On Two Types of Surfaces of General Type with Vanishing Geometric Genus

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Notation and Terminology

If S is a complex manifold we put

$\mathcal{O}_S$: the structure sheaf of S,

Θ_S: the sheaf of germs of holomorphic vector fields on S,

Ω_S^p: the sheaf of germs of holomorphic p-forms on S,

K_S: the canonical bundle of S (i.e. the bundle of holomorphic m-forms, $m = \dim_{\mathbb{C}} S$) or also the canonical divisor on S.

If F is any holomorphic vector bundle on S we let $\mathcal{O}_S(F)$ be the corresponding sheaf of germs of holomorphic sections. For any coherent sheaf G on S we put

$h^p(G) = \dim_{\mathbb{C}} H^p(S, G)$,

$\chi(G) = $ Euler characteristic of G.

We use the standard notations

$$p_g(S) = h^0(K_S) \qquad \text{(geometric genus)},$$
$$P_m(S) = h^0(K_S^{\otimes m}) \quad m \geq 2 \quad \text{(\textit{m}-th plurigenus)},$$
$$q(S) = h^0(\Omega_S^1) \qquad \text{(irregularity)}.$$

If $S \subset \mathbb{P}^N$ we abbreviate $\mathcal{O}_S(F \otimes H^l) = \mathcal{O}_S(F(l))$ for H the hyperplane bundle of $\mathbb{P}^N$.

If D is any divisor on S let $|D|$ be the corresponding linear system. If $|D| \neq \phi$ we can pick a basis $d_0, \ldots, d_n$ for $h^0(D)$ and define the rational map $\Phi_D: S \to \mathbb{P}^n$ by assigning to the point x the point $\Phi_D(x)$ in $\mathbb{P}^n$ with homogeneous coordinates $(d_0(x), \ldots, d_n(x))$. We abbreviate $\Phi_{mK} = \Phi_m$ and we say that S is of *general type* if Φ_m is birational onto its image for big m. Finally if S is a surface we let c_1^2 and c_2 be the degrees of the Chern classex c_1^2, resp. c_2 and $\text{Tor}(S)$ the torsion subgroup of the Neron Severi-group.

§ 0. Introduction

The classical criterion of Castelnuovo for the rationality of a complex algebraic surface S states that S is rational (i.e. birationally equivalent to $\mathbb{P}^2$) if and only if $P_2 = q = 0$. It is therefore natural to ask for the existence of non-rational surfaces with $p_g = q = 0$. Perhaps the oldest example is due to Enriques and this surface still bears his name. See Kodaira, [8] for more details. Here we only remark that c_1^2 is zero for this surface. Campedelli seems to be the first to give examples of non-rational surfaces with $p_g = q = 0$, but $c_1^2 > 0$, $P_2 \neq 0$. See [6]. From the general classification theory of Kodaira [8] it follows that such a surface is of general type. Later on Godeaux [7] constructed other such surfaces. Until very recently no other surfaces of general type with $p_g = q = 0$ were known.[1] The study of the torsion groups however led M. Reid to the construction of some new examples, see [10]. It turns out that his ideas can be applied fruitfully to compare the Campedelli surface and one of the Godeaux surfaces. Apart from giving a modern treatment of the surfaces in question this is the main purpose of this paper. We find that the Campedelli surface can be deformed into some Godeaux surface. We derive this from a more precise result concerning the deformations of the Campedelli surface (Theorem 6).

In order to motivate the computations for the Godeaux surface in question we constantly compare it with a much simpler Godeaux surface.

We give the relevant constructions in § 1. The multi-canonical mappings are studied in § 2. We confirm a conjecture of Bombieri, namely that Φ_3 is birational for the Campedelli surface. (See Bombieri, [2]). In § 3 we give the main result connecting the Campedelli and Godeaux surfaces. As a by-product we find the fundamental group of the Campedelli surface. In § 4 we study the deformations of the various surfaces.

§ 1. Three Examples

Example 1. We follow Bombieri, [1] Section 7. The group $\mathbb{Z}_5$ operates on $\mathbb{P}^3$ as follows. Let $\varepsilon = \exp(2\pi i/5)$, put

$$e(x_1, x_2, x_3, x_4) = (x_1, \varepsilon x_2, \varepsilon^2 x_3, \varepsilon^3 x_4)$$

This action has as fixed points precisely the coordinate vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$. Choose any e-invariant smooth surface S' of degree 5 not passing through the coordinate vertices. The group $\mathbb{Z}_5$ then operates freely and without fixed points on S', hence the quotient S is smooth. We have $p_g(S') = h^0(K_{S'}) = h^0(H_{S'}) = 4$ and $q(S') = 0$. Because $\chi(\mathcal{O}_{S'}) = 5\chi(\mathcal{O}_S)$, one has that $\chi(\mathcal{O}_S) = 1$. Now $q(S) = 0$, hence $p_g(S) = 0$. To calculate $c_1^2(S)$, observe that $\chi(\mathcal{O}_S) = \frac{1}{5}\chi(\mathcal{O}_{S'})$,

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and $c_2(S) = \frac{1}{5}c_2(S')$ — the latter being equal to the Euler-Poincaré characteristic of the surface. So Riemann-Roch gives that

$$c_1^2(S) = \frac{1}{5}c_1^2(S') = \frac{1}{5}(H_{S'} \cdot H_{S'}) = \frac{1}{5}\deg S' = 1.$$

Remark 1. M. Reid [10] points out how the construction of the Godeaux-surfaces is related to the torsion group. Let S be any surface with invariants $p_g = q = 0$, $c_1^2 = 1$, $\mathrm{Tor}(S) = \mathbb{Z}_5$. Denote the elements of $\mathbb{Z}_5$ by $0, 1, 2, 3$, and 4. Reid shows that $h^0(K_S + i) \neq 0$ $(i = 1, \ldots, 4)$. Choose non-zero sections $x_i \in H^0(K_S + i)$ $(i = 1, \ldots, 4)$, then any 5-th degree monomial in the x_i gives a section of $5K_S$. By calculation of the respective numbers one shows that there is exactly one relation between these monomials. If $\pi \colon Y \to X$ is the cyclic unbranched covering of degree 5 associated to $\mathbb{Z}_5$ and one denotes $y_i = \pi^*(x_i)$ then the y_i $(i = 1, \ldots, 4)$ form a basis for $H^0(Y, K_Y)$ and they define the birational map $\Phi_1 \colon Y \to \mathbb{P}^3$. The image is a 5-th degree surface Y_1 in $\mathbb{P}^3$. If Y is non-singular one recovers a Godeaux surface. We shall use such a construction in case of the Campedelli surface.

Example 2 (cf. Godeaux, [7]). The group $\mathbb{Z}_8$ operates in a similar way on $\mathbb{P}^6$ with fixed points the coordinate vertices. We put $G = \mathbb{Z}_8$. Let T be the complete intersection of 4 quadrics in $\mathbb{P}^6$ chosen in the following way:

(1) no quadric passes through the coordinate vertices

(2) the ideal generated by the 4 quadrics is G-invariant. As before the quotient T is a smooth surface and calculation of the invariants gives:

$$p_g(T) = q(T) = 0, \qquad c_1^2(T) = 2.$$

We shall show how to construct such a T explicitly. We choose 4 general quadrics F_1, F_2, F_3, F_4 such that $e(F_\nu) = \varepsilon^{2\nu - 1}$. F_ν $(\nu = 1, \ldots, 4)$, where $\varepsilon = \exp(2\pi i/8)$. We put $H = F_1 + F_2 + F_3 + F_4$. We take $H, e(H), e^2(H)$, and $e^3(H)$ for the four quadrics. Then (2) is trivially fulfilled. We observe that F_1 is a linear combination of

$$
\begin{array}{llll}
x_1^2, x_3 x_7, x_4 x_6 & \text{and} & x_5^2; & F_2 \text{ of} \\
x_2^2, x_1 x_3, x_5 x_7 & \text{and} & x_6^2; & F_3 \text{ of} \\
x_3^2, x_1 x_5, x_2 x_4 & \text{and} & x_7^2; & \text{finally } F_4 \text{ of} \\
x_4^2, x_1 x_7, x_2 x_6 & \text{and} & x_3 x_5.
\end{array}
$$

So it is possible to let all squares appear in H and hence in $e(H)$, $e^2(H)$ and $e^3(H)$, so that (1) is fulfilled. Finally we have the equality of ideals

$$(H, e(H), e^2(H), e^3(H)) = (F_1, F_2, F_3, F_4)$$

and it is easy to see that T is smooth in general.

Example 2 bis. A similar construction is possible for the groups $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. See Reid, [10]. Because we want to compare the Campedelli surface X with torsion $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ we give the construction of X explicitly. Denote the elements of $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by $000, 100, 010, 001, 011, 101, 110$ and 111. Any non-zero section of $H^0(K + 100)$ is denoted by x_1, similarly

$$x_2 \in H^0(K + 010), \qquad x_3 \in H^0(K + 001), \qquad y_1 \in H^0(K + 011), \qquad y_2 \in H^0(K + 101),$$
$$y_3 \in H^0(K + 110), \quad \text{and} \quad z \in H^0(K + 111).$$

All squares x_i^2, y_i^2 $(i = 1, 2, 3)$ and z^2 belong to $H^0(2K)$ and there must be 4 relations among them. Let $\pi: Y \to X$ be the G-cyclic covering of the surface in question, $\pi^* x_i = \xi_i$, $\pi^* y_i = \eta_i$ $(i = 1, 2, 3)$ and $\pi^* z = \zeta$. One finds that $H^0(K_Y)$ has the ξ_i, η_i, and ζ as a basis and the map Φ_1 gives a birational map of Y onto a complete intersection Y_1 of 4 quadrics in $\mathbb{P}^6$. This map is biholomorphic provided $|K_Y|$ has no base points and Y_1 is smooth. The group G acts on $\mathbb{P}^6$ as follows:

$$(100)(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \zeta) = (-\xi_1, \xi_2, \xi_3, \eta_1, -\eta_2, -\eta_3, -\zeta),$$
$$(010)(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \zeta) = (\xi_1, -\xi_2, \xi_3, -\eta_1, \eta_2, -\eta_3, -\zeta,$$
$$(001)(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \zeta) = (\xi_1, \xi_2, -\xi_3, -\eta_1, -\eta_2, \eta_3, -\zeta).$$

We see that the fixed set of G is 3-dimensional, hence X can be chosen disjoint from this set and smooth by a construction similar to the one in Example 2.

Our third example will be a double covering of the projective plane branched along a very specific 10-th degree curve.

First we need some preliminaries on blown-up planes. Let $P = \mathbb{P}^2$ and blow up P at the points $p_1, \dots, p_n$. Let P_1 be the resulting surface and let $\pi_1: P_1 \to P$ be the blowing-down map. We put $E_i = (\pi_1)^{-1}(p_i)$ $(i = 1, \dots, n)$ and we blow up P_1 at the point $l_i \in E_i$ $(i = 1, \dots, n)$. We obtain P_2 and a map $\pi_2: P_2 \to P_1$. Put $F_i = (\pi_2)^{-1}(l_i)$, $i = 1, \dots, n$ and denote the proper transform on P_2 of the curves E_i by the same characters. Finally put $\pi = \pi_1 \circ \pi_2$. Note the following intersection numbers on P_2.

$$\begin{aligned} &(E_i, E_i) = -2 \\ &(F_i, F_i) = -1 \\ &(F_i, E_i) = (\pi^* H, \pi^* H) = 1 \\ &(E_i, \pi^* H) = (F_i, \pi^* H) = 0. \end{aligned} \tag{1}$$

From *now on we use $\pi^* H$ and H interchangeably*. We shall frequently use the following easily verified formula for the canonical divisor K_2 on P_2:

$$K_2 \sim (-3H) + \sum_{i=1}^{n} (E_i + 2F_i). \tag{2}$$

Next, assume that we are given a curve $C' \subset P$ of degree $2m$ with certain singularities $p_1, \dots, p_n$. Suppose that it is possible to blow up P at $p_1, \dots, p_n$ and then P_1 at $l_1, \dots, l_n$ etc., obtaining a surface P_k such that

(1) the strict transform $\tilde{C} \subset P_k$ of C' is smooth,

(2) the divisor $\tilde{C}$ is linearly equivalent to $2F - \sum_{i=1}^{s} G_i$, where G_i are certain smooth curves which are all disjoint from $\tilde{C}$ and from each other.

Then we can construct a double covering Q of P_k having $C = \tilde{C} + \sum_{i=1}^{s} G_i$ as branch locus. This we see as follows. Let $\{U_\alpha\}$ be a coordinate covering of P_k such that C is given by the local equations $\{c_\alpha = 0\}$. Let $\{f_{\alpha\beta}\}$ be a system of transition functions for the bundle $[F]$ corresponding to F, then, because of (2) $c_\alpha = f_{\alpha\beta}^2 c_\beta$ on $U_\alpha \cap U_\beta$. Hence we find a submanifold $Q \subset [F]$ by the equations $\omega_\alpha^2 = c_\alpha$, where ω_α is a fibre coordinate of $[F]|U_\alpha$. Because of (1) and (2), Q is smooth and is a double cover of P_k, branched along C.

We apply this remark in case our singularities are triple points of type $(3, 3)$, i.e. the three branches have a common tangent, which separates into three different lines after blowing up once. Assume that $p_1, \ldots, p_n$ are such singularities. Blowing up P at these points we let $l_i \in E_i$ correspond to the common tangent of the branches of C at p_i and blowing up P_1 at these points we obtain P_2. One easily derives the following linear equivalence on P_2, connecting the total transform and strict transform of C':

$$\pi^* C' \sim \tilde{C} + \sum_{i=1}^{n} (3 E_i + 6 F_i)$$

Using (1) one sees that $(\tilde{C}, E_i) = 0$ $(i = 1, \ldots, n)$, hence C does not meet E_i and we may form the non-singular divisor $C = \tilde{C} + \sum_{i=1}^{n} E_i$. Let us define:

$$F = m H - \sum_{i=1}^{n} (E_i + 3 F_i). \tag{3}$$

One sees that $C \sim 2F$ and we may form the double covering Y' of P_2 branched along C. Let $\rho: Y' \to P_2$ be the branching map. The curves $\rho^{-1}(E_i) = A_i$ are rational and moreover $(A_i, A_i) = \frac{1}{4}(\rho^* E_i, \rho^* E_i) = \frac{1}{4} \cdot 2 \cdot -2 = -1$. Hence the A_i are exceptional and may be blown down. We let Y be the blown down surface and $\sigma: Y' \to Y$ the blowing down map. Now we are ready to give our third example, due to Campedelli, [6].

Example 3. One constructs a 10-th degree curve in P with six singularities of type $(3, 3)$ as follows. Let $\mathbb{C}^2 \subset P$ obtained from P by deleting the line at infinity. Take in $\mathbb{C}^2$ two concentric circles C_1 and C_2 together with an ellipse C_3 touching them both. Together with the two circle points at infinity $C_1 \cup C_2 \cup C_3$ has six points of type $(2, 2)$. It is easy to construct a 4-th degree curve C_4 touching this curve precisely at those six points. We let $C' = C_1 \cup C_2 \cup C_3 \cup C_4$. For later use we remark that *the six points do not lie on a conic*. The surface Y' constructed by the above procedure has *precisely* 6 exceptional curves and we let Y be as before. The surface Y is the Campedelli surface.

In order to explain why $p_g = q = 0$, $c_1^2 = 2$, one has to compare this example with the case where the curve has no singularities. We call this the corresponding *smooth case.* We shall prove below that in the smooth case corresponding to the Campedelli surface, one has $p_g = 6$, $q = 0$ and $c_1^2 = 8$. We shall see that throwing in a triple point of type $(3, 3)$ has the effect of lowering c_1^2 by 1. If the triple points are in general position, each time we insert a triple point p_g lowers 1 and q stays invariant. In our case "general position" exactly means that the six points $p_1, \ldots, p_6$ do not lie on a conic. Hence $p_g = 0$, $q = 0$, $c_1^2 = 2$. Now we give the details.

We need two formulas which hold in case of a double covering $\rho: Z \to X$ branched along a non-singular curve $C \subset X$ with $C \sim 2F$:

$$K_Z \sim \rho^* (K_X + F), \tag{4}$$

$$H^p(Z, \rho^*(G)) \cong H^p(X, G) \oplus H^p(X, G \otimes F^{-1}) \quad \text{for any vector bundle } G$$
on X and any integer p. $\tag{5}$

The first formula is easily verified using the construction of Z as given below formula (2). Formula (5) can be proved as follows: From the Leray spectral sequence it follows that $H^p(Z, \rho^* G) \cong H^p(X, \rho_* \rho^* G)$ and

$$\rho_* \rho^* G \cong G \otimes \rho_* \rho^* \mathcal{O}_X \cong G \otimes (\mathcal{O}_X \oplus \mathcal{O}_X(-F)).$$

The last equality comes from the splitting of $\rho_* \rho^* \mathcal{O}_X = \rho_* \mathcal{O}_Z$ in pieces invariant and anti-invariant under sheet-interchange.

From (2), (3) and (4) one sees ($m = 5$, $n = 6$):

$$K_{Y'} \sim \rho^* \left(2H - \sum_{i=1}^{6} F_i \right) \tag{6}$$

as a consequence one sees that $c_1^2(Y') = (K_{Y'}, K_{Y'}) = -4$, hence $c_1^2(Y) = -4 + 6 = 2$.

Now use (5) and (6) to compute

$$p_g(Y) = p_g(Y') = h^0 \left(\rho^* \left(2H - \sum_{i=1}^{6} F_i \right) \right).$$

We find that it is equal to $h^0 \left(P_2, 2H - \sum_{i=1}^{6} F_i \right)$, i.e. the number of quadrics through $p_1, \dots, p_6$, hence equals zero. Hence $p_g(Y) = 0$.

Finally one must compute

$$q(Y) = q(Y') = h^1(\mathcal{O}_{Y'}) = h^1(P_2, \mathcal{O}_{P_2}) + h^1(P_2, \mathcal{O}_{P_2}(-F)) = h^1(\mathcal{O}_{P_2}(-F)).$$

We now use

Lemma 1. $h^1(\mathcal{O}_{P_2}(-F)) = 0$.

Proof. By Serre-duality $h^1(\mathcal{O}_{P_2}(-F)) = h^1(\mathcal{O}_{P_2}(F + K_2)) = h^1 \left(2H - \sum_{i=1}^{6} F_i \right)$ by (2) and (3).

The exact sequence on P_2

$$0 \to 2H - \sum_{i=1}^{6} F_i \to 2H \to \bigoplus_{i=1}^{6} \mathcal{O}_{F_i} \to 0$$

gives in cohomology:

$$0 \to H^0 \left(2H - \sum_{i=1}^{6} F_i \right) \to H^0(2H) \to \mathbb{C}^6 \to H^1 \left(2H - \sum_{i=1}^{6} F_i \right) \to 0$$

Now $h^0 \left(2H - \sum_{i=1}^{6} F_i \right)$ is the number of conics through the six points $p_1, \dots, p_6$, hence equals zero. Note $h^0(2H) = 6$, so by the above sequence $h^1 \left(2H - \sum_{i=1}^{6} F_i \right) = 0$.

Corollary. $q(Y) = 0$.

So we have proved the following proposition.

Proposition 2. *For the Campedelli-surface Y (Example 3) we have* $p_g(Y) = q(Y) = 0$, $c_1^2(Y) = 2$.

Remark 2. This does *not* imply that Y is of general type. However, in § 2 we prove that $P_2(Y) = 3$, and then Kodaira's classification theory [8] implies that Y is of general type.

Remark 3. If there exists a 10-th degree curve with six points $(p_1, \ldots, p_6)$ of type $(3, 3)$ lying on a conic the same calculation shows that for the resulting double cover (after blowing down exceptional curves), we get $p_g = q = 1$, $c_1^2 = 2$. It is not clear whether such a curve exists.

Remark 4. One can prove that throwing in double points or triple points which resolve into double points or simple points after one blowing up do *not* change the invariants p_g, q, and c_1^2.

§ 2. The Multicanonical Mappings

We recall the results in Bombieri [1] for the case of the Godeaux surface of Example 1. The calculations will be entirely similar for the second example and will be given only for the surface with torsion group $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of Example 2 bis.

Theorem 1. *For the Godeaux surface S of Example 1 the following holds:*

$|2K_S|$ *has 4 base points,*

$|3K_S|$ *has 2 base points and Φ_3 is not holomorphic at those points.*

Φ_4 *is not biholomorphic at the 4 base points of $|2K_S|$.*

Proof. From Remark 1 one sees that the sections of mK_S are given by monomials $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ with $i_1 + 2i_2 + 3i_3 + 4i_4 \equiv 0 \bmod 5$. So a basis for $|mK_S|$ is represented by:

$$(x_1 x_4, x_2 x_3) \quad \text{for } m = 2,$$
$$(x_1 x_2^2, x_1^2 x_3, x_2 x_4^2, x_3^2 x_4) \quad \text{for } m = 3,$$
$$(x_1 x_3^3, x_1^2 x_4^2, x_1^3 x_2, x_1 x_2 x_3 x_4, x_2^2 x_3^2, x_2^3 x_4, x_3 x_4^3) \quad \text{for } m = 4.$$

Using this the assertions of the theorem can easily be verified. We refer to Bombieri, [1] for details.

Theorem 2. *For the Godeaux surface X of Example 2 bis the following holds:*

Φ_2 *is a holomorphic $8:1$ map onto $\mathbb{P}_2$ branched along 7 lines in general position.*

Φ_3 *is holomorphic and birational.*

Proof. From the way X is constructed we see that $H^0(X, mK_X)$ is generated by monomials

$$x_1^{n_1} x_2^{n_2} x_3^{n_3} y_1^{m_1} y_2^{m_2} y_3^{m_3} z^r \quad (n_1 + n_2 + n_3 + m_1 + m_2 + m_3 + r = m)$$

with

$$n_1 + n_2 + n_3 + r \equiv 0 (2),$$
$$n_1 + m_1 + n_2 + m_2 \equiv 0 (2),$$
$$n_1 + m_1 + n_3 + m_3 \equiv 0 (2).$$

We find the following explicit generators for low m:

$$x_1^2, x_2^2, x_3^2, y_1^2, y_2^2, y_3^2, z^2 \quad \text{for } m=2,$$

$$x_1 x_2 y_3, x_1 x_3 y_2, x_2 x_3 y_1, x_1 y_1 z, x_2 y_2 z, x_3 y_3 z, y_1 y_2 y_3 \quad \text{for } m=3.$$

Because 4 relations between the 7 squares Φ_2 maps X holomorphically onto the intersection $\mathbb{P}$ of 4 hyperplanes in $\mathbb{P}^6$ branched along the intersection of the 7 coordinate hyperplanes with $\mathbb{P}$. Because $P_2(X)=7$ there are no relations among the given generators for $H^0(X, 3K_X)$. One easily checks that Φ_3 is holomorphic and is everywhere $1:1$, hence Φ_3 is a holomorphic birational map.   Q.E.D.

Remark. If $\Phi_3(X)$ is non-singular it follows from the above proof that Φ_3 is biholomorphic. We have not checked this since the calculation it involves seems discouragingly large.

Theorem 3. *For the Campedelli surface Y we have*

(i) Φ_2 *is an* $8:1$*-holomorphic map onto* $\mathbb{P}^2$ *of which the set-theoretic branch locus consists of 7 lines. The position of these lines is as follows: three of the lines have their intersection points on three other lines and the 7-th line is arbitrary.*

(ii) Φ_3 *is a birational holomorphic map. On Y the 3 curves coming from the original quadrics forming part of the branch locus of $\rho: Y' \to \mathbb{P}_2$ are rational curves with self-intersection -2. They are the only such curves on Y and every Φ_m $(m \geqq 2)$ blows down these 3 curves into 3 normal rational double points.*

Proof. (i) We assume that projective coordinates (x, y, z) are chosen in such a way that

$$C_1 \equiv (x^2 + y^2 - r^2 z^2 = 0),$$
$$C_2 \equiv (x^2 + y^2 - R^2 z^2 = 0),$$
$$C_3 \equiv (R^2 x^2 + r^2 y^2 - R^2 r^2 z^2 = 0).$$

Hence the six points of type $(3,3)$ on C' are the points $(\pm r, 0, 1)$, $(0, \pm R, 1)$ and $(1, \pm i, 0)$. A basis for the 4-th degree curves touching the quadrics at these 6 points is given by

$$x^2 C_1, y^2 C_2 \text{ and } z^2 C_3.$$

Let $\tau: P \to P$ be the corresponding rational map, i.e.

$$\tau(x, y, z) = (x^2 C_1, y^2 C_2, y^3 C_3).$$

The map τ becomes well defined as a holomorphic map on P_2. In the following we thus let $\bar{\tau}: P_2 \to P$ be the holomorphic map corresponding to τ. By an easy computation one shows

(1) The map $\bar{\tau}$ is generically $4:1$.

(2) The set-theoretic banch locus of $\bar{\tau}$ consists of the three cordinate axes $u=0$, $v=0$, $w=0$ plus the lines l_1, l_2, l_3, see Fig. 1. Here (u, v, w) are projective coordinates of P, the target of $\bar{\tau}$.

A more accurate analysis shows that the proper transforms of C_1, C_2, and C_3 on P_2 map onto the points p_1, p_2 resp. p_3; the six curves F_i map in pairs onto the lines l_j and finally the curves E_i map onto the points q_j.
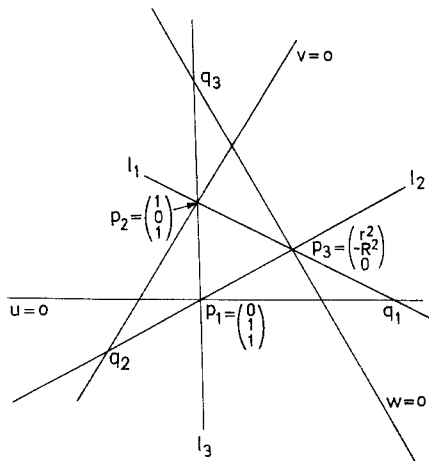
Fig. 1

In the rest of the proof we delete ρ^* in front of divisors on Y' coming from P_2.

To see the relation with Φ_2 we note that $2K_{Y'} \sim 4H - \sum_{i=1}^{6} 2F_i$ and the sections of the latter correspond in a $1-1$ way with the quartics touching the quadrics C_1, C_2, C_3 at the points $p_1, \ldots, p_6$. First of all this shows that $P_2 = 3$ and secondly one derives a commutative diagram

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ \rho\ } & P_2 \\[4pt]
\sigma \downarrow & \Psi \searrow & \downarrow \bar{\tau} \\[4pt]
Y & \xrightarrow[\Phi_2]{} & P .
\end{array}
\tag{7}
$$

Because $\Psi(A_i)$ is a point we see that Φ_2 is indeed holomorphic.

To verify the statement about the branch locus of Φ_2 we remark that the branch locus of Ψ consists of the branch locus of $\bar{\tau}$, shown in Fig. 1, and the image under $\bar{\tau}$ of the branch locus of ρ, hence consists of the points p_1, p_2 and p_3 and a general line in P.

(ii) To prove that Φ_3 is birational we must prove that Φ_3 separates points on a Zariski-dense set of Y. We can equally well prove that the three-canonical map of Y' has this property. Now $3K_{Y'} \sim 6H - \sum_{i=1}^{6} 3F_i$ and the sections of the latter are linear combinations of the ones invariant under sheet interchange and the ones anti-invariant under this operation. Recall the discussion above (5).

The invariant sections correspond to 6-tics having contact of order three with the conics C_1, C_2 and C_3 at the points $p_1, \ldots, p_6$. The anti-invariant sections correspond to sections of $3K_{Y'} - F = H + \sum_{i=1}^{6} F_i$, and we have $h^0\left(H + \sum_{i=1}^{6} F_i\right) = h^0(H) = 3$. We see that the invariant sections separate the fibres of the map ρ

away from the branch locus and because there are antiinvariant sections they separate the 2 points in the fibre. To see that Φ_3 is holomorphic, notice that $|3K|$ does not have base points, because the antiinvariant sections of $3K_{Y'}$ are sections of $H + \sum_{i=1}^{6} F_i$, hence do not have base points on Y'.

Let $\tilde{C}_i$ be the proper transform on P_2 of the curves C_i $(i = 1, 2, 3)$ and set $D_i = \rho^{-1}(\tilde{C}_i)$. Then one has on Y' that

$$(D_i, D_i) = \tfrac{1}{2}(\tilde{C}_i, \tilde{C}_i) = \tfrac{1}{2}((C_i, C_i) - 8) = -2.$$

Hence D_i $(i = 1, 2, 3)$ is a rational curve with self-intersection -2 and the same holds for $\sigma(D_i)$, $i = 1, 2, 3$. It is easy to see that on Y there are no other such curves.

In fact, any such curve D must be blown down to a point under the holomorphic map Φ_2 $((D \cdot 2K) = 0)$ and the discussion under (i) shows that only the D_i $(i = 1, 2, 3)$ are blown down to points, hence D must be one of them. The other assertions now follow from Mumford [9].

§ 3. The Torsion Group and a Godeaux-Type Construction for the Campedelli Surface

The object of this section is to analyse the torsion group of the Campedelli surface and from this to obtain another construction for it.

Lemma 2. *The torsion group of Y contains $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a subgroup.*

Proof. We number the curves E_i and F_i in such a way that those arising from blowing up the two points *not* lying on C_i $(i = 1, 2, 3)$ we call E_{2i}, E_{2i-1}, resp. F_{2i} and F_{2i-1}. We put $D_i = \rho^{-1}(C_i), (i = 1, 2, 3)$, $A_j = \rho^{-1}(E_j)$ and $B_j = \rho^{-1}(F_j) (j = 1, \dots, 6)$. On P_2 we have the linear equivalences:

$$\tilde{C}_i \sim 2H - \sum_{j \neq i}(E_{2j} + E_{2j-1} + 2F_{2j} + 2F_{2j-1}) \quad i = 1, 2, 3.$$

Notice that on Y' one has

$$\rho^* \tilde{C}_i \sim 2D_i, \quad \rho^* E_j \sim 2A_j, \quad \text{and} \quad \rho^* F_j \sim B_j$$

we find:

$$2D_i \sim 2\{H - \sum_{j \neq i}(A_{2j} + A_{2j-1} + B_{2j} + B_{2j-1})\} \quad i = 1, 2, 3.$$

Now suppose that

$$D_i \sim H - \sum_{j \neq i}(A_{2j} + A_{2j-1} + B_{2j} + B_{2j-1}) \quad i = 1, 2, 3.$$

Then $h^0(Y', D_i) \neq 0$ implies that

$$h^0\big(H - \sum_{j \neq i}(A_{2j} + A_{2j-1} + B_{2j} + B_{2j-1})\big) \neq 0.$$

But the last number equals

$$h^0\Big(H - \sum_{j \neq i} B_{2j} + B_{2j-1}\Big) = h^0\Big(Y', \rho^*\big(H - \sum_{j \neq i}(F_{2j} + F_{2j-1})\big)\Big)$$
$$= h^0\Big(P_2, H - \big(\sum_{j \neq i} F_{2j} + F_{2j-1}\big)\Big)$$

by formula (5). But this is the number of lines through the four points p_j lying on C_i, hence equals zero. This contradiction shows that

$$D_i - H + \sum_{j \neq i}(A_{2j} + A_{2j-1} + B_{2j} + B_{2j-1}) = g_i \qquad i = 1, 2, 3 \tag{8}$$

are non-zero $\mathbb{Z}_2$-torsion elements. Obviously the g_i are numerically independent, hence the g_i generate a $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ torsion subgroup of $\mathrm{Tor}(Y')$, hence of $\mathrm{Tor}(Y)$.

Lemma 3. $h^0(K + g_i) = 1$ $(i = 1, 2, 3)$, $h^0(K + g_i + g_j) = 1$ $(i \neq j, \ i, j = 1, 2, 3)$ and $h^0(K + g_1 + g_2 + g_3) = 1$.

Proof. On Y', $K + g_i = H + D_i + \sum_{j \neq i}(A_{2j} + A_{2j-1}) - (B_{2i} + B_{2i-1})$ as we see from (6) and (8). Because all D_i and A_j have negative self-intersection:

$$h^0\Big(H + D_i + \sum_{j \neq i}(A_{2j} + A_{2j-1}) - B_{2i} - B_{2i-1}\Big)$$
$$= h^0(H - (B_{2i} + B_{2i-1}))$$
$$= \text{the number of lines through the two points } p_{2i} \text{ and } p_{2i-1}, \text{ hence equals 1.}$$

The other equalities can be checked similarly. Q.E.D.

We enumerate the elements in G and in $H^0(Y, K + g)$ for $g \neq 0, g \in G$ as in example 2 bis. We let $\pi: Z \to Y$ be the cyclic unbranched covering of Y corresponding to G and we enumerate the sections of $H^0 Z, K_Z)$ corresponding to sections of $H^0(Y, K + g)$ for $g \neq 0, g \in G$ again in the same way as in example 2 bis.

Lemma 4. $p_g(Z) = 7$, $q(Z) = 0$.

Proof. Because $\chi(\mathcal{O}_Z) = 8$ we only need to prove that $q(Z) = 0$. But $q(Z) = h^1(\mathcal{O}_Z) = h^1(Y, \pi_* \mathcal{O}_Z) = \sum_{g \in G} h^1(Y, \mathcal{O}_Y(g))$.

Now for $g \neq 0$, $h^1(Y, \mathcal{O}_Y(g)) = 0$. Indeed $h^0(Y, \mathcal{O}_Y(g)) = 0$, because g is a torsion element, $h^2(Y, \mathcal{O}_Y(g) = h^0(Y, \mathcal{O}_Y(K_Y - g)) = h^0(Y, K_Y + g) = 1$ by Lemma 3. Then $h^1(Y, \mathcal{O}_Y(g)) = 0$ by the Riemann Roch formula.

Corollary. *The elements* $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \zeta$ *form a basis of* $H^0(Z, K_Z)$. *The canonical ring* $R = \sum_m H^0(Z, m K_Z)$ *of* Z *is generated by these elements. There are four relations between the squares* $\xi_1^2, \xi_2^2, \xi_3^2, \eta_1^2, \eta_2^2, \eta_3^2$ *and* ζ^2 *and any other relation in* R *comes from these. Hence* $\mathrm{Spec}(R) = \Phi_1(Z) = \bar{Z}$ *is a complete intersection of 4 quadrics in* $\mathbb{P}^6$ *of the form* $\sum_{i=1}^{3} a_i \xi_i^2 + \sum_{i=1}^{3} b_i \eta_i^2 + c \zeta^2$. *Moreover* $\bar{Z}$ *has 24 rational double points, namely the points coming from the curves* $\sigma(D_j)$ $(j = 1, 2, 3)$ *on* Y.

Proof. Because $p_g(Z)=7$ and the elements ξ_i $(i=1,2,3)$ and ζ are independent, they form a basis of $H^0(Z,K_Z)$. Because $P_2(Y)=3$ and $x_1^2, x_2^2, x_3^2, y_1^2, y_2^2, y_3^2, z^2$ are all sections of $2K_Y$ there must be 4 relations between them, hence there are 4 relations between the corresponding elements of $H^0(Z,2K_Z)$. Because $P_2(Z)=\chi(\mathcal{O}_Z)+\frac{1}{2}\cdot 2\cdot 1\cdot c_1^2(Z)=8+16=24$ and there are $\binom{6+2}{2}=28$ quadratic monomials generated by the ξ_i $(i=1,2,3)$, η_j $(j=1,2,3)$ and ζ these 4 relations are the only ones. Likewise one may prove that all the relations between higher order monomials come from these 4 ones. This proves that Spec $R=\Phi_1(Z)$ and the assertion about the double points now follow from Mumford [9].

This corollary shows that we have proved the following theorem:

Theorem 4. *The Campedelli surface Y has $G=\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ in its torsion group. It can be realized as follows. Let Y_1 be the surface Y with the three curves D_i $(i=1,2,3)$ blown down. The group G acts on $\mathbb{P}^6$ (see Example 2) and Y_1 is the quotient under G of an intersection of 4 quadrics in $\mathbb{P}^6$ each of which is G-invariant (i.e. a linear combination of the quadrics of the homogeneous coordinates in $\mathbb{P}^6$) and does not meet the fixed locuses of the generators of G. The intersection of the 4 quadrics has precisely 24 ordinary double points and no other singular points and they are distributed over three G-orbits.*

Corollary. $\pi_1(Y)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$, hence $\mathrm{Tor}(Y)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$.

Proof. We prove that Z is simply connected. First of all there is a family of deformations $\{\bar{Z}_t\}(t\in\Delta)^2$ such that for $t\neq 0$, $\bar{Z}_t$ is a complete intersection of 4 quadrics in $\mathbb{P}^6$ without singular points and $\bar{Z}_0\approx\bar{Z}$. Now one applies the results of Brieskorn [4] to see that there is a family of deformations $\{Z_t\}$ $(t\in\Delta)$ such that each Z_t is a minimal resolution $\bar{Z}_t$, i.e. $Z_t\approx\bar{Z}_t$ for $t\neq 0$ and $Z_0\approx Z$. This means that Z_t and Z are the same topologically, hence $\pi_1(Z)=\pi_1(Z_t)=0$.

§4. Deformations of Godeaux Surfaces and Campedelli's Surface

Theorem 5. *The Godeaux surfaces with torsion groups $\mathbb{Z}_5$, $\mathbb{Z}_8$, $\mathbb{Z}_4\oplus\mathbb{Z}_2$ and $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ all have smooth local moduli spaces. The first one has dimension 8, the last three have dimension 6.*

Proof. Because the Godeaux surfaces are of general type $h^0(\Theta)=0$ for them. If we can prove that $h^2(\Theta)=0$ as well we find first of all that

$$h^1(\Theta)=-\chi(\Theta)=10\chi(\mathcal{O})-2c_1^2$$

hence equals 8 in the first case and 6 in the other three cases. Then the statement of the local moduli spaces is a well known consequence.

We shall prove that $h^2(\Theta)=0$ for the last three surfaces, the proof that $h^2(\Theta)=0$ for the first one being similar but simpler. If $T=Q_1\cap Q_2\cap Q_3\cap Q_4$ is the complete intersection of 4 quadrics $Q_i(i=1,2,3)$ in $\mathbb{P}^6$ the statement $h^2(T,\Theta_T)=0$ will

---

2 As usual Δ is a small disc around the origin in $\mathbb{C}$.

imply $h^2(\Theta)=0$ for the last three Godeaux surfaces. So we shall prove that $h^2(T,\Theta_T)=0$. Remark that dually one must prove that $h^0(\Omega^1_T\otimes K_T)=0$. We use the exact sequences:

$$0\to\overset{4}{\bigoplus}\mathcal{O}_T(-1)\to\Omega^1_{\mathbb{P}^6}(1)\,|\,T\to\Omega^1_T\otimes K_T\to 0$$

and

$$0\to\Omega^1_{\mathbb{P}^6}(1)\,|\,T\to\overset{7}{\bigoplus}\mathcal{O}_T\to\mathcal{O}_T(1)\to 0.$$

The first one gives that $h^0(\Omega^1_T\otimes K_T)=h^0(\Omega^1_{\mathbb{P}^6}(1)\,|\,T)$. From the second one we have the inequality:

$$h^0(\Omega^1_{\mathbb{P}^6}(1)\,|\,T)\leqq 7-7+h^1(\Omega^1_{\mathbb{P}^6}(1)\,|\,T).$$

This reduces the proof to showing that $h^1(\Omega^1_{\mathbb{P}^6}(1)\,|\,T)=0$. In fact we have a slightly more general vanishing theorem.

Lemma 5. $h^1(\Omega^1_{\mathbb{P}^6}(l)\,|\,T)=0$ for $l\neq -8$.

Proof. We put $Q_1\cap Q_2\cap Q_3=T_1$ and $Q_1\cap Q_2=T_2$. We have exact sequences of the form:

$$0\to\Omega^1_{\mathbb{P}^6}(k)\to\Omega^1_{\mathbb{P}^6}(k+2)\to\Omega^1_{\mathbb{P}^6}(k+2)\,|\,Q_1\to 0,$$
$$0\to\Omega^1_{\mathbb{P}^6}(m)\,|\,Q_1\to\Omega^1_{\mathbb{P}^6}(m+2)\,|\,Q_1\to\Omega^1_{\mathbb{P}^6}(m+2)\,|\,T_2\to 0,$$
$$0\to\Omega^1_{\mathbb{P}^6}(n)\,|\,T_2\to\Omega^1_{\mathbb{P}^6}(n+2)\,|\,T_2\to\Omega^1_{\mathbb{P}^6}(n+2)\,|\,T_1\to 0$$

and

$$0\to\Omega^1_{\mathbb{P}^6}(r)\,|\,T_1\to\Omega^1_{\mathbb{P}^6}(r+2)\,|\,T_1\to\Omega^1_{\mathbb{P}^6}(r+2)\,|\,T\to 0.$$

The first one and Bott's vanishing theorem (Bott, [3]) a special case of which states that $h^1(\Omega^1_{\mathbb{P}^6}(k))=0$ for $k\neq 0$, gives that $h^j(\Omega^1_{\mathbb{P}^6}(k)\,|\,Q_1)=0$ all k for $j\geqq 2$, but $h^1(\Omega^1_{\mathbb{P}^6}(k)\,|\,Q_1)=0$ for $k\neq -2$. Now we proceed by induction.   Q.E.D.

Theorem 6. *For the Campedelli surface Y one has that $h^2(Y,\Theta_Y)=0$, hence Y has a smooth moduli space of dimension 6. Three dimensions come from the fact that the construction of the Campedelli surface depends on 3 parameters. The remaining three parameters come from independently letting vanish the three curves $D_i(i=1,2,3)$.*

Proof. We closely follow the computation of the proof of Burns and Wahl, [5], Theorem (4.2). As before we let Z_1 be the intersection of 4 quadrics in $\mathbb{P}^6=P$ given by $F_i=0(i=1,2,3)$. Let Z be the minimal resolution of singularities. Note that G operates on Z and Z_1, the respective quotients being Y and Y_1. The group G also operates on $H^1(\Theta_Z)$ and $H^1(\Theta_{Z_1})$. For any G-vector space H, we denote by H_G the G-invariant subspace of it. Notice that $H^1_G(\Theta_Z)\cong H^1(\Theta_Y)$ and $H^1_G(\Theta_{Z_1})\cong H^1(\Theta_{Y_1})$.

We let p,q,r be the three singular points of Y_1 and we use the same characters for three points on Z_1 in the 3 different G-orbits. We have the following exact

sequences:

$$0 \to \Theta_{Z_1} \to \Theta_P | Z_1 \to N'_{Z_1} \to 0,$$

$$0 \to N'_{Z_1} \to \overset{4}{\underset{g \in G}{\bigoplus}} \mathcal{O}_{Z_1}(2) \to \bigoplus (\mathbb{C}_{g(p)} + \mathbb{C}_{g(q)} + \mathbb{C}_{g(r)}) \to 0,$$

$$0 \to \mathcal{O}_{\mathbb{P}^6} \xrightarrow[(F_1, F_2, F_3, F_4)]{} \overset{4}{\bigoplus} \tilde{I}(2) \to N'_{Z_1} \to 0,$$

$$0 \to \tilde{I}(2) \to \mathcal{O}_P(2) \to \underset{g \in G}{\bigoplus} (\mathbb{C}_{g(p)} + \mathbb{C}_{g(q)} + \mathbb{C}_{g(r)}) \to 0.$$

The meaning of these sequences is precisely analogous to the corresponding sequences in the proof of Burns and Wahl, Theorem (4.2) ([5]). We let G operate on the corresponding cohomology groups to get

$$h^1(\Theta_Y) = h_G^1(\Theta_{Z_1}) + 3 = h_G^0(N'_{Z_1}) - 3.$$

Here we use that $H^0(N'_{Z_1}) \to H^1(\Theta_{Z_1})$ is surjective because all locally trivial deformations of Z_1 take place in $\mathbb{P}^6$ for reasons similar to the ones given in Burns and Wahl, [5].

Then

$$h_G^0(N'_{Z_1}) = 4 h_G^0(\mathcal{O}_{Z_1}(2)) - 3 + 4 h_G^1(\tilde{I}(2))$$
$$= 9 + 4 h_G^1(\tilde{I}(2)).$$

Hence $h^1(\Theta_Y) = 6 + 4 h_G^1(\tilde{I}(2))$. The last sequence shows that $h_G^1(\tilde{I}(2)) = 3 - s$, where s is the rank of the 3×7 matrix formed by the values of the seven G-invariant squares at the three points p, q and r. Because p, q and r are in general position $s = 3$ and $h^1(\Theta_Y) = 6$. Then, by Riemann Roch $h^2(\Theta_Y) = 0$ and Y is not obstructed, in fact the local moduli space of Y is smooth and of dimension 6. The curves D_i can be independently made to vanish and contribute each 1 parameter to the moduli space, as follows from Burns and Wahl [5]. The remaining 3 parameters then obviously come from the construction of the Campedelli surface.   Q.E.D.

Corollary. *The Campedelli surface can be deformed into a Godeaux surface with* $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *as a torsion group.*

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