On the rank of non-rigid period maps in the weight one and two case

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1. Introduction

A variation of polarized Hodge structures over a quasi-projective smooth complex manifold S can be thought of as a holomorphic horizontal locally liftable map

$$f : S \to \Gamma\backslash D,$$

where Γ is the monodromy-group of the variation and D is a period domain (see section 2). In this note we find an upperbound on the rank of period maps which admit a non-trivial deformation in the case of weight one and two (our techniques only apply to these weights). See section 3 for precise results. Suffices to say that this upperbound is sharp and the bound can be attained using families of projective varieties.

2. Preparations

In order to make this note as self contained as possible, we shall recall a few relevant facts from [P]. First, we repeat the definition of a polarized Hodge structure of weight w. We start with

- A free $\mathbb{Z}$-module $H := H_{\mathbb{Z}}$ of rank N,
- A Hodge vector $\mathbf{h} = (h^0, h^1, \ldots, h^w) \in \mathbb{N}^{w+1}$ with $h^j = h^{w-j}$, $j = 0, \ldots w$ and $\sum_{j=1}^{w} h^j = N$,
- An integral $\mathbb{Z}$-bilinear form Q, which is $(-1)^w$-symmetric and which has signature $\sum_{j=1}^{w}(-1)^j h^j$.

A Hodge structure of weight w on H with Hodge vector $\mathbf{h}$ is a direct sum decomposition $H_{\mathbb{C}} := H \otimes \mathbb{C} = \oplus_{j=0}^{w} H^{j,w-j}$ with $H^{j,w-j} = \overline{H}^{w-j,j}$ and $\dim H^{j,w-j} = h^j$. The form Q polarizes this Hodge structure if

i. $Q(H^{j,w-j}, H^{w-k,k}) = 0$ for $j \neq k$.

ii. $(-1)^{j(j+1)/2}(\sqrt{-1})^{-w} Q(h, \bar{h}) > 0$ if $h \neq 0$.

The Hodge structures on H with Hodge vector $\mathbf{h}$ polarized by Q are parametrized by points of a period domain $D = D(\mathbf{h}, Q)$. It is a homogeneous domain for the action of the Lie group $G_{\mathbb{R}}$ of isometries (with respect to Q) of the vector space $H_{\mathbb{Z}} \otimes \mathbb{R}$. The domain D is open in its compact dual $\check{D}$, a projective variety homogeneous under the

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group G of isometries of Q acting on $H_{\mathbf{C}}$. So, if we fix a reference Hodge structure $F \in D$ and we let B be the isotropy group with respect to G we have the principal fibration

$$B \to G \to \check{D}.$$

The tangent bundle of $\check{D}$ is the associated bundle under the adjoint representation of B on Lie$G$/Lie$B$. To define the horizontal tangent bundle, observe that the choice of the reference Hodge structure F corresponding to the decomposition $\oplus_{i=0}^{w} H^{i,w-i}$ induces a weight zero Hodge structure on

$$\mathfrak{g} = \mathrm{Lie}G$$

by setting

$$\mathfrak{g}^{j,-j} = \{X \in \mathfrak{g} \mid XH^{i,w-i} \subset H^{i+j,w-i-j}\}.$$

The horizontal tangent space $T_F^{\mathrm{hor}}(\check{D})$ is given by Lie$B$ + $\mathfrak{g}^{-1,1}$/Lie$B$. There is an almost canonical identification

$$\iota : \mathfrak{g}^{-1,1} \xrightarrow{\cong} T_F^{\mathrm{hor}}(\check{D})$$

which one gets by sending $X \in \mathfrak{g}^{-1,1}$ to $\exp(tX) \cdot F|_{t=0}$. In the sequel we will always use this identification, e.g when we write down Lie-brackets of tangent vectors.

Suppose that we are given a quasi-projective smooth complex variety S and a representation $\sigma : \pi_1(S) \to G_{\mathbf{Z}}$ whose image, the monodromy group, is denoted by Γ. From it we can form a locally constant system H_S on S with fibres isomorphic to (H, Q). A polarized variation of Hodge structures of type $\mathbf{h}$ over S polarized by Q is given by a so-called <u>period map</u>, i.e. a holomorphic map $f : S \to \Gamma\backslash D$ which comes from a σ-equivariant holomorphic map $\tilde{f}$ from the universal cover $\tilde{S}$ to D which is horizontal i.e whose derivative sends tangents along $\tilde{S}$ to horizontal tangents along D, i.e for any $s \in S$, setting $F = \tilde{f}(s)$ we have:

> Using the identification ι, the subset $(d\tilde{f})T_s(\tilde{S})$ of $T_F^{\mathrm{hor}}(D)$ defines a subspace $\mathfrak{a}$ of $\mathfrak{g}^{-1,1}$.

In fact, it follows [C-T, Proposition 5.2] that $\mathfrak{a}$ is an abelian subspace.

Our next topic is the <u>curvature</u> of the natural $G_{\mathbf{R}}$-invariant metric $\langle\, , \rangle$ on D which on horizontal tangents is given by $\langle X, Y\rangle = -\mathrm{Trace}(XY^*)$. The asterisk means that one takes the transpose conjugate with respect to Q and the natural complex structure on $\mathfrak{g}$. For a proof of the next Lemma, see [P, section 1].

(2.1) **Lemma** *The holomorphic bisectional curvature tensor H at F evaluated on commuting non zero horizontal vectors X and Y of length one is equal to*

$$-\langle [X^*, Y], [X^*, Y]\rangle$$

and hence is non-positive.

Finally, we need to recall some facts related to (small) deformations of period maps.

(2.2) **Definition** *A deformation of a period map $f : S \to \Gamma\backslash D$ consists of a locally liftable horizontal map $\mathbf{f} : S \times T \to \Gamma\backslash D$ extending f in the obvious way.*

Every deformation of a period map f has its associated infinitesimal deformation $\delta \in H^0(f^*T(\Gamma\backslash D))$. Now, since $\mathbf{f}$ is itself horizontal, using [C-T, Proposition 5.2] again, it follows that any two vectors tangent to $S \times T$ at (s,t) map to two commuting (horizontal) tangents in the tangent space to $\Gamma\backslash D$ at $\mathbf{f}(s,t)$. So we can apply the curvature estimates not only to tangents which are images of tangents to S under the period map, but also to those tangents in $\Gamma\backslash D$ which correspond to values of the sections in $f^*T(\Gamma\backslash D)$ which are infinitesimal deformations of f. Indeed these values give certain tangent vectors to $\Gamma\backslash D$. We conclude that for all $s \in S$, $X \in T_s(S)$ and any infinitesimal deformation $\delta \in H^0(S, f^*T(\Gamma\backslash D))$ the holomorphic bisectional curvature $H(X, \delta(u))$ for the induced metric connection on $f^*T(\Gamma\backslash D)$ is non-positive. We now invoke

(2.3) **Lemma** *Suppose U, M are manifolds, $f : U \to M$ a holomorphic map and $\delta \in H^0(U, f^*(M)$. Fix a Riemannian metric g on M, inducing one on $f^*T(M)$ denoted by the same letter. Assume that*
(i) The function $G(u) := g(\delta(u), \delta(u)$ is bounded.
(ii) U does not admit bounded plurisubharmonic functions.
(iii) For all u, $X \in T_u(U)$ the holomorphic bisectional curvature $H(X,Y)$ of the metric connection ∇ for g in the directions X and $Y := \delta(u)$ is non-positive,
then
δ is a flat section and $H(X,Y) = 0$.

This follows immediately from the formula (we normalize so that X and Y have length one):

$$\partial_X \bar{\partial}_{\bar{X}} G|_u = g(\nabla_X(\delta), \nabla_X(\delta))|_u - H(X,Y).$$

From this lemma we can infer that the infinitesimal deformations are flat sections of the bundle $f^*\mathrm{End}T^{\mathrm{hor}}(\Gamma\backslash D)$ (see [P, Theorem 3.2]) and so, upon taking values at F, we get a subspace of $\mathfrak{g}^{-1,1}$. Recalling that $\mathfrak{a}$ corresponds to the full tangent space to S at a point of S, by lemma 2.1 this formula also shows :

(2.4) **Corollary** *The tangent space to deformations of a period map $f : S \to \Gamma\backslash D$ is contained in*

$$\mathfrak{b} := \{Y \in \mathfrak{g}^{-1,1} \mid [\, Y^*, \mathfrak{a}] = 0\}.$$

3. The results

Let us now introduce for any $Y \in \mathfrak{g}^{-1,1}$ the following notation

$\mathfrak{a}(Y) :=$ maximal abelian subspace $\mathfrak{a}'$ of $\mathfrak{g}^{-1,1}$ with $[Y^*, X] = 0 \ \forall X \in \mathfrak{a}'$.
$a(Y) := \dim \mathfrak{a}(Y)$
$a = a(\mathfrak{g}^{-1,1}) := \max a(Y)$ (maximum over $Y \in \mathfrak{g}^{-1,1}$, $Y \neq 0$).

Clearly $a(Y)$ is an upper bound for the rank of a period map which admits non trivial deformations in the direction of Y and so a bounds the rank of period maps deformable in any direction. Consequently, any period map of rank $\geq a+1$ has to be rigid.

In this note we determine the number a as a function of the Hodge numbers, but only for weight one and two. The result can be summarized as follows

(3.1) **Theorem** *(i) In weight one with Hodge vector* (g, g) *one has*

$$a = \frac{1}{2}g(g-1).$$

There exists a quasi-projective variation of rank a *which has exactly 1 deformation parameter.*
(ii) In weight two with Hodge vector (p, q, p) *we have*

$$a = \begin{cases} 1 & \text{if } p = 1 \\ q-1 & \text{if } p = 2 \\ (p-1)[\frac{1}{2}(q-1)] + \epsilon & \text{if } p \geq 3 \end{cases}$$

where $\epsilon = 1$ *if* q *is even and* $\epsilon = 0$ *if* q *is odd.*
There exists a quasi-projective variation of rank a *which has exactly 1 deformation parameter.*
(iii) Any period map having rank $\geq a+1$ *is rigid.*

Remark. The variations of rank a can all be constructed from 2-cohomology of projective families of smooth complex algebraic varieties. See the remark at the end of section 6.
Remark. Malcev's technique in principle only gives non-trivial bounds in weights one and two, because with this technique one cannot exploit the fact that the deformation tangent vectors commute with the tangent vectors to the base of the parameter space - after suitable identifications with endomorphisms of H. The method however works also for certain very degenerate sequences of Hodge numbers, e.g. in the even weight case if all Hodge numbers $h^{w-2j-1,2j+1}$ vanish. The result in this case is almost identical; one has to view the number a in the preceding theorem as a function of one, resp. two variables for weight one, resp two and substitute $g = h^{m,m-1}$, resp $p = h^{m-1,m+1}$, $q = h^{m,m}$ if the weight is $2m-1$, resp. $2m$.

Examples

- Any family of g-dimensional polarized abelian varieties having $\frac{1}{2}g(g-1)+1$ or more moduli is rigid.

- Any family of K3-surfaces or Enriques surfaces, whose period map has rank two or more is rigid.

Remark. Sunada in [S] considers holomorphic maps from a compact complex variety to a smooth compact quotient of a bounded symmetric domain by a discrete group. His results are formulated somewhat differently, but it covers the two cases of the preceding theorem, where D is a bounded symmetric domain (but Sunada's techniques need that S be projective and smooth). More recently Noguchi in [N] used techniques from hyperbolic geometry to arrive at the bounds of the two previously given Examples.

4. A variation of Malcev's theorem

In this section we derive our main technical tool, which is a variation of [C-K-T, Theorem 3.1].

(4.1) **Theorem** *Let $\mathfrak{g}$ be a the complexification of a real semi-simple Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Assume that there exists an ordering of the roots relative to some Cartan subalgebra such that complex conjugation maps the root space for a positive root α to the root space of $-\alpha$. Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{g}$ which is a direct sum of positive root spaces and let $\mathfrak{a}$, $\mathfrak{b}$ two abelian subspaces of $\mathfrak{s}$ such that $[\mathfrak{a}, \overline{\mathfrak{b}}] = 0$, where the bar denotes complex conjugation. Then $\mathfrak{s}$ contains two abelian subspaces $\lambda(\mathfrak{a}), \lambda(\mathfrak{b})$ which are direct sums of positive root spaces with* $\dim \lambda(\mathfrak{a}) = \dim \mathfrak{a}$, $\dim \lambda(\mathfrak{b}) = \dim \mathfrak{b}$ *and such that* $[\lambda(\mathfrak{a}), \overline{\lambda(\mathfrak{b})}] = 0$.

Proof: One has to modify the proof of [C-K-T, Theorem 3.1] slightly. Let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordering of the positive roots. We let X_j be a root vector for the root α_j. We can find a basis $\{A_1, \ldots, A_a\}$, resp. $\{B_1, \ldots, B_b\}$ of $\mathfrak{a}$, resp. $\mathfrak{b}$ such that

$$
\begin{aligned}
A_j &= X_{k_j} + \text{linear combin. of root vectors for roots} > \alpha_{k_j} \\
& \qquad\qquad 1 \le k_1 < \ldots < k_a \le n, \\
B_j &= X_{l_j} + \text{linear combin. of root vectors for roots} < \alpha_{l_j} \\
& \qquad\qquad n \ge l_1 > \ldots > l_b \ge 1,
\end{aligned}
$$

Since $0 = [A_i, A_j] = [X_{k_i}, X_{k_j}]+$ a linear combination of root vectors for roots $> \alpha_{k_i} + \alpha_{k_j}$, it follows that $[X_{k_i}, X_{k_j}] = 0$ and similarly we find that $[X_{l_i}, X_{l_j}] = 0$. Finally, since complex conjugation is assumed to reverse the sign of the roots, we find $[A_i, \overline{B_j}] = [X_{k_i}, \overline{X_{l_j}}]+$ root spaces belonging to roots $> \alpha_{k_i} - \alpha_{l_j}$ we can also conclude that $[X_{k_i}, \overline{X_{l_j}}] = 0$. In this last argument $k_i - l_j$ can become 0 and then the corresponding vector $[X_{k_i}, \overline{X_{l_j}}] = 0$ need not be a root vector, but possibly lies in the Cartan subalgebra. For given A_i and B_j this happens at most once and does not affect the argument. We take now for $\lambda(\mathfrak{a})$, resp. $\lambda(\mathfrak{b})$ the space spanned by the X_{k_i}, resp. X_{l_i}, i.e the space of the *leading root vectors*, resp. the *terminal root vectors*. ■

We apply Malcev's theorem (Theorem 4.1) to the real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ introduced in section 2. It is shown in [C-K-T, Section 5] that a Cartan subalgebra exists which is of Hodge type $(0,0)$ and that there exists an ordering of the roots such that for each $p>0$, resp $p<0$ the Hodge component $\mathfrak{g}^{p,-p}$ is a direct sum of root vectors of positive roots , resp. negative roots and the complex conjugate of a root vector in $\mathfrak{g}^{p,-p}$ belongs to $\mathfrak{g}^{-p,p}$ so that we can indeed apply Malcev's theorem with $\mathfrak{s}=\mathfrak{g}^{-1,1}$. Now $a(Y)$ is the dimension of the largest abelian subspace $\mathfrak{a}'$ consisting of vectors commuting with $Y^*=-\overline{Y}$. The previous theorem allows us to assume that Y is a root vector and so we obtain:

(4.2) **Corollary** *We have* $a:=\max a(Y)$ *(maximum over root vectors* $Y\in\mathfrak{g}^{-1,1}, Y\neq 0$*).*

5. Bounds for the rank of non-rigid period maps

We recall some conventions from [C-K-T]. If we choose any basis for $H\otimes\mathbb{C}$ we decompose it in blocks according to the Hodge decomposition, where blocks range from $(0,0)$ (left upper corner) to $(w,w), w=1$ or 2 (the lower right corner). A matrix A placed in block (p,q) is denoted by $A[p,q]$. E_{ij} denotes a matrix with 1 in position (i,j) and no other non zero entries. In the course of deriving an upper bound for $a(\mathfrak{g}^{-1,1})$ we repeat the computations from [C-K-T] for a good Hodge frame, a corresponding Cartan subalgebra and root vectors for $\mathfrak{g}^{-1,1}$.
We first compute a in the weight one case.

(5.1) **Lemma** $a=\frac{1}{2}g(g-1)$

Proof: There is a *Hodgeframe* for $H\otimes\mathbb{C}$, i.e. a basis of $H\otimes\mathbb{C}$ consisting of a basis $\{e_1,\dots,e_g\}$ for $H^{0,1}$ and its complex conjugate for $H^{1,0}$ such that the matrix for $\sqrt{-1}Q$ is equal to $I_g[0,1]-I_g[1,0]$. Introduce for $k=1,\dots g$ the diagonal matrices $Y_k:=E_{kk}[0,0]-E_{kk}[1,1]$. These form a basis of the Cartan subalgebra of $\mathfrak{g}$. The root vectors spanning $\mathfrak{g}^{-1,1}$ are the $\frac{1}{2}g(g+1)$ symmetric matrices $Y_{ij}=E_{ij}[1,0]+E_{ji}[1,0]$ since

$$[Y_{ij},Y_k]=(\delta_{ik}+\delta_{jk})Y_{ij}.$$

Now for every symmetric $g\times g$-matrix X the condition $[\,X[1,0],Y_{ij}]=0$ is equivalent to X having zero i-th row (and column) and zero j-th row (and column). If $i=j$ we find $\frac{1}{2}g(g-1)$ for the maximal dimension of spaces of such X. ∎

Now we treat the case of weight two. A Hodge frame, in this case consists of a basis for $H^{2,0}$, its conjugate for $H^{0,2}$ and a real basis for $H^{1,1}$ such that the matrix for Q has the form $-I_p[0,2]+I_q[1,1]-I_p[2,0]$. For our purposes however it is better to use a different frame. Starting from such a Hodge frame we modify the middle part, say $\{f_1,\dots,f_q\}$ as follows. If $q=2t$, we take $\{f_1+\sqrt{-1}f_{t+1},\dots,f_t+\sqrt{-1}f_{2t},\ f_1-\sqrt{-1}f_{t+1},\dots,f_t-\sqrt{-1}f_{2t}\}$. In this case $Q=-I_p[0,2]+M[1,1]-I_p[2,0]$, where $M=\begin{pmatrix}0_t & I_t\\ I_t & 0_t\end{pmatrix}$.
If $q=2t+1$ we do essentially the same except that we retain one real basis vector for

$H^{1,1}$ and it take as our last basis vector for $H^{1,1}$. This modifies M in the preceding formula slightly ; it becomes $M = \begin{pmatrix} 0_t & I_t & 0 \\ I_t & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

We have

(5.2) **Lemma** *If $q = 2t$ we have*

$$a = \begin{cases} 1 & \textit{if } p = 1 \\ q-1 & \textit{if } p = 2 \\ (p-1)(t-1)+1 & \textit{if } p \geq 3 \textit{ and } q \geq 2 \end{cases}$$

Proof: First observe that in case $p = 2$ the bound from [C-K-T] gives the result by subtracting off one from their bound, allowing for the extra deformation parameter. The other less trivial bounds are obtained as follows.

The diagonal matrices

$$Y_k(0) = E_{kk}[0,0] + E_{kk}[2,2],\ k = 1, \ldots, p$$

and

$$Y_k(1) = (E_k - E_{t+k})[1,1],\ k = 1, \ldots, t$$

give a basis for the Cartan subalgebra and the matrices

$$Y_{ij} = E_{ij}[1,0] + E_{ji}[2,1]M,\ i = 1, \ldots, p,\ j = 1, \ldots, q$$

give a basis for the root vectors in $\mathfrak{g}^{-1,1}$ since

$$[Y_k(j), Y_{ij}] = (\delta_{kj} - \delta_{t+j\ k})Y_{ij}.$$

The complex conjugate of Y_{ij} is equal to $\overline{Y}_{ij} = E_{ji}M[0,1] + E_{ij}[1,2]$ and if we have $X' = X[1,0] + X^T M[2,1] \in \mathfrak{g}^{-1,1}$, the condition that $[X', \overline{Y}_{ij}] = 0$ means that X has zeros in rows $i, t+i$ and column j except in the entry (i,j). In other words, the problem reduces to the abelian subspace problem for Hodge numbers $p-1,\ 2t-1,\ p-1$, and the main theorem of [C-K-T] tells us this maximum is $(p-1)(t-1)+1$ if $p \geq 3$ and it is of course zero if $p = 1$. Taking into account the possibly non-zero entry (i,j) yields the desired upper bound. ■

(5.3) **Lemma** *If $q = 2t+1$ we have*

$$a = \begin{cases} 0 & \textit{if } q = 1 \\ 1 & \textit{if } p = 1 \\ q-1 & \textit{if } p = 2 \\ (p-1)t & \textit{if } p \geq 3 \textit{ and } q \geq 3 \end{cases}$$

Proof: The only change with the previous case is that there is an extra element $Y_q = E_{q,q}[1,1]$ in the Cartan subalgebra which leads to extra root vectors

$$Y_{qi} = E_{qi}[1,0] + E_{iq}M[2,1],$$

as one can easily check. The new root vectors however do not change any of the computations we did in the case where q is even. ∎

6. Construction of non-rigid period maps of maximal rank

We introduce some basic variations.

1) A weight one variation.
We have the tautological variation $\mathbb{A}_g$ of weight one over $\mathfrak{h}_g$. If we take a torsion free subgroup Γ of finite index in $Sp_g\mathbb{Z}$ not containing -Id, this variation descends to a variation on $\Gamma\backslash\mathfrak{h}_g$, which quasi-projective by [B-B]. This variation we denote by $\overline{\mathbb{A}}_g$.

2) A variation of weight 2 with $p = 1$.
Let H be a lattice with form Q of signature $(2,q)$ and consider the tautological variation $\mathbb{B}_q$ of weight two over

$$B_q := \{[F] \in \mathbb{P}(H \otimes \mathbb{C}) \mid Q(F,F) = 0, Q(F,\overline{F}) > 0\}.$$

As in 1) this variation descends to a variation $\overline{\mathbb{B}}_q$ over a suitable quasi-projective smooth quotient of B_q.

3) A variation of weight 2 with Hodge numbers $\{p, 2q, p\}$.
Over

$$B_{p,q} = \{Z \in \mathbb{C}^{p,q} \mid \overline{Z}^T Z < I_{q,q}\}$$

there exists a variation of weight 2 and this also descends to a variation $\overline{\mathbb{B}}_{p,q}$ over a suitable quasi-projective quotient (see [C-K-T, Section 7]).

The construction of a variation realizing the bound in Lemma 5.1 is easy. One takes the variation $\overline{\mathbb{A}}_{g-1}$ and takes the direct sum with a constant Hodge structure with Hodge numbers $h^{0,1} = h^{1,0} = 1$. This actually has 1 deformation parameter (compare with the variation $\mathbb{A}_1$).

In case of weight two we use the following remark repeatedly. The tensorproduct of $\overline{\mathbb{A}}_1$ with a fixed weight one Hodge structure with Hodge vector $(1,1)$ gives a weight two variation with Hodge numbers $(1,2,1)$ over a smooth quasi-projective curve and his has 1 deformation parameter. Let us denote this variation with $\overline{\mathbb{B}}'$. The construction for the bound in Lemma 5.2 proceeds as follows. For $p = 1$ and $q \geq 2$ we take the variation $\overline{\mathbb{B}}'$ and take the direct sum with $q-2$ copies of the trivial Hodge structure of pure type $(1,1)$. For $p = 2$ and $q \geq 2$ we take $\overline{\mathbb{B}}_{q-2}$ which has a parameter space of dimension $q-2$

and Hodge numbers $\{1, q-2, 1\}$. Now take the direct sum with $\overline{\mathbb{B}}'$. In total we have a base of dimension $q-1$ and 1 deformation parameter. If $p \geq 3$ a similar construction applies: instead of $\overline{\mathbb{B}}_{q-2}$ one takes $\overline{\mathbb{B}}_{p-1,t-1}$ and then proceeds as before.
In case of odd q (Lemma 5.3) the constructions are similar. The last construction needs a modification: one starts with $\overline{\mathbb{B}}_{p-1,t}$ and takes the direct sum with a constant Hodge structure with Hodge numbers $\{1,1,1\}$. If we view it as a fibre of the 1-parameter variation $\mathbb{B}_1$ it is clear that also here we have an extra deformation parameter.

Remark All of these variations occur as variations on primitive 2-cohomology of projective families of smooth algebraic varieties. For the weight one variation this is trivial, and for $\mathbb{B}_q$ one can take families of K3-surfaces with Picard number q for $q \leq 19$ and products of these for higher values of q. For $\overline{\mathbb{B}}_{p,q}$ one can realize them using a generalized Prym construction [C-S].

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