,6}$ it does give $\mathrm{CH}_0^{\mathrm{hom}}(Y)=0$.

3.2. *Extending linear cycles in pencils*

Lemma 11. *Let $P=\mathbb{P}^{n+r}\subset P'=\mathbb{P}^{n+r+1}$, $L\subset P'$ a linear space of dimension $k+1$, and let $M=L\cap P$ be the corresponding linear space of dimension k in P. Let $X\subset P$ be an n-dimensional smooth complete intersection containing M.*

(i) *Suppose that $k < \frac{1}{2}n$; then there exist a smooth complete intersection Y in P' of the same multidegree as X, which contains L and for which $Y \cap P = X$.*
(ii) *If $k < \frac{1}{2}(n-1)$ the general hyperplane of P' passing through L cuts Y in a smooth hypersurface.*

Proof. (i) Put $N = n + r$. and choose homogeneous coordinates

$$\{X_0, \dots, X_N, X_{N+1}\}$$

in P' so that P is given by $X_{N+1} = 0$. Let X be given by the equations $F_i(X_0, \dots, X_N) = 0$, $i = 1, \dots, r$. Suppose that $\deg F_i = d_i$. Let $G_i = 0$ be a hypersurface of degree $d_i - 1$ in P' containing L and consider the complete intersections inside P' given by equations $F_i + X_{N+1}G_i = 0$, $i = 1, \dots, r$. They all contain L which is in fact the base locus of the linear system of these complete intersections. The generic member of this system can only be singular at points on L. We will show that specific members cannot have singularities at all along L which therefore implies that for generic choices of G_i the resulting complete intersection satisfies the demands.

To find such a complete intersection, pick any codimension r linear space of P, say given by the linear equations $H_1 = \cdots = H_r = 0$ and put $G_i = H_i K$ where $K(X_0, \dots, X_{N+1})$ is of degree $d_i - 2$ and $K = 0$ contains L. Suppose that $q = (p_0, \cdots, p_N, p_{N+1}) \in L$ is a singularity on the corresponding complete intersection. Let $p = (q_1, \dots, q_N, 0) \in M$. Then $p \in X$ (since X contains M). Let ∇ be the gradient with respect to $(X_0, \dots, X_N)$. Then the vectors

$$\nabla_p F_i + q_{N+1} K(q) \nabla_p H_i, \quad i = 1, \dots, r$$

have to be dependent, say $-q_{N+1}K(q)\nabla_p \sum_i \lambda_i H_i = \nabla_p \sum \lambda_i F_i$. This implies that the hyperplane $\sum_i \lambda_i H_i = 0$ contains T_pX since the latter is the common annihilator of $\nabla_p F_1, \dots, \nabla_p F_r$ while any hyperplane given by an equation $H = 0$ is the annihilator of $\nabla_p H$. So we have found a hyperplane of P which contains M and T_pX. It suffices to find a hyperplane for which this is not possible:

Claim. *There exists a hyperplane $H \subset P$ passing through M which does not contain any of the tangent planes to X at points $p \in M$.*

To prove this, fix a linear $(N-k-1)$-dimensional subspace M' of P disjoint from M. Fix a hyperplane H containing M. Such a hyperplane varies in the projective space $(M')^*$ of hyperplanes in M'. Let $I = \{(p, H) \in M \times (M')^* \mid T_pX \subset H\}$. The variety I has dimension $k + (r-1)$ since the projection $I \to M$ is surjective with fibers of dimension $(r-1)$ since $T_pX \cap M'$ has codimension r so the hyperplanes containing it form a projective space of dimension $(r-1)$. It suffices to show that the projection $I \to (M')^*$ is not surjective. For that it suffices to observe that $\dim I = k + (r-1) < \dim(M')^* = N - k - 1 = n + r - k - 1$ since $2k < n$.

(ii) The argument is similar: Replace M by L, M' by a linear subspace $L' \subset P'$ disjoint from L with $\dim L' = N - k - 1$. The hyperplanes H containing L vary in the projective space $(L')^*$ of hyperplanes in L' and if $H \cap Y$ is singular at $q \in L$ one has $T_q(Y) \subset H$. Consider $J = \{(q,H) \in L \times (L')^* \mid T_qY \subset H\}$. Since $\dim J = k+1+(r-1) = k+r$ and $\dim J = k+r < n+r-k-1 = \dim(L')^*$ the projection $I' \to (L')^*$ is not onto and if H is a hyperplane corresponding to a point $[H]$ not in the image, it cannot contain any tangent plane of Y and the intersection is smooth. □

Remark 12. (1) The first assertion in Lemma 11 can be generalized to quasi-smooth weighted complete intersections with respect to weighted linear subspaces.

(2) The proof of the second assertion shows that it has nothing to do with complete intersections: it is valid for any smooth $Y \subset P'$ containing a linear space L which satisfies $\dim L < \frac{1}{2} \dim Y$.

4. Normal Functions of Pencils for which the Generalized Hodge Conjecture Holds

Consider the middle cohomology groups for a smooth $(2m+1)$-fold X. By [14, Example 12.11] the generalized Hodge conjecture is guaranteed by the following condition on m-cycles:

$$\mathrm{GHC}_m(X) \quad \begin{cases} \text{There exist } S \text{ and } \alpha \in \mathrm{Corr}_m(S,X) \text{ such that its} \\ \text{Abel–Jacobi map } \mathrm{Alb}(S) \to J^{m+1}(X) \text{ is onto.} \end{cases}$$

It is implicitly assumed that the relative cycle defining α restricts on the fibers $\{s\} \times X$ to cycles which are homologous to zero. By Lefschetz theorem on the hyperplane section and functoriality, one can always suppose that $S = C$, a smooth curve and then $\alpha \in \mathrm{CH}_{m+1}(C \times X)$ and the induced map $\alpha^1 : J^1(C)_{\mathbb{Q}} \to J^{m+1}(X)_{\mathbb{Q}}$ is onto. It follows that $\alpha^1 \circ {}^{\mathsf{T}}\alpha^1$ is a non-zero endomorphism of $J^{m+1}(X)_{\mathbb{Q}}$. Suppose that

$$\mathrm{End}(J^{m+1}(X)_{\mathbb{Q}}) = \mathbb{Q} \cdot \mathrm{id}, \tag{4}$$

then one has

$$\alpha^1 \circ {}^{\mathsf{T}}\alpha^1 = r(\alpha) \cdot \mathrm{id}, \quad r(\alpha) \neq 0. \tag{5}$$

The main result is:

Proposition 13. *Let $Y \subset \mathbb{P}^{N+1}$ be a smooth $(2m+2)$-fold with*

$$\mathrm{CH}^{\mathrm{hom}}_{m+1}(Y) = 0$$

and let $j : X \hookrightarrow Y$ be a smooth hyperplane section. Suppose that $H^{2m+1}(X)$ has Hodge level ≤ 1, i.e.

$$H^{2m+1}(X) = H^{m,m+1}(X) + H^{m+1,m}(X),$$

and that, moreover, (4) holds as well as assumption $\mathrm{GHC}_m(X)$. Then there exists $r \in \mathbb{Q}^*$ such that for any $d' \in \mathrm{CH}^{\mathrm{hom}}_m(X) \cap j^* \mathrm{CH}_{m+1}(Y)$ one has $\alpha \circ {}^{\mathsf{T}}\alpha(d') = rd'$.[b]

[b] *A priori* r depends on d', but the proof shows that this is in fact not the case.

To prove this proposition, one lets X vary in a hyperplane pencil. So let $L_{N-1} \subset \mathbb{P}^{N+1}$ a linear subspace of codimension 2. The linear pencil of hyperplanes passing through L_{N-1} is parametrized by a line $L \subset \mathbb{P}^{N+1}$ disjoint from L_{N-1}. The corresponding hyperplane sections are denoted by $X_t = Y \cap H_t$, $t \in L$. Set $\tilde{Y} = \{(y,t) \in Y \times L \,|\, y \in X_t\}$ and let $\tilde{f} : \tilde{Y} \to L$ be induced by the second projection. Let $U \subset L$ be the Zariski-open subset such that X_t is smooth for $t \in U$. Set $Y' := \tilde{f}^{-1}U$, $f = \tilde{f}|Y' : Y' \to U$. Start with any $(m+1)$-cycle Z on Y whose intersection Z_t with a smooth fiber X_t is homologous to zero on that fiber. Then $t \mapsto [Z_t] \in J^{n-m}(X_t)$ defines the *normal function* ν_Z associated to Z with cohomology invariant $\delta(\nu_Z) \in H^1(U, R^{2m+1}f_*\mathbb{Q})$. In the case under consideration, the Leray spectral sequence for f gives an isomorphism

$$\tau : \ker(H^{2m+2}(Y') \to H^0(U, R^{2m+1}f_*\mathbb{Q})) \xrightarrow{\cong} H^1(U, R^{2m+1}f_*\mathbb{Q}) \tag{6}$$

and it is well known [22, (3.9)] that τ maps $[Z]_{Y'}$, the cohomology class of Z on Y, which by assumption is in the left-hand side, to the cohomology invariant $\delta(\nu_Z)$. So $[Z]_{Y'} = 0$ is zero precisely when $\delta(\nu_Z) = 0$.

The starting observation is:

Lemma 14. *In the above setting, suppose that $\delta(\nu_Z) = 0$. Then for all $t \in U$ one has $Z|X_t = 0$ in $\mathrm{CH}_m^{\mathrm{hom}}(X_t)$.*

Proof. The cycle Z pulls back to $\tilde{Y}$ and restrict to a cycle on Y' which is denoted by the same symbol. Since τ (see (6)) is an isomorphism, its cohomology class $[Z]_{Y'}$ as a class on Y' vanishes. Let Σ be the union of the singular fibers so that $Y' = \tilde{Y} \backslash \Sigma$ and let $k : Y' \hookrightarrow \tilde{Y}$ be the inclusion. There is a commutative diagram

$$\begin{array}{ccccccc}
\ker(k_*) & \to & \mathrm{CH}_{m+1}(\tilde{Y}) & \xrightarrow{k_*} & \mathrm{CH}_{m+1}(Y') & \to 0 \\
\downarrow{\scriptstyle \mathrm{cl}_1} & & \downarrow{\scriptstyle \mathrm{cl}_2} & & \downarrow{\scriptstyle \mathrm{cl}_3} & \\
\left(H^{2m+2}_{\Sigma}(\tilde{Y})\right)_{\mathrm{alg}} & \to & H^{2m+2}_{\mathrm{alg}}(\tilde{Y}) & \to & H^{2m+2}(Y', \mathbb{Q}). &
\end{array}$$

The subscript “alg” stands for the subspace generated by classes of algebraic cycles. In particular, cl_1 surjective, and the assumption $\mathrm{CH}^{\mathrm{hom}}_{m+1}(Y) = 0$ implies that cl_2 is an isomorphism on the subgroup of cycles coming from Y. The diagram then shows that $[Z] \in \mathrm{CH}_{m+1}(\tilde{Y})$ is supported on the singular fibers. The restriction of $[Z]$ to a smooth fiber therefore vanishes. □

Proof of Proposition 13. Take an $(m+1)$-cycle Z' on Y which restricts to a cycle on X representing d'. Assume that Z' is chosen so that it is transversally intersected by any smooth member X_t of the pencil in an m-cycle ${Z'}_t$ homologous to zero on X_t. The assumption $\mathrm{GHC}_m(X_t)$ provides a family of curves C_t, $t \in U$ and correspondences α_t on $C_t \times X_t$. Possibly after some base change $U' \to U$ these fit into a global relative family $C_{U'} \to U'$ and a relative correspondence $\alpha_{U'}$ from $C_{U'}$ to Y'/U'. It induces a global endomorphism of the family of intermediate Jacobians $J^m(X_t)$ which must be a homothety since this is the case for generic t.

Hence the numbers $r(\alpha_t) \in \mathbb{Q}$ from (5) for those $t \in U$ for which (4) holds, are independent of t, say $r(\alpha_t) = r(\alpha)$. The Abel–Jacobi maps for the family Z'_t yield the normal function $\nu_{Z'}$ while the normal function for the $(m+1)$-cycle $\alpha{\circ}^{\mathsf{T}}\alpha(Z')$ is just $t \mapsto \alpha^1{\circ}^{\mathsf{T}}\alpha^1(\nu_{Z'}(t))$ which is equal to $r(\alpha)\nu_{Z'}(t)$. Lemma 14 applied to the difference $Z = \alpha{\circ}^{\mathsf{T}}\alpha(Z') - r(\alpha) \cdot Z'$ then completes the proof. $\square$

5. Controlling Chow Groups using Pencils of Hyperplane Sections

The aim is to show the following result:

Theorem 15. *Let* $W \subset \mathbb{P}^{N+1+k}$ *be a fixed* $(2m+2+k)$*-fold. Suppose moreover, that*

(1) $\mathrm{CH}^{\mathrm{hom}}_m(W) = 0$;
(2) $\mathrm{CH}^{\mathrm{hom}}_0(Y) = \mathrm{CH}^{\mathrm{hom}}_{m+1}(Y) = 0$ *for every smooth codimension* k *linear section* Y *of* W;
(3) *for any smooth hyperplane section* X *of* Y *one has* $h^{m-1,m-1}(X) = h^{m,m}(X) = 1$, $H^{2m+1}(X)$ *has Hodge level* ≤ 1, *and, moreover condition* $GHC_m(X)$ *holds*;
(4) *any smooth linear section* B *of* Y *of codimension* 2 *has the* $(m-1)$*st reflexive vanishing property with respect to correspondences*;

Then, provided X *is sufficiently general, and the technical condition below (Assumption 17) holds, one has* $\mathrm{CH}^{\mathrm{lin}}_{m-1}(X) \cap \mathrm{CH}^{\mathrm{hom}}_{m-1}(X) = 0$.

For $m = 1$, this can be simplified since Proposition 6 states that for 0-cycles on surfaces with $q = 0$ the reflexive vanishing holds which yields:

Corollary 16. *Let* W *be a smooth* $(4+k)$*-fold in* $\mathbb{P}^{N+1+k}$, *and let* $Y \supset X \supset B$ *smooth linear sections of dimensions* 4, 3, 2 *respectively. Suppose that Assumption* 17 *holds. Suppose moreover, that*

(1) $\mathrm{CH}^{\mathrm{hom}}_1(W)) = 0$;
(2) $\mathrm{CH}^{\mathrm{hom}}_0(Y) = \mathrm{CH}^{\mathrm{hom}}_2(Y) = 0$;
(3) $\mathrm{GHC}_1(X)$ *holds*;
(4) $h^{3,0}(X) = 0 = h^{1,0}(B)$ *and* $h^{1,1}(X) = 1$.

Then, provided X *is sufficiently general, one has* $\mathrm{CH}^{\mathrm{hom}}_0(X) = 0$.

The technical assumption alluded to above is inspired by Lemma 11:

Assumption 17. Let $s = m-1$ or $s = m$. Let $M \subset X$ be a given linear space of dimension s. Then

(1) there exists a linear space $L = L_{s+1} \subset \mathbb{P}^{N+1+k}$ of dimension $(s+1)$ and a smooth codimension k linear section Y of W containing L such that X is a hyperplane section of Y with $X \cap L = M$;
(2) the generic hyperplane section X' of Y containing L is smooth.

Proof of Theorem 15. Consider a class of the form $c = \deg(X)\cdot[M] - [H^{m+2} \cap X]$, with M a linear space of dimension $(m-1)$ contained in X and H a hyperplane. By Hypothesis (3) of the theorem such c belong to $\mathrm{CH}^{\mathrm{hom}}_{m-1}(X)$ and these classes generate $\mathrm{CH}^{\mathrm{lin}}_{m-1}(X) \cap \mathrm{CH}^{\mathrm{hom}}_{m-1}(X)$. So it suffices to prove that $c = 0$. Apply Assumption 17 as follows. Starting with the linear space $M \subset X$ one finds Y containing L as in (1), and a smooth X' as in (2). Next, one applies Assumption (1) to (X', L) to find (Y', L'). The situation can be summarized in the diagram

$$\begin{array}{ccc} & X' & \xrightarrow{j'} & Y' \\ & \Big\downarrow k & & \\ X \xrightarrow{j} & Y, & & \end{array}$$

where all arrows are inclusions of hyperplane sections $Y, Y' \subset W$, a $(2m+2+k)$-fold $W \subset \mathbb{P}^{N+1+k}$. By construction one has $j^*[L] = [M]$. Put $d = \deg(Y)\cdot[L] - [H^{m+2} \cap Y]$. By Hypothesis (3) of the theorem and Lefschetz's hyperplane theorem $h^{m,m}(Y) = h^{m,m}(X) = 1$ and hence $d \in \mathrm{CH}^{\mathrm{hom}}_m(Y)$. Clearly $j^*d = c$. The smooth hyperplane section X' of Y contains L and hence L defines a class $[L] \in \mathrm{CH}_m(Y')$ supported on X' and so $d = k_*[d']$ for some $d' \in \mathrm{CH}^{\mathrm{lin}}_m(X') \cap \mathrm{CH}^{\mathrm{hom}}_m(X')$ and since $L' \cap X' = L$, for $e' = \deg(Y')[L'] - [H^{m+1} \cap Y'] \in \mathrm{CH}_{m+1}(Y')$ one has $(j')^*e' = d'$.[c] The main result of the previous section, Proposition 13 can then be applied to the class $d' \in \mathrm{CH}^{\mathrm{hom}}_m(X')$ and to the inclusion $j' : X' \hookrightarrow Y'$ (instead of j). Hence

$$\alpha \circ {}^{\mathsf{T}}\alpha(d') = rd'.$$

The varieties X and X' fit in a pencil of hyperplanes of $\mathbb{P}^{N+1}$ passing through a certain linear subspace $L_{N-1} \subset \mathbb{P}^{N+1}$ of codimension 2 and parametrized by a line $\mathbb{P}^1 \subset \mathbb{P}^{N+1}$ disjoint from L_{N-1}. The base locus $B = Y \cap L_{N-1}$ is smooth and $\tilde{Y} = \{(y,t) \in Y \times \mathbb{P}^1 \mid y \in X_t\}$ is the blow-up of Y in B with blow-down map $\sigma : \tilde{Y} \to Y$ induced by the second projection. The first projection $p : \tilde{Y} \to \mathbb{P}^1$ is the pencil. It gives a natural embedding of X into $\tilde{Y}$ as some fiber of p. For the smooth members $X_t, t \in U$ of the pencil the property $\mathrm{GHC}_m(X_t)$ holds. Hence there is a relative correspondence $\alpha_U \in \mathrm{Corr}(C_U, \tilde{f}^{-1}(U))$ extending to $\tilde{\alpha} \in \mathrm{Corr}(C_{\mathbb{P}^1}, \tilde{Y})$. Let $q : C_{\mathbb{P}^1} \to \mathbb{P}^1$ be the natural projection. By construction

$$\tilde{\alpha} \circ {}^{\mathsf{T}}\tilde{\alpha} | X_{p(y)} = \alpha_{q(y)} \circ {}^{\mathsf{T}}\alpha_{p(y)}.$$

Denote the proper transform in $\tilde{Y}$ of a class of a cycle on Y by placing a tilde over the class. The above equation then shows:

$$\tilde{\alpha} \circ {}^{\mathsf{T}}\tilde{\alpha}(\tilde{d}) = r\tilde{d}, \quad d = k_*d'. \tag{7}$$

Composing $\tilde{\alpha}$ with the correspondence from $\sigma : \tilde{Y} \to Y$ given by the blow down morphism one obtains $\tilde{\alpha}^{(1)} = \sigma_* \circ \tilde{\alpha} \in \mathrm{Corr}_m(C_{\mathbb{P}^1}, Y)$. Composing it with the correspondence from $\tilde{Y}$ to B defined by inclusion of the exceptional divisor $E \hookrightarrow \tilde{Y}$

[c]Since $h^{m+1,m+1}(Y') \neq 1$ in general, there is no *a priori* reason why e' should be homologous to zero; one can only say that the class is primitive.

followed by $\sigma|E$, one gets the correspondence $\tilde{\alpha}^{(2)} \in \mathrm{Corr}_{m-1}(C_{\mathbb{P}^1}, B)$. The induced homomorphisms

$$\begin{aligned} \tilde{\alpha}^{(1)} &: \mathrm{CH}_0(C_{\mathbb{P}^1}) \to \mathrm{CH}_m(Y) \\ \tilde{\alpha}^{(2)} &: \mathrm{CH}_0(C_{\mathbb{P}^1}) \to \mathrm{CH}_{m-1}(B) \end{aligned}$$

come from the decomposition

$$\mathrm{CH}_m(\tilde{Y}) \cong \mathrm{CH}_m(Y) \oplus \mathrm{CH}_{m-1}(B). \tag{8}$$

On the other hand, any class in $\mathrm{CH}_m(\tilde{Y})$ such as $\tilde{d}$ can be decomposed according to the decomposition (8): $\tilde{d} = d + s$ and (7) implies

$$rj^*d = j^* \circ \tilde{\alpha}^{(1)} \circ {}^{\mathsf{T}}\tilde{\alpha}^{(1)} d + j^* \circ \tilde{\alpha}^1 \circ {}^{\mathsf{T}}\tilde{\alpha}^{(2)} s.$$

Put $u = {}^{\mathsf{T}}\tilde{\alpha}^{(1)} d \in \mathrm{CH}_0(C_{\mathbb{P}^1})$. Since d is supported on X', also $d' := \tilde{\alpha}(u) \in \mathrm{CH}_m(\tilde{Y})$ is supported on X' so that $j^*\sigma_* d'$ is supported on $X' \cap X = B$. In other words — with $i : B \hookrightarrow X$ the inclusion — one can write

$$j^*\sigma_* d' = i_* s', \quad r' \in \mathrm{CH}_{m-1}(B). \tag{9}$$

This means that

$$d' = \sigma_* d' + s'$$

is the decomposition of d' according to (8). Hence (9) translates into

$$j^* \tilde{\alpha}^{(1)}(u) = i_* \tilde{\alpha}^{(2)}(u), \quad u = {}^{\mathsf{T}}\tilde{\alpha}^{(1)} d$$

and hence

$$rj^*d = i_* \circ \tilde{\alpha}^{(2)} \circ {}^{\mathsf{T}}\tilde{\alpha}^{(1)} d + j^* \circ \tilde{\alpha}^1 \circ {}^{\mathsf{T}}\tilde{\alpha}^{(2)} s.$$

One deduces:

Observation. Let $d \in \mathrm{CH}_m^{\mathrm{hom}}(Y)$ as above. Then the $(m-1)$-cycle j^*d on X is rationally equivalent to zero provided

$$\tilde{\alpha}^{(2)} \circ {}^{\mathsf{T}}\tilde{\alpha}^{(1)} | \, \mathrm{CH}_m^{\mathrm{hom}}(Y) = 0 \tag{10}$$

$$\tilde{\alpha}^{(1)} \circ {}^{\mathsf{T}}\tilde{\alpha}^{(2)} | \, \mathrm{CH}_{m-1}^{\mathrm{hom}}(B) = 0. \tag{11}$$

Hypothesis (1) of Theorem 15 guarantees that (10) holds. Indeed, let B be the base locus of the pencil of hyperplane section on Y. One has the inclusions

$$B \hookrightarrow Y = W \cap \mathbb{P}^{N+1} \hookrightarrow W.$$

Consider the $(k+1)$-plane P of codimension $k+1$ linear sections sections of W through B. This gives a family of n-folds parametrized by P, all of which are in the same family as X:

$$\begin{array}{ccc} \alpha_P/P & \longrightarrow & \mathrm{Bl}_B W \\ \big\downarrow & & \\ C_P/P & & \end{array}$$

and hence

$$\begin{array}{ccc} \mathrm{CH}^{\mathrm{hom}}_m(W) & \xrightarrow{\tilde\alpha_P^{(2)}\circ{}^{\mathsf T}\tilde\alpha_P^{(1)}} & \mathrm{CH}^{\mathrm{hom}}_{m-1}(B) \\ \uparrow & & \| \\ \mathrm{CH}^{\mathrm{hom}}_m(Y) & \xrightarrow{\tilde\alpha^{(2)}\circ{}^{\mathsf T}\tilde\alpha^{(1)}} & \mathrm{CH}^{\mathrm{hom}}_{m-1}(B). \end{array}$$

This diagram together with $\mathrm{CH}^{\mathrm{hom}}_m(W) = 0$ implies that $\tilde\alpha^{(2)}\circ{}^{\mathsf T}\tilde\alpha^{(1)} = 0$.

Now (11) follows from (10), using the other assumptions. Indeed, since $\mathrm{CH}^{\mathrm{hom}}_0(Y) = 0$, by Corollary 3 there is a smooth variety V of dimension $2m$, and a $\beta \in \mathrm{Corr}_{-1}(Y, V)$ such that ${}^{\mathsf T}\beta\circ\beta|\,\mathrm{CH}^{\mathrm{hom}}_m(Y) = \mathrm{id}$. Consider the correspondence $\alpha = \tilde\alpha^{(2)}\circ{}^{\mathsf T}\tilde\alpha^{(1)}\circ{}^{\mathsf T}\beta \in \mathrm{Corr}_0(V, B)$. This correspondence acts trivially on $\mathrm{CH}^{\mathrm{hom}}_{m-1}(V)$ by assumption. Hence, by the reflexive vanishing property for B, the transpose $\beta\circ\alpha^{(1)}\circ{}^{\mathsf T}\tilde\alpha^{(2)}$ acts trivially on $\mathrm{CH}^{\mathrm{hom}}_{m-1}(B)$. One concludes that $\tilde\alpha^{(1)}\circ{}^{\mathsf T}\tilde\alpha^{(2)}|\,\mathrm{CH}^{\mathrm{hom}}_{m-1}(B) = 0$. □

6. Complete Intersections with Small Chow Groups

6.1. *Complete intersections of very small multidegree*

For complete intersections X one proceeds as follows. As explained at the start of Sec. 3.2, given any linear subspace L contained in X, a generic complete intersections of sufficiently small multidegree Lemma 11 guarantees the Assumption 17. By Proposition 8, one has $\mathrm{CH}_k(X) = \mathrm{CH}^{\mathrm{lin}}_k(X)$, $k \le m$ for sufficiently small multidegrees. The same multidegrees work also for Y. So Theorem 15 can be applied. Moreover at the same time for very small multidegrees conditions (1) and (2) of this Theorem will be verified. So one is left with conditions (3) and (4).

In the special case $m = 1$, i.e. X is a threefold, one needs to see that $\mathrm{CH}^{\mathrm{hom}}_2(Y) = 0$. This follows from Corollary 4 applied to Y and the fact that $H^3(Y) = 0$ by the Lefschetz hyperplane theorem. The latter also implies $H^1(B) = 0$ and $b_2(X) = 1$. So, for complete intersection threefolds the situation further simplifies:

Corollary 18. *Let Y be a generic smooth complete intersection four-fold and let X be a smooth hyperplane section. Suppose that*

(1) $\mathrm{CH}^{\mathrm{hom}}_0(Y) = 0$;

(2) $\mathrm{GHC}_1(X)$ *holds*;

(3) $h^{3,0}(X) = 0$.

Then $\mathrm{CH}^{\mathrm{hom}}_0(X) = 0$.

It is easy to see that this does not give new cases since Proposition 8 for 0-cycles is Roitman's theorem which is optimal. For weighted complete intersections the situation is more fortunate.

6.2. *Weighted complete intersections*

For a complete intersection of multidegree $(d_1, \ldots, d_r)$ in $\mathbb{P}(a_0, \ldots, a_{n+r})$ the canonical bundle K is given by the formula $K = \mathcal{O}(\sum d_j - \sum a_i)$ and so X is of general type as soon as $\sum d_j - \sum a_i > 0$, even if the canonical line bundle has no sections. For *quasi-smooth* complete intersections Steenbrink has shown [18] that the cohomology groups carry pure Hodge structures so that there is an intermediate Jacobian, an Abel–Jacobi map etc. Moreover, since such a complete intersection is the quotient of a smooth complete intersection in $\mathbb{P}^{n+r}$, working with rational coefficients, there is a good intersection theory for the Chow groups (see [6]). Proposition 9 shows that the various Chow groups vanish for quasi-smooth complete intersections of very small degrees.

We expect that Corollary 18 also holds in the weighted situation. For the moment, I can at least show that everything works as it should in the following:

Example 19. See [15, Secs. 2.9–2.10] for the geometry of this example. Let $X = X_{6,6,6}$ be a generic complete intersection of type $(6,6,6)$ in $P = \mathbb{P}(2^4, 3^3)$. One can take for $Y = Y_{6,6}$ a complete intersection of type $(6,6)$ in P and $W = P$.

The system $|\mathcal{O}(6)|$ embeds P in $\mathbb{P}^{25}$ as the linear join of the Veronese embedded $\mathbb{P}^3$, say $v_3(\mathbb{P}^3) \subset P(\mathcal{O}(3)) = \mathbb{P}^{19}$ and $\mathbb{P}^2$, say $v_2(\mathbb{P}^2) \subset P(\mathcal{O}(2)) = \mathbb{P}^5$. This description makes it quite easy to verify the (appropriate modification of) Assumption 17 in this case: one just has to makes judicious choices for Y, X', Y' as follows. Any point $p \in P \subset \mathbb{P}^{25}$ is on a line L joining two points $q = v_3(a)$ and $r = v_2(b)$. If $p \in X = P \cap H_1 \cap H_2 \cap H_3$, for generic choices of H_i the point r does not belong to X. The hyperplanes containing $H_1 \cap H_2 \cap H_3$ form a $\mathbb{P}^2$ and so one can find two independent hyperplanes H_1' and H_2' in this web passing through r. For generic choices $H_1' \cap H_2'$ will meet P transversally and hence $Y = P \cap H_1' \cap H_2'$ is quasi-smooth. By Remark 12, the general hyperplane H_3' through L will also meet Y transversally and this gives X'. Taking a generic point $s \in v_2(\mathbb{P}^2)$ for instance, the join with L is a plane $M \subset \mathbb{P}$ and one can find two independent hyperplanes H_1', H_2'' in the web containing $H_1' \cap H_2' \cap H_3'$ that pass through s. These contain M and, again by Remark 12 for generic choices $H_1'' \cap H_2''$ meets P transversally in a quasi-smooth Y'.

The three-fold X is of general type since $K_X = \mathcal{O}_X(1)$ but K_X has no sections and so $p_g(X) = 0$. The condition $\mathrm{GHC}_1(X)$ holds due to Ortland's constructions. In fact, any weighted homogeneous polynomial of degree 6 can be written $F = C(Y_1, Y_2, Y_3, Y_4) + Q(Z_1, Z_2, Z_3)$ where C is cubic in the first four variables Y_j and Q is quadratic in the remaining ones, Z_j. To $X = V(F_1, F_2, F_3)$ then is associated a web $F_\lambda = \sum_{i=1}^3 \lambda_i F_i$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{P}^2$. Whence a web of cubics C_λ in $\mathbb{P}^3$ and conics Q_λ in $\mathbb{P}^2$. The conics degenerate into a pair of lines (L, L') along a degree 3 curve $E \subset \mathbb{P}^2$ which is smooth for generic X and all cubic surfaces contain 27 lines L_k. The pairs (L, L_k), (L', L_k) define weighted planes $P' = \mathbb{P}(2,2,3,3) \subset P$ entirely contained in some F_λ and $P' \cap X$ is a curve of genus 2. The parameter space of those planes is a curve B which is generically $(54:1)$ onto E and whence a correspondence from B to X. Ortland has verified that for generic X the Abel–Jacobi map $J(B) \to J^1(X)$ is onto.

Also, $K_Y = \mathcal{O}_Y(-4)$, and hence Y is Fano. By Proposition 9, one sees that $\mathrm{CH}_0^{\mathrm{hom}}(Y) = 0$.

One could replace X by any toroidal resolution $f : \hat{X} \to X$. One still has $\mathrm{CH}_0^{\mathrm{hom}}(\hat{X}) = 0$, since the isolated singularities are replaced by trees of weighted projective spaces. The canonical bundle f^*K_X+ effective divisor remains big, and $p_g(\hat{X}) = 0$ as well so that $\hat{X}$ is a smooth threefold of general type giving an example where the Bloch conjecture is true.

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