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C. A. M. PETERS

The local Torelli-theorem. II : cyclic branched coverings

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The Local Torelli-Theorem.
II: Cyclic Branched Coverings (*).

C. A. M. PETERS (**)

Introduction.

In this part we continue our investigations in [6] concerning the local Torelli-theorem. Our calculations were inspired by a paper of Wavrik ([5]). For convenience we recall in § 1 some of his results. The conditions 1.6 are crucial for the calculations.

We consider two cases in more detail, e.g. 1) cyclic branched coverings of projective n-space $\mathbb{P}_n$, and 2) cyclic branched coverings of the Hirzebruch-surfaces Σ_n. In both cases some degree-like condition for the branch locus ensures that the conditions 1.6 are fulfilled.

In § 2 the local Torelli-theorem is reduced to some question about the branch locus. Whereas in case 1) this question is solved in a direct way, in case 2) we reduce it—by means of duality—to a question on linear series on the branch curve. The latter question is seen to be a consequence of a trivial fact on polynomials in two variables.

In this way we arrive at our main results, e.g. Theorem 3.1 and 3.4, which state that the local Torelli-theorem holds in the two cases mentioned (apart from some degree-like condition for the branch locus).

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1. – Preliminaries.

We shall frequently use notations and results from part I. We refer to them by placing a « I » in front of the figures. For convenience of the

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(**) Math. Inst. Leiden, Wassenaarse weg 80, Leiden, The Netherlands.

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reader we shall give the notations and crucial results (without proofs). See [6] for details.

We employ the following (partly standard) notations and conventions: If W is a complex manifold, V a submanifold of W and F any holomorphic vector bundle on W, we set:

$F|V$: the restriction of F to V;

moreover:

O_W: the trivial bundle on W;

T_W: the holomorphic tangent bundle on W;

Ω^d_W: the bundle of holomorphic d-forms on W;

K_W: the canonical bundle on W, i.e. Ω^n_W, where $n = \dim W$;

$N_{V/W}$: the normal bundle of V in W, i.e. the quotient bundle arising in the exact sequence

I(0.1) $$0 \to T_V \xrightarrow[i]{} T_W|V \xrightarrow[j]{} N_{V/W} \to 0 .$$

Dualising this we find ($p = \operatorname{codim}_W V$):

I(0.2) $$K_V \cong K_W \otimes \Lambda^p N_{V/W} \text{ (adjunction formula)}.$$

We shall often identify a holomorphic vector bundle with its sheaf of holomorphic sections.

In case $W = \mathbf{P}_n$ we drop the subscripts W in the notations O_W, T_W, etc. If $V \subset \mathbf{P}_n$ is a complete intersection of s hypersurfaces V_k of degree n_k ($k = 1, \ldots, s$) we put: $V = V(n_1, \ldots, n_s) = V(1, \ldots, s)$.

H denotes the hyperplane bundle on $\mathbf{P}_n$, and $F(k) = F \otimes H^k$. Note that the normal bundle of $V(1, \ldots, s)$ in $\mathbf{P}_n$ is isomorphic to $\bigoplus_{k=1}^{s} O_V(n_k)$, so by I(0.2):

I(0.3) $$K_V \cong O_V(n_1 + \ldots + n_s - n - 1).$$

Therefore we put $\lambda = \sum_{k=1}^{s} m_k - (n+1)$.

The submodule of the polynomial ring $\mathbf{C}[\xi_0, \ldots, \xi_n]$ consisting of homogeneous polynomials of degree k is denoted as σ_k. Notice that we may identify $H^0(\mathbf{P}_n, O(k))$ and σ_k after the choice of a fixed system of homogeneous coordinates on $\mathbf{P}_n$:

$$i_k\colon \sigma_k \xrightarrow{\sim} H^0(\mathbf{P}_n, O(k)) .$$

We abbreviate:

$\partial_k = \partial/\partial\xi_k F$ for any $F \in \mathbf{C}[\xi_0, \ldots, \xi_n]$;

CAM: connected compact complex manifold;

PAM: Kähler CAM;

PAM: projective manifold.

I(3.2) $\quad H^p(\mathbf{P}_n, T(k) = 0$ except for

(i) $p = n,\ k < -n-2$;

(ii) $p = 0,\ k > -2$;

(iii) $p = n-1,\ k = -n-1$.

The following results apply to the case where V and W are CAM's and $V \subset W$ is a smooth hypersurface.

I(1.9*a*) *If* $H^1([V]) = H^1(T_W|V) = H^0(T_V) = 0$ then *the Kuranishi space of* V *is smooth of dimension* $\dim H^1(T_V)$.

In order to explain the next results we observe the following. Sequence I(0.1) and the same sequence tensored with K_V can be placed in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & T_V \arrow[r] \arrow[d, "s"] & T_W|V \arrow[r] \arrow[d, "s"] & N_V \arrow[r] \arrow[d, "s"] & 0 \\
0 \arrow[r] & T_V \otimes K_V \arrow[r, "i\otimes 1"] & T_W|V \otimes K_V \arrow[r, "j\otimes 1"] & N_{V/W} \otimes K_V \arrow[r] & 0
\end{tikzcd}$$

where s is multiplication with $s \in H^0(K_V)$. In cohomology this gives rise to the obvious diagram a portion of which reads:

$$(*) \qquad \begin{tikzcd}
H^0(T_W|V) \arrow[r] \arrow[d, "\cup s"] & H^0(N_{V/W}) \arrow[d, "\cup s"] \\
H^0(T_W|V \otimes K_V) \arrow[r, "(j\otimes 1)_*"'] & H^0(N_{V/W} \otimes K_V) .
\end{tikzcd}$$

Now we can state the other two results we need:

I(2.3) *Assume W is a* CAKM *and $V \subset W$ is a smooth hypersurface. If the Kuranishi space of V is smooth and moreover $H^0(T_V) = H^0(T_V \otimes K_V) = H^1(W|V) = 0$,* then *Torelli holds locally if and only if the next condition is fulfilled:*

Suppose for $v \in H^0(N_{V/W})$ we have that $v \cup s \in \operatorname{Im}(j \otimes 1)_*$ for all $s \in H^0(K_V)$ then $s \in \operatorname{Im} j_*$.

I(5.2) (special case) *Let $W = \mathbb{P}_n$ and V a smooth hypersurface of degree m defined by $F = 0$. Notice that in $(*)$ $H^0(N_{V/W}) \cong H^0(O_V(m))$ and there is a natural surjection:*

$$H^0(O_{\mathbb{P}_n}(m)) \xrightarrow{r} H^0(O_V(m)) .$$

Using the isomorphism i_m, we have that the inverse image of $\operatorname{Im}(j_*)$ *under $r \circ i_m$ is the module generated by the partial derivatives of F, and likewise for* $\operatorname{Im}(j \otimes 1)_*$.

We now return to the situation we are interested in. Consider the following situation. Let W be a CAKM [1] of dimension m; G a holomorphic vector bundle on W and F a fixed line bundle on W. Form the projective line bundle F associated to $F \oplus O_W$. The section of $\hat{F}$, defined by the subbundle O_W of $F \oplus O_W$ is denoted as s_∞: Choosing some bundle isomorphism $\hat{F} \xrightarrow{\sim} \hat{F} \setminus s_\infty$ we may consider F as being embedded in $\hat{F}$, so that we obtain a commutative diagram:

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d, "\pi/F"] & \hat{F} \arrow[d, "\pi"] \\ W \arrow[r, equal] & W \end{tikzcd}$$

We shall frequently drop the subscript W in such notations as K_W, O_W, etc. Moreover we set $\hat{K} = K_{\hat{F}}$, $\hat{O} = O_{\hat{F}}$, etc. Finally, let L^{-1} be the tautologous subline bundle of $\pi^*(F \oplus 0)$, i.e. $L \cong [s_\infty]$ [2].

[1] We need the Kählerian structure not untill 1.9.
[2] As usual we set [...] to denote the line bundle corresponding to the divisor... .

Recall the following facts concerning the cohomology groups $H^p(\hat{F}, \pi^* G \otimes L^t)$, $t \in Z$, cf. [5], proposition 2.1 and 2.2,

LEMMA 1.1(*a*). *$H^p(\hat{F}, \pi^* G \otimes L^t) \cong \bigoplus_{s=0}^{t} H^p(W, G \otimes F^{-s})$ in case $t \geq 0$.*

(*b*) *$H^p(\hat{F}, \pi^* G \otimes L^{-t}) \cong \bigoplus_{s=1}^{t-1} H^{p-1}(W, G \otimes F^s)$ in case $t \geq 2$, $p \geq 1$. If $t = 1$ or $p = 0$ this group is zero.*

To state the next lemma we shoose a covering $\{U_\alpha\}_{\alpha \in A}$ of W such that $F|U_\alpha$ is trivial. We let x_α be the fibre coordinate of F above U_α: Choose homogeneous fibre coordinates $(\xi_\alpha : \eta_\alpha)$ of F above U_α in such a way, that with respect to the embedding $F \subset \hat{F}$ we have $x_\alpha = \xi_\alpha / \eta_\alpha$. Inspection of the proof of [5], Proposition 2.1 shows that in case $p = 0$ lemma 1.1(*a*) can be formulated thus:

LEMMA 1.1(*c*). *Let $\{U_\alpha\}$, ξ_α, η_α as above. If $\gamma \in H^0(\pi^* G \otimes L^t)$, $t \geq 0$, then locally on $\pi^{-1}(U_\alpha)$ we have*

$$\gamma_\alpha = \sum_{i=0}^{t} \gamma_{i,\alpha} \xi_\alpha^i \eta_\alpha^{t-i}$$

where $\{\gamma_{i,\alpha}\}_{\alpha \in A}$ forms a section γ_i of $H^0(G \otimes F^{-i})$ $(i = 0, \dots, t)$.

Conversely, any collection $\{\gamma_\alpha\}_{\alpha \in A}$ of this kind defines a unique section γ of $G \otimes L^t$.

Next, we need some well known facts about $\hat{T}$, which follow for instance from [2], § 13:

LEMMA 1.2. *Let T_π be the «bundle along the fibres of π», i.e. the kernel of the surjective bundle map $\hat{T} \xrightarrow[\pi_*]{} \pi^* T$. Then T_π fits naturally in the exact sequence*

$$0 \to L^{-1} \to \pi^*(F \oplus 0) \to T_\pi \otimes L^{-1} \to 0 .$$

COROLLARY 1.3. (i) $T_\pi \cong \pi^* F \otimes L^2$; (ii) $\hat{K} \cong \pi^*(K \otimes F^{-1}) \otimes L^{-2}$.

We shall now give a definition of a k-cyclic covering, which is adequate for our purposes. It follows from [5], theorem 1.1 that our definition is equivalent to the usual one, given in [5].

DEFINITION. *A map $\nu : V \to W$ is called a k-cyclic covering of W, if there exists a line bundle F on W and a holomorphic section $\Phi : W \to F^k$ which is transverse to the zero section of F^k such that moreover*

(i) *V can be identified with the inverse image of $\Phi(W)$ under the k-th power $\otimes^k : F \to F^k$.*

(ii) *There is a commutative diagram*

$$\begin{tikzcd} V \arrow[d, "\nu"] & \subset & F \arrow[d, "\pi"] \\ W & = & W \end{tikzcd}$$

where the inclusion $V \subset F$ *is obtained by means of the identification mentioned in* (i).

The set of zeros of Φ is the branch locus of ν, denoted as C. Because of transversality, C is a non-singular submanifold of W.

REMARK 1.4. From this definition it follows that, in terms of the covering $\{U_\alpha\}$, $V \subset \hat{F}$ is the submanifold given by $\Phi_\alpha \eta_\alpha^k - \xi_\alpha^k = 0$, where $\{\Phi_\alpha\}_{\alpha\in A}$ gives the section Φ of F^k. Hence $[V] \cong F^k \otimes L^k$ and by 1.1(*c*) the collection $\{\Phi_\alpha \eta_\alpha^k - \xi_\alpha^k\}_{\alpha\in A}$ defines a section v of $\pi^* F^k \otimes L^k$ corresponding to $(\Phi, 0, \dots, 0, -1)$ in $\bigoplus_{s=0}^{k} H^0(F^k \otimes F^{-s})$.

It will be convenient to collect the following statements in a lemma:

LEMMA 1.5. (i) $[V] \cong \pi^* F^k \otimes L^k$;

(ii) $[C] \cong F^k$;

(iii) $K_V \cong \pi^*(F^{k-1} \otimes K)|V$.

PROOF. Only part (iii) needs to be proved. The adjunction formula (I, 0.2) gives $K_V \cong \pi^*(F^{k-1} \otimes K)|V \otimes L^{k-2}|V$. Because $V \subset F$ and $L|F$ is trivial, we obtain (iii).

Our computations will be based on the following assumptions:

ASSUMPTIONS 1.6.

(*a*) $H^0(F^{-\alpha}) = 0$ *for* $\alpha > 0$;

(*a*)′ $H^1(F^\alpha) = 0$ *for all* α;

(*b*) $H^0(T \otimes F^{-\alpha}) = 0$ *for* $\alpha > 0$;

(*b*)′ $H^1(T \otimes F^{-\alpha}) = 0$ *for* $\alpha \geqslant 0$;

(*c*) $H^0(K) = 0$;

(*c*)′ $H^1(K \otimes F^\alpha) = 0$ *for* $\alpha \geqslant 0$;

(*d*) $H^0(T \otimes K) = 0$;

(*e*) F *and* $K \otimes F^{k-1}$ *are ample.*

As one observes from inspecting 1.1, 1.3 and 1.5 these assumptions imply the vanishing of certain cohomology groups on $\hat{F}$. We will denote this by the same characters.

LEMMA 1.7:

(a) $H^s(T_\pi \otimes [V]^{-1}) = 0$ *for* $s = 0, 1$ *and* $s = 2$ *if* $k > 2$;

(a)′ $H^1(\hat{O}) = H^1(T_\pi) = H^1([V]) = H^2([V]^{-1}) = H^2(\pi^* F \otimes [V]^{-1}) = 0$;

(b) *and* (b′) $H^s(\pi^* T \otimes [V]^{-1}) = 0$ *for* $s = 0, 1, 2$;

(c) $H^1(\hat{K}) = 0$ *for* $s = 0, 1, 2$;
$H^s(T_\pi \otimes \hat{K}) = 0$ *for* $s = 0, 1$;

(c)′ $H^1(\hat{K} \otimes [V]) = H^1(\pi^* K \otimes [V]) = 0$ *and* $H^1(\pi^* K \otimes [V] \otimes F^\alpha) = 0$ *for* $\alpha \geqslant 0$;

(d) $H^s(\pi^* T \otimes \hat{K}) = 0$ *for* $s = 0, 1$.

By means of the next two diagrams we may derive the vanishing of certain cohomology groups on V.

LEMMA 1.8. *On* $\hat{F}$ *we have two exact diagrams:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_\pi \otimes [V]^{-1} & \to & \hat{T} \otimes [V]^{-1} & \to & \pi^* T \otimes [V]^{-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_\pi & \to & \hat{T} & \xrightarrow{\pi_*} & \pi^* T & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_\pi | V & \to & \hat{T} | V & \xrightarrow{\pi_* | V} & \pi^* T | V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{1}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_\pi \otimes \hat{K} & \to & \hat{T} \otimes \hat{K} & \to & \pi^* T \otimes \hat{K} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_\pi \otimes \hat{K} \otimes [V] & \to & \hat{T} \otimes \hat{K} \otimes [V] & \to & \pi^* T \otimes \hat{K} \otimes [V] & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_\pi | V \otimes K_V & \to & \hat{T} | V \otimes K_V & \to & \pi^* T | V \otimes K_V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2}$$

PROOF. Diagram (2) can be obtained from (1) by means of tensoring with $\hat{K} \otimes [V]$ and noticing that $K_V \cong (\hat{K} \otimes [V])|V$.

Exactness of (1) can be seen as follows: the middle horizontal sequence is exact by definition (cf. 1.2); the first horizontal one is obtained on tensoring with $[V]^{-1}$ the second one, and, finally the last horizontal sequence is the restriction to V of the middle one; exactness of the vertical sequences is a standard fact.

PROPOSITION 1.9. *On V we have*

(*a*) $H^0(T_V) = H^1(O_V) = H^1(K_V) = H^0(T_V \otimes K_V) = 0$;

(*b*) $H^1(\hat{T}|V) = 0$.

PROOF. (*a*) $H^0(T_V)$ is dual to $H^n(\Omega^1_V \otimes K_V) = H^{1\,n}(K_V)$. Because $K_V = \pi^*(K \otimes F^{k-1})|V$ is ample by 1.6(*e*), a vanishing theorem due to Nakano ([4]) implies that $H^{1,n}(K_V) = 0$. Hence $H^0(T_V) = 0$. Because V is Kähler $\dim H^0(T_V \otimes K_V) = \dim H^0(\Omega^1_V) = \dim H^1(O_V)$, so, to complete the proof of (*a*) we need only to see that $H^1(O_V) = H^1(K_V) = 0$. This we do by means of the exact cohomology sequence of

$$(2A) \qquad 0 \to G \otimes V^{-1} \xrightarrow{\cdot v} G \to G|V \to 0$$

where we substitute $G = \hat{O}$, resp. $\hat{K} \otimes [V]$. In fact by (1.7) and the fact that K_V is the restriction to V of $\hat{K} \otimes [V]$ we may easily derive the vanishing of $H^1(O_V)$ and $H^1(K_V)$.

(*b*) Here we need to investigate the cohomology diagram of (1). We separate two cases

(1) $k = 2$

$H^2(\pi^* T \otimes [V]^{-1}) = 0$ (see 1.7(*b*)), so part of the cohomology diagram of (1) reads:

$$\begin{tikzcd} & & H^1(\pi^* T) \arrow[d] \\ H^1(T_\pi|V) \arrow[r] & H^1(\hat{T}|V) \arrow[r] & H^1(\pi^* T|V) \arrow[d] \\ & & 0 \end{tikzcd}$$

Now $H^1(\pi^* T) = H^1(T) = 0$ (1.6), so $H^1(\pi^* T|V) = 0$, and we need to show only that $H^1(T_\pi|V) = 0$. Because $T_\pi|V \cong \pi^* F|V$ (substitute $k = 2$ into 1.3), we must show $H^1(\pi^*F|V) = 0$. Use the cohomology sequence of (2A) with $G = F$ and remark that $H^1(F) = H^2(\pi^*F \otimes [V]^{-1}) = 0$ by 1.7. This establishes part (b) in case (1)

(2) $k > 2$

We have

$$H^1(T) = H^1(T_\pi) = H^2(\pi^* T \otimes [V]^{-1}) = 0\,,$$

and moreover $H^2(T_\pi \otimes [V]^{-1}) = 0$ in this case (cf. 1.7(a)). So, by the cohomology diagram of (1), $H^1(T) = H^2(\hat{T} \otimes [V]^{-1}) = 0$, and hence, by the same diagram, $H^1(\hat{T}|V) = 0$, as asserted.

THEOREM 1.10. *Under the assumptions* (1.6), *V has a modular variety of dimension* $\dim H^1(T_V)$, *i.e. the Kuranishi space for V is smooth.*

PROOF. Apply I.1.9(a); the vanishing of $H^1([V])$ is mentioned in 1.7(a)$'$, whereas $H^1(\hat{T}|V) = H^0(T_V) = 0$ by lemma 1.9. (Q.E.D.)

In the next section we reduce the proof of the local Torelli-theorem to a problem about bundles on the branch locus $C \subset W$. To this end we consider for $s = 2, \ldots, k$ the next diagram, of which the rows are exact:

$$\text{(3)}\qquad \begin{array}{ccccccccc} 0 \to & T_C \otimes F^{s-k}|C & \to & (T \otimes F^{s-k})|C & \xrightarrow[j_s^1]{} & ([C] \otimes F^{s-k})|C \to 0 \\ & \downarrow{\cdot\varphi} & & \downarrow{\cdot\varphi} & & \downarrow{\cdot\varphi} \\ 0 \to & T_C \otimes (F^{s-1} \otimes K)|C & \to & (T \otimes F^{s-1} \otimes K)|C & \xrightarrow[j_s^2]{} & ([C] \otimes F^{s-1})|C \to 0 \end{array}$$

where $\varphi \in H^0((F^{k-1} \otimes K)|C)$.

The conditions (1.6) imply that

$$\text{(3}A\text{)}\qquad H^0(T_C \otimes F^{s-k}|C) = H^0(T_C \otimes (F^{s-1} \otimes K)|C) = 0\,.$$

Indeed, $H^0(T_C \otimes F^{s-k}|C)$ is dual to $H^{n-1}(\Omega_C^1 \otimes K_C \otimes F^{k-s}|C)$, so, because $K_C \otimes F^{k-s}|C \cong (K \otimes F^{2k-s})|C$ is ample (1.6(e)), a vanishing theorem of Nakano [4] gives that this group is zero. A similar reasoning applies to $T_C \otimes (F^{s-1} \otimes K)/C$.

By (3A), the cohomology diagram of (3) reads:

$$\text{(4)}_s\qquad \begin{array}{ccccc} 0 \to H^0((T \otimes F^{s-k})|C) & \xrightarrow[j^1_{\infty *}]{} & H^0([C] \otimes F^{s-k}|C) \to & & H^1(T_C \otimes F^{s-k}|C) \\ \downarrow{\cup\varphi} & & \downarrow{\cup\varphi} & & \downarrow{\cup\varphi} \\ 0 \to H^0((T \otimes F^{s-1} \otimes K)|C) & \xrightarrow[j^2_{s*}]{} & H^0([C] \otimes F^{s-1} \otimes K|C) & \to & H^1) T_C \otimes (F^{s-1} \otimes K)|C)\,. \end{array}$$

We shall meet the next condition for $s = 2, \ldots, k$:

Suppose $\alpha_s \in H^0([C]\otimes F^{s-k}|C)$ is such that

$$(5)_s \qquad \alpha_s \cup \varphi \in \operatorname{Im} j^2_{s*} \quad \text{for all } \varphi \in H^0((F^{k-1}\otimes K)|C)$$

then $\alpha_s \in \operatorname{Im} j^1_{s*}$.

Compare now the bundles $[C]\otimes F^{s-k}$, $[C]\otimes F^{s-1}\otimes K$ and $F^{k-1}\otimes K$ with their restrictions to C by means of the standard exact sequences

$$(5A) \qquad 0 \to G\otimes [C]^{-1} \to G \xrightarrow[r]{} G|C \to 0\,.$$

By means of (1.6) we find:

(i) $H^0([C]\otimes F^{s-k}) \xrightarrow[r^1_s]{\sim} H^0(([C]\otimes F^{s-k})|C)$ in case $s<k$.

$$0 \to \mathbf{C} \xrightarrow[\Phi]{} H^0([C]) \xrightarrow[r^1_k]{} H^0([C]|C) \to 0 \quad \text{in case } k=s\,.$$

(ii) $0 \to H^0(F^{s-1}\otimes K) \xrightarrow[\Phi]{} H^0(F^{s-1}\otimes K\otimes [C]) \xrightarrow[r^2_s]{} H^0(F^{s-1}\otimes K\otimes [C]|C) \to 0$

(iii) $H^0(F^{k-1}\otimes K) \xrightarrow[\varrho]{\sim} H^0((F/^{-1}\otimes K)|C)$

where r^1_s, r^2_s, and ϱ come from the restriction map r in the above exact sheaf sequence with appropriate bundles substituted for G.

We state for later reference:

DEFINITION 1.11. $\mathfrak{I}^\alpha_s$, $\alpha=1,2$; $s=2,\dots,k$, *is defined to be the inverse image under* r^α_s *of* $\operatorname{Im} j^\alpha_{s*}$ (cf. $(4)_s$).

Remark that the exact sequences (5A) with $G=T\otimes F^{s-k}$ and conditions 1.6 imply that $H^0((T\otimes F^{s-k})|C)=0$ in case $s<k$ so in fact $\mathfrak{I}^1_s=0$ if $s<k$. From these remarks one can easily see that the next lemma holds:

LEMMA 1.12. (*a*) *Condition* $(5)_s$ *is, for* $s<k$, *equivalent to the next assertion*:

Let $A\in H^0([C]\otimes F^{s-k})$. *Suppose* $A\cup Z\in \mathfrak{I}^2_s$ *for all* $Z\in H^0(F^{k-1}\otimes K)$ *then* $A=0$.

(*b*) *Condition* $(5)_k$ *is equivalent to the next assertion*:
Let $A\in H^0([C])$. *Suppose* $A\cup Z\in\mathfrak{I}^2_k$ *for all* $Z\in H^0(F^{k-1}\otimes K)$ *then* $A\in\mathfrak{I}^1_k$.

In case C is a curve we shall be able to verify the next condition:
The cupproduct

$$(6)_s \qquad H^0(F^{k+1-s}|C\otimes K_C)\otimes H^0(F^{-1}|C\otimes K_C) \to H^0(K_C^2\otimes F^{k-s}|C)$$

is onto

LEMMA 1.13. *Condition* $(6)_s$ *implies* $(5)_s$.

PROOF. Condition $(6)_s$ is (by the appendix of part I) dual to:
The cupproduct

$$H^1(T_C \otimes F^{s-k}|C) \otimes H^0(F^{-1}|C \otimes K_C) \to H^1(F^{s-k-1}|C \otimes K_C \otimes T_C)$$

is non-degenerate in the first factor.

Now use that $K_C \cong (F^k \otimes K)|C$ (adjunction formula) together with the fact that K_C and T_C are dual bundles to see that the above cupproduct in fact is the cupproduct:

$$H^1(T_C \otimes F^{s-k}|C) \otimes H^0((F^{k-1} \otimes K)|C) \to H^1(T_C \otimes (F^{s-1} \otimes K)|C)\,.$$

So the assertion $(6)_s$ is dual to:

Suppose that $\beta_s \in H^0(T_C \otimes F^{s-k}|C)$. *If* $\beta_s \cup \zeta = 0$ *for all* $\zeta \in H^0((F^{k-1} \otimes K)|C)$ *then* $\beta_s = 0$.

By $(4)_s$ one can easily derive that this assertion in fact implies $(5)_s$.

2. – Reduction to a problem about the branch locus.

In this sectjon we shall prove that, in order to establish the local Torelli-problem for V, it is sufficient to prove $(5)_s$ for all $s = 2, \dots, k$.

We study the situation considered at the end of § 1 where we substitute V for V and $\hat{F}$ for W. By II, 1.9 and II, 1.10 theorem I, 2.3 applies, and we get:

LEMMA 2.1. *Consider the commutative diagrams with exact rows*:

$$\begin{array}{ccc} 0 \to H^0(\hat{T}|V) & \xrightarrow{j} & H^0([V]|V) \\ \downarrow{\cdot s} & & \downarrow{\cdot s} \\ 0 \to H^0(\hat{T}|V \otimes K_V) & \xrightarrow{j\otimes 1} & H^0([V]|V \otimes K_V)\,, \end{array} \qquad \textit{with } s \in H^0(K_V)\,.$$

Torelli holds locally for V if and only if the next condition holds:

Suppose $\nu \in H^0([V]|V)$ *and* $\nu \cup s \in \operatorname{Im}(j \otimes 1)_*$ *for all* $s \in H_0(K_V)$ *then* $\nu \in \operatorname{Im} j_*$.

From this lemma we see that we have to determine j_* and $\operatorname{Im}(j \otimes 1)_*$: To this end we first study the groups $H^0([V]|V)$, $H^0([V]|V \otimes K_V)$, $H^0(\hat{T}|V)$ and $H^0(\hat{T}|V \otimes K_V)$ in more detail.

Consider the exact sequences $(2A)$, where we substitute $G = [V]$, resp. $\hat{K} \otimes [V]$. Because of the ad^{p}unction formula $(\hat{K} \otimes [V]^2)|V \cong$ $\cong K_V \otimes [V]|V$. Lemma 1.7 gives that $H^1([V]) = H^1([V] \otimes \hat{K}) = 0$. Moreover $[V] \cong F^k \otimes L^k$ (1.5), so, by 1.1(a): $H^0([V]) \cong \bigoplus_{s=0}^{k} H^0(F^s)$. Remark 1.4 tells us that in cohomology, the induced map

$$(\cdot v)_*: H^0(\hat{O}) \to H^0([V]) \cong \bigoplus_{s=0}^{k} H^0(F^s)$$

is given by $(\cdot v)_*(c) = (-c, 0, \ldots, 0, c\Phi)$, if $c \in H^0(\hat{O}) = \mathbf{C}$.

Similar considerations apply to the case $G\ \hat{K} \otimes [V]^2$.

We collect these facts in a lemma:

LEMMA 2.2. *We have commutative diagrams with exact rows:*

$$\begin{array}{ccccccc} 0 \to \mathbf{C} & \longrightarrow & H^0([V]) & \xrightarrow{r_*} & H^0([V]|V) & \to 0 \\ \| & & \downarrow \wr & & \| & \\ 0 \to \mathbf{C} & \xrightarrow{a} & \bigoplus_{s=0}^{k} H^0(F^s) & \xrightarrow{b} & H^0([V]|V) & \to 0 \end{array} \tag{7}$$

where $a(c) = (-c, 0, \ldots, 0, c\Phi)$

$$\begin{array}{ccccccc} 0 \to & H^0(K \otimes [V]) & \longrightarrow & H^0(K \otimes [V]^2) & \xrightarrow{r_{2*}} & H^0(K_V \otimes [V]|V) & \to 0 \\ & \downarrow \wr & & \downarrow \wr & & \| & \\ 0 \to & \bigoplus_{s=1}^{k-1} H^0(F^s \otimes K) & \xrightarrow{a_2} & \bigoplus_{s=1}^{2k-1} H^0(F^s \otimes K) & \xrightarrow{b_2} & H^0(K_V \otimes [V]|V) & \to 0 \end{array} \tag{8}$$

with

$$a_2(\varphi_1, \ldots, \varphi_{k-1}) = (-\varphi_1, -\varphi_2, \ldots, -\varphi_{k-1}, 0, \varphi_1\Phi, \varphi_2\Phi, \ldots, \varphi_{k-1}\Phi),$$
$$\varphi_\mu \in H^0(F^\mu \otimes K).$$

Next, we study the cohomology diagrams of (1) and (2). By means of 1.7 we get:

LEMMA 2.3. *There are commutative diagrams with exact rows:*

$$\begin{array}{ccccccc} 0 \to & H^0(T_\pi) & \xrightarrow{i_*} & H^0(\hat{T}) & \xrightarrow{\pi_*} & H^0(\pi^* T) & \to 0 \\ & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\ 0 \to & H^0(T_\pi|V) & \longrightarrow & H^0(\hat{T}|V) & \longrightarrow & H^0(\pi^* T|V) & \to 0 \end{array} \tag{9}$$

$$\begin{array}{ccccccc} 0 \to & H^0(\hat{K} \otimes [V] \otimes T_\pi) & \to & H^0(\hat{T} \otimes \hat{K} \otimes [V]) & \to & H^0(\pi^* T \otimes \hat{K} \otimes [V]) & \to 0 \\ & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\ 0 \to & H^0(T_\pi|V \otimes K_V) & \to & H^0(\hat{T}|V \otimes K_V) & \to & H^0(\pi^* T|V \otimes K_V) & \to 0 \end{array} \tag{10}$$

To obtain diagram morphisms from (7) to (8), (9) to (10) resp. we need the next result, which can be obtained by means of the exact sequence (2A), with $G = \hat{K} \otimes [V]$ and the results of 1.7.

LEMMA 2.4. *By means of bundle restriction we obtain a commutative diagram*

$$\begin{array}{ccc} H^0(\hat{K} \otimes [V]) & \underset{\mu}{\xrightarrow{\sim}} & H^0(K_V) \\ \big\downarrow \wr & & \big\| \\ \bigoplus_{s=1}^{k-1} H^0(F^s \otimes K) & \underset{\nu}{\xrightarrow{\sim}} & H^0(K_V) \end{array}$$

In the next two propositions we make explicit the diagram morphisms from (7) to (8) and part of those from (9) to (10) which arise when we consider the cupproduct with an element $s \in H^0(K_V)$. Here we make essential use of the following fact—to be established with 1.1(c):

Fact: *The isomorphisms* $H^0(\pi^* G \otimes L^t) \cong \bigoplus_{s=0}^{t} H^0(G \otimes F^{-s})$ *are compatible with the forming of cupproduct, whenever those isomorphisms are given with respect to the fixed covering* $\{U_\alpha\}_{\alpha \in A}$ *as given in* 1.1(c).

PROPOSITION 2.5. *There is a commutative diagram with exact columns*:

$$\begin{array}{ccc} & 0 & & 0 \\ & \big\downarrow & & \big\downarrow \\ \mathbf{C} \otimes & \bigoplus_{s=1}^{k-1} H^0(F^s \otimes K) & \xrightarrow[\text{cup}]{} & \bigoplus_{s=1}^{k-1} H^0(F^s \otimes K) \\ & \big\downarrow a \otimes 1 & & \big\downarrow a_2 \\ \bigoplus_{s=0}^{k} H^0(F^s) \otimes & \bigoplus_{s=1}^{k-1} H^0(F^s \otimes K) & \xrightarrow[\text{cup}]{} & \bigoplus_{s=1}^{2k-1} H^0(F^s \otimes K) \\ & \big\downarrow b \otimes \nu & & \big\downarrow b_2 \\ & H^0([V]|V) \otimes H^0(K_V) & \xrightarrow[\text{cup}]{} & H^0([V]|V \otimes K_V) \\ & \big\downarrow & & \big\downarrow \\ & 0 & & 0 \end{array}$$

PROPOSITION 2.6. *There is a commutative diagram*:

$$\begin{array}{ccccc} H^0(\hat{T}) & \otimes & H^0(\hat{K} \otimes [V]) & \xrightarrow[\text{cup}]{} & H^0(\hat{T} \otimes \hat{K} \otimes [V]) \\ \big\downarrow \wr & & \big\downarrow \wr\, \nu & & \big\downarrow \wr \\ H^0(\hat{T}|V) & \otimes & H^0(K_V) & \xrightarrow[\text{cup}]{} & H^0(\hat{T}|V \otimes K_V) \end{array}$$

Using 2.5 and 2.6 we see from 2.1:

PROPOSITION 2.7. *Torelli holds locally for V if the next condition is fulfilled:*
Let $(\nu_0, \ldots, \nu_k) \in \bigoplus_{s=0}^{k} H^0(F^s)$. *If for all* $(\varkappa_1, \ldots, \varkappa_{k-1}) \in \bigoplus_{s=0}^{k} H^0(F^s \otimes K)$ *we have that*

$$b_2\{(\nu_0, \ldots, \nu_k)\cdot(\varkappa_1, \ldots, \varkappa_{k-1})\} \in \operatorname{Im}(j \otimes 1)_*,$$

then $b(\nu_0, \ldots, \nu_k) \in \operatorname{Im} j_*$.

PROPOSITION 2.8. (*a*) $\operatorname{Im} j_* = b\big(H^0(0) \oplus H^0(F) \oplus \Im_k^1\big)$

(*b*) $\operatorname{Im}(j \otimes 1)_* = b_2\big(H^0(F \otimes K) \oplus \ldots \oplus H^0(F^k \otimes K) \oplus \Im_2^2 \oplus \ldots \oplus \Im_k^2\big)$
where we refer to 1.11 *for the definition of* $\Im_s^\alpha$, $\alpha = 1, 2$, $s = 2, \ldots, k$.

We shall postpone the proof of 2.8 untill the end of this section and we shall first outline how 2.7 and 2.8 reduce the Torelli problem to a verification of condition $(5)_s$. First, we see from 2.8 that no condition is needed for ν_0 and ν_1 in 2.7. Secondly, if we look at the condition in 2.7 with $\varkappa_1 = \ldots = \varkappa_{k-2} = 0$ we see that for $\nu_s \in H_0(F^s)$, $s = 2, \ldots, k$ we obtain:

$$b_2(\nu_s \cdot \varkappa_{k-1}) \in b_2(\Im_s^2) \quad \text{for all } \varkappa_{k-1} \in H^0(F^{k-1} \otimes K)$$

and in order that $b(\nu_s) \in \operatorname{Im} j_*$ for $s = 2, \ldots, k$ we need that $\nu_2 = .. = \nu_{k-1} = 0$ and that $b(\nu_k) \in b(\Im_k^1)$.

Now, to see that $(5)_s$ implies the condition of 2.7 we need a lemma, which we will prove in the course of the proof of 2.8:

LEMMA 2.9. *With the notations of* 1.11 *we have:*

$$H^0(F^k) \cap \operatorname{Ker} b = \operatorname{Ker} r_k^1$$
$$H^0(F^s \otimes K) \cap \operatorname{Ker} b_2 = \operatorname{Ker} r_s^2 \qquad (s = k+1, \ldots, 2k-1)\,.$$

From this we easily see that:

$$b(\nu_k) \in b(\Im_k^1) \Leftrightarrow \nu_k \in \Im_k^1\,,$$
$$b_2(\nu_s \cdot \varkappa_{k-1}) \in b_2(\Im_k^2) \Leftrightarrow \nu_s \cdot \varkappa_{k-1} \in \Im_s^2\,.$$

The above remarks show that we have:

THEOREM 2.10. *Torelli holds locally if the next condition is fulfilled:*
Let $\nu_s \in H^0(F^s)$, $s = 2, \ldots, k$. *If for all* $\varkappa_{k-1} \in H^0(F^{s-1} \otimes K)$ *we have* $\nu_s \varkappa_{k-1} \in \Im_s^2$, *then* $\nu_2 = \ldots = \nu_{k-1} = 0$ *and* $\nu_k \in \Im_k^1$.

COROLLARY 2.11. *Torelli holds locally for V if $(5)_s$ holds for V, $s = 2, \ldots, k$.*

COROLLARY 2.12. *In case V is a surface, Torelli holds locally for V if $(6)_s)$ holds, $s = 2, \ldots, k$.*

PROOF (of both corollaries). Apply 1.12, resp. 1.12 and 1.13 to 2.10. We now proceed to prove 2.8.

PROOF OF 2.8. We shall only prove part (a), because the proof of (b) is entirely similar.

Because of 2.3 $\dim(\operatorname{Im} j_*) = \dim H^0(\hat{T}|V) = \dim H^0(\hat{T}) = \dim H^0(T_\pi) + \dim H^0(T) = \dim\{H^0(0) \oplus H^0(F) \oplus \Im_k^1\}$, hence it is sufficient to prove:

$$b(H_0(0) \oplus H^0(F) \oplus \Im_k^1) \subset \operatorname{Im} j_* . \tag{11}$$

We do this in three steps

Step 1. *There is a commutative diagram*:

$$\begin{tikzcd} H^0(O) \oplus H^0(F) \arrow[r, "\Im"] \arrow[d, "\wr"] & \bigoplus_{s=0}^{k} H^0(F^s) \arrow[ddd, "b"] \\ H^0(T_\pi) \arrow[d, "i_*"] & \\ H^0(\hat{T}) \arrow[d, "r_*", "\wr"'] & \\ H^0(\hat{T}|V) \arrow[r, "i_*"] & H^0([V]|V) \end{tikzcd} \tag{12}$$

with $\Im(c, \Psi) = (-kc, -k\Psi, 0, \ldots, 0)$.

We can see this as follows. Employ the notation of 1.1(c). Let $(c, \psi) \in H^0(0) \oplus H^0(F)$ be represented by $\{(c, \Psi_\alpha)\}_{\alpha\in A}$. By 1.1$(c)$ the collection $\{c\eta_\alpha + \Psi_\alpha \xi_\alpha\}_{\alpha\in A}$ defines the section, say $\overline{(c, \Psi)}$ of T_π corresponding to (c, Ψ). Using the fibre coordinate x_α above U_α we see that $i_*\overline{(c, \Psi)}$ is given on $\pi^{-1}(U_\alpha)$ by $(c + \Psi_\alpha x_\alpha)\,\partial/\partial x_\alpha$.

Because $\Phi_\alpha - x_\alpha^k = 0$ is a local equation for V in $\pi^{-1}(U_\alpha)$, by definition of the map j_* we find that $j_* r_* i_*\overline{(c, \Psi)}$ is given locally on $\pi^{-1}(U_\alpha)$ by the restriction to V of:

$$(c + \Psi_\alpha x_\alpha)\,\partial/\partial x_\alpha\{\Phi_\alpha - x_\alpha^k\} = -k(cx_\alpha^{k-1} + \Psi_\alpha x_\alpha^k) .$$

Again by 1.1(c) this collection corresponds to $(-kc, -k\Psi, 0, \ldots, 0) \in \bigoplus_{=0}^{k} H^0\ F^s)$.

So we have proved that $b(\Im(c, \Psi)) = j_* r_* i_* \overline{(c, \Psi)}$, whence the commutativity of (12).

Step 2. Embed W in F as the zero section. Then $C = V \cap W$ and we get a natural commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \hat{O} & \to & [V] & \to & [V]|V & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O & \to & [C] & \to & [C]|C & \to & 0 \end{array}$$

In cohomology we thus get:

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{C} & \xrightarrow[a]{} & \bigoplus_{s=0}^{k} H^0(F^s) & \xrightarrow[b]{} & H^0([V]|V) & \to & 0 \\ & & \| & & \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle \chi} & & \\ 0 & \to & \mathbf{C} & \to & H^0(F^k) & \xrightarrow[r_k^1]{} & H^0([C]|C) & \to & 0 \end{array} \tag{13}$$

Step 3. There is a commutative diagram

$$\begin{array}{ccccc} H^0(\hat{T}|V) & \xrightarrow[j_*]{} & H^0([V]|V) & \xleftarrow[b]{} & \bigoplus_{s=0} H^0(F^s) \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle p_k} \\ H^0(T|C) & \xrightarrow[j_{k*}^1]{} & H^0([C]|C) & \xleftarrow[r_k^1]{} & H^0(F^k) \end{array} \tag{14}$$

where $\Psi = r_C \cdot (\pi_*|V)$, the composition of $(\pi_*|V) : \hat{T}|V \to \pi^* T|V$ and the restriction $r_C : \pi^* T|V \to \pi^* T|C = T|C$.

Due to (13), to see this, we need only to prove the commutativity of the left hand square and the surjectiveness of Ψ. Now the first assertion follows from the cohomology of the commutative diagram:

$$\begin{array}{ccc} \hat{T}|V & \xrightarrow[j]{} & [V]|V \\ \downarrow{\scriptstyle \pi_*|V} & & \big| \\ \pi^* T|V & & \big| \\ \downarrow{\scriptstyle r_C} & & \downarrow \\ T|C & \xrightarrow[j_k^1]{} & [C]|C \end{array}$$

The second assertion follows by inspecting the cohomology diagram of:

$$\begin{array}{ccccccc} 0 \to & \hat{T} \otimes [V]^{-1} & \to & \hat{T} & \xrightarrow{r_1} & \hat{T}|V & \to 0 \\ & \downarrow & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \Psi} & \\ 0 \to & \pi^*(T \otimes F^{-k}) & \to & \pi^* T & \xrightarrow{r_2} & T|C & \to 0 \end{array}$$

where r_2 and r_1 are the restriction maps. We leave the details to the reader.

We now combine these steps. Remark that (2) gives that $b\big(H^0(O) \oplus \oplus H^1(F)\big) \subset \operatorname{Im} j_*$. From (13) and (14) we may easily derive a proof of lemma 2.9. Moreover (14) gives that $\chi(\operatorname{Im} j_*) = \operatorname{Im} j_k^1$, and together with 2.9 this implies $b\big(H^0(F^k)\big) \cap \operatorname{Im} j_* = b(\mathfrak{I}_k^1)$. These two facts together constitute a proof of 2.8(a).

3. – Applications.

THEOREM 3.1. *Let V be a k-cyclic covering of $\mathbb{P}_n$, branched along a hypersurface of degree mk. Let $\lambda := m(k-1) - n - 1$. Torelli holds locally for V if $\lambda > 0$, $m \neq 1$ and moreover $m \neq 3$ in case $n = 2$* (*).

PROOF. In this case $F = H^k$ and $K = H^{-n-1}$, so $F^{k-1} \otimes K = H^\lambda$ is ample by assumption. We may easily verify the other conditions of 1.6 when we use I, 3.2. Applying 2.10 in this case, we see—with the help of I, 5—that it is sufficient to prove the next assertion:

Let $\Phi_s \in \sigma_{ms}$ $(s = 2, \dots, k)$. If $\Phi_s \cdot \varkappa$ lies in the ideal $\mathfrak{i}$ generated by $\partial_0 \Phi, \dots, \partial_n \Phi$ for all $\varkappa \in \sigma^\lambda$, then $\Phi_s = 0$ for $s = 2, \dots, k-1$ and $\Phi_k \in \mathfrak{i}$.

Because $\mathfrak{i}$ is generated by polynomials of degree $mk - 1$ and $ms > mk - 1$ in case $s = 2, \dots, k-1$ we may as well prove that $\Phi_s \in \mathfrak{i}$ for all s.

This however follows easily from I.6.3, or, alternatively by an application of the following theorem of Macaulay's ([3]):

THEOREM. *Let $\mathfrak{i} = (F_0, \dots, F_n)$ be such that $\sqrt{\mathfrak{i}} = \mathfrak{m}$. Then with $\sigma = \sum_{\mathfrak{i}=0}^{n} (\deg F_{\mathfrak{i}} - 1)$ we have for $t = 1, \dots, \sigma$: $[\mathfrak{i} : \mathfrak{m}^t] = \mathfrak{i} + \mathfrak{m}^{\sigma+1-t}$.*

Indeed, substitute $t = \lambda$, $F_i = \partial_i \Phi$, so $\sigma = (mk-2)(n+1)$ and $\sigma + 1 - t = n(mk-1) + m > ms$ $(s = 2, \dots, k)$, hence $\Phi_s \in \mathfrak{i}$. (Q.E.D.)

(*) After this paper was written the author saw that K. I. Kĭi proved this also, see: *A local Torelli theorem for cyclic coverings of P^n with positive canonical class*, Math. USSR Sbornik, **21** (1973), pp. 145-155 (translation 1974).

Next, we consider a Hirzebruch surface Σ_n ($n = 0, 1, \dots$). This is by definition $\hat{H}^n$, where H is the hypersection bundle on $\mathbf{P}_1$: (Notation of § 1). So there is a $\mathbf{P}_1$-bundle structure on Σ_n. Let $\Sigma_n \underset{\omega}{\to} \mathbf{P}_1$ be this fibration. Any bundle on Σ_n then is of the form $\omega^* H^a \otimes L^b$, where a and b are integers. Such a bundle we denote by (a, b). We may apply 1.1(a) to this situation to get:

LEMMA 3.2:

$$(a) \quad \left.\begin{aligned} &H^0((a, b)) = 0 \quad \textit{if } a < 0 \\ &H^1((a, b)) = 0 \quad \textit{if } a - nb \geqslant 2 \\ &H^2((a, b)) = 0 \end{aligned}\right\} \textit{ in case } b \geqslant 0 .$$

$$(b) \quad \left.\begin{aligned} &H^0((a, b)) = 0 \\ &H^1((a, b)) = 0 \quad \textit{for } a + nb < 0 \\ &H^2((a, b)) = 0 \quad \textit{for } a \geqslant -2 \end{aligned}\right\} \textit{ in case } b < 0 .$$

(c) *In case* $a \geqslant 0$, $b \geqslant 0$, *we have that* $H^0((a, b)) \cong \bigoplus_{s=0}^{b} \sigma_{a-ns}$.

We define the natural numbers μ and ν by $F = (\mu, \nu)$. Suppose

$$\mu \geqslant n\nu \quad \text{and} \quad \nu \geqslant 3 . \tag{15}$$

We want to verify conditions 1.6. The first two follow immediately from 3.2 and (15). To establish the remaining ones we need the usual sequence for the tangent bundle of a $\mathbf{P}_1$-bundle (compare 1.2):

$$0 \to (n, 2) \to T \to (0, 2) \to 0 . \tag{16}$$

In particular we see that

$$K = (-n-2, -2) . \tag{17}$$

Then, applying 3.2 to (16) and (17) one may verify 1.6(c) up to (e).

We now may apply 2.12 to see that we only have to verify $(6)_s$ for $s = 2, \dots, k$. However this follows directly from the next lemma, the proof of which we delete:

LEMMA 3.3. *There is a commutative diagram:*

$$\begin{array}{ccccc}
\bigoplus_{s=0}^{b} \sigma_{a-ns} \otimes \bigoplus_{s=0}^{b'} \sigma_{a'-ns} & & \xrightarrow{\text{product}} & & \bigoplus_{s=0}^{b+b'} \sigma_{a+a'-ns} \\
\| \wr & & & & \| \wr \\
H^0((a,b)) \otimes H^0((a',b')) & & \xrightarrow{\text{cup}} & & H^0((a+a', b+b')) \\
\downarrow{\scriptstyle r_1 \otimes r_2} & & & & \downarrow{\scriptstyle r_3} \\
H^0((a,b)|C) \otimes H^0((a',b')|C) & & \xrightarrow{\text{cup}} & & H^0((a+a', b+b')|C)
\end{array}$$

where r_j, $j = 1, 2, 3$ *are the restrictions and* a, a', b, b' *are non-negative integers.*

ꞡecause the bundles mentioned in $(5)_s$ in this case are all of the type (a, b) for some non-negative integers a, b, we may use 3.3 to see that $(5)_s$ indeed is fulfilled. Hence:

THEOREM 3.4. *Let* Σ_n *be a Hirzebruch surface,* $n = 0, 1, \ldots$. *Let* V *be a* k*-cyclic covering of* Σ_n *such that* (15) *holds. Then the local Torelli-theorem holds for* V.

REMARK 3.5. This implies in particular that V has a modular variety of dimension $\dim H^1(T_V)$ in this case (cf. 1.10).

REMARK 3.6. In this case we find examples of surfaces with ample canonical bundle for which the local Torelli theorem holds; one may prove that the surfaces considered are simply-connected, so this provides new examples for which the conjecture stated in [1], problem 6.1 is true.

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