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Rigidity for variations of Hodge structure and Arakelov-type finiteness theorems

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0. Introduction

The theorem of Arakelov referred to in the title states that there are at most finitely many non-isomorphic, non-isotrivial families of curves over a fixed curve with singular fibres over a fixed set of points. (See [A]). Isotrivial means that the family becomes birationally a product after a finite base change. If one replaces the fibres by higher dimensional varieties this is no longer true: Take a product of a family of abelian varieties with non-constant periods and a fixed abelian variety – for a less trivial example see [F]. In [F], Faltings also formulates a Hodge theoretic condition for families of principally polarized abelian varieties over curves which when substituted for "isotrivial" in Arakelov's theorem renders a true statement. In [P2] one finds a version of Faltings' proof which is purely Hodge theoretic. In the present note the infinitesimal deformation space T for a period mapping with source a quasi-projective manifold M is computed, thereby obtaining rigidity results. It should be stressed that one restricts oneself not only to deformations through variations of Hodge structures but also one demands that the resulting local system on $M \times T$ carries a compatible variation of Hodge structure. In order to explain the results assume that that there be given local system H_M of free $\mathbb{Z}$-modules on M carrying a polarized variation of Hodge structure. The local system $End(H_M)$ then carries a polarized weight zero Hodge structure and by [S, Corollary 7.23] the $\mathbb{Z}$-module End H_M of global sections of $End(H_M)$ carries a weight zero Hodge structure.[1]

THEOREM. *A variation of Hodge structure underlying H_M is rigid if* $(\mathrm{End}\, H_M \otimes \mathbb{C})^{(-1,1)} =$ *or if the period map associated to the variation of Hodge structure is regularly tangent.* $) = 0$,

See Section 3 for the precise meaning of *regularly tangent*. It suffices to say here that it is a sort of linear algebraic genericity condition which can be verified locally at the tangent spaces of M; "most" period maps are regularly tangent.

[1] Endomorphisms are assumed to be skew with respect to polarisations.

In order to derive results for families of varieties some form of Torelli must hold. See Theorem 3.8 for a geometric translation of the main theorem.

In Section 4 the infinitesimal point of view is combined with Deligne's results from [D2]. Under additional assumptions rigidity is shown to imply a finiteness result of Arakelov type, which generalizes Faltings' main result in [F], but which does not imply Arakelov's theorem itself. See Section 4 for precise results.

The main ingredient for rigidity is a curvature calculation based on results in [G-S]. It is presented in Section 1. In Section 2 the definition of a polarized variation of Hodge structure is recalled and the class of deformations alluded to before is introduced with the concept *family of variations of Hodge structure with fixed monodromy*. The corresponding infinitesimal deformations are studied; Schmid's asymptotic analysis ([S]) is used to show that these infinitesimal variations, viewed as sections of a certain Hodge bundle extend to sections of their quasi-canonical extensions. In Section 3 the main results are stated and proven.

In closing, I want to mention that Sunada[2] has some results about holomorphic maps of a *compact* complex manifold into a hermitian symmetric domain (see [Su]). There the maximal number of parameters with which such a map can be deformed, keeping source and target fixed is evaluated. Similar calculations can be done in the present set-up, which generalizes Sunada's set-up. This will be treated in a forthcoming paper.

1. Curvature for Griffiths period domains

We recall briefly Griffiths' construction of the classifying spaces for polarized Hodge structures [G].

We let H be a finite dimensional real vector space, w an integer and $\{h^{p,w-p}, p \in \mathbb{Z}\}$ a family of nonnegative integers with $h^{p,w-p} = h^{w-p,p}$ and $\sum_p h^{p,w-p} = \dim H$. Let ψ be a non degenerate bilinear form on H such that $(-1)^w\psi$ is symmetric. *A weight w Hodge structure* on H *with Hodge numbers* $\{h^{p,w-p}\}$ is a direct sum decomposition $H_{\mathbb{C}} := H \otimes \mathbb{C} = \oplus_{p\in\mathbb{Z}} H^{p,w-p}$ with $H^{p,w-p} = \bar{H}^{w-p,p}$ and $\dim H^{p,w-p} = h^{p,w-p}$.

Introduce the Hermitian form

$$\phi(x, y) := (-1)^p i^w \psi(x, \bar{y}) \quad \forall x\, y \in H^{p,w-p}. \tag{1.1}$$

The Hodge structure is *polarized by ψ* if first of all $\psi(H^{p,w-p}, H^{w-r,r}) = 0$ for $p \neq r$ and if secondly ϕ is positive definite. We let $D = D(H, \psi, \{h^{p,w-p}\})$ be the set of all ψ-polarized Hodge structures on H with Hodge numbers $\{h^{p,w-p}\}$. The map

[2] Cf. also J. Noguchi, Inv. Math. 93, 15–34 (1988), §4.

which associates to every point in D its *associated Hodge filtration* $F^p := \oplus_{i \geqslant p} H^{i,w-p}$ identifies D with an open subset of a manifold $\check{D}$ of flags $\{F^p\}$ for which F^p is the Ψ-orthogonal complement of F^{w-p+1}. We call D a *period domain.*

The manifold D is homogeneous under the action of $G := \{g \in \mathrm{Gl}(H);\ \psi(gx, gy) = \psi(x, y)\ \forall x, y \in H\}$ while $\check{D}$ is homogeneous under the action of $G_{\mathbb{C}} := \{g \in \mathrm{Gl}(H_{\mathbb{C}});\ \psi(gx, gy) = \psi(x, y) \forall x, y \in H_{\mathbb{C}}\}$. For $F = \{F^p\} \in D$ the stabilisor in $G_{\mathbb{C}}$ is $B := \{g \in G_{\mathbb{C}} \mid g(F^p) = F^p\}$. The choice of $F \in D$ induces on the Lie algebra of G a weight zero Hodge structure with

$$\mathfrak{g}^p := H^{p,-p}(\mathfrak{g}) = \{f \in \mathrm{Lie}(G_{\mathbb{C}});\ f(H^{i,w-i}) \subseteq H^{i+p,w-i-p}\}$$

This Hodge structure is polarized by

$$\Psi(X, Y) := -\mathrm{Trace}(X \cdot Y^T). \tag{1.2}$$

Clearly we have

$$[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j} \tag{1.3}$$

We put

$$\mathfrak{g}^+ = \bigoplus_{i>0} \mathfrak{g}^i, \quad \mathfrak{g}^- := \bigoplus_{i<0} \mathfrak{g}^i \tag{1.4}$$

and we observe that the Lie algebra of B can be identified with $\mathfrak{g}^+ \oplus \mathfrak{g}^0$ and hence the tangent space of $\check{D}$ at F can be identified with $\mathfrak{g}^-$. The tangent bundle $T(\check{D})$ coincides with the vector bundle associated to the principal bundle $B \to G_{\mathbb{C}} \to G_{\mathbb{C}}/B$ by the adjoint representation of B on $\mathrm{Lie}(G_{\mathbb{C}})/\mathrm{Lie}(B)$. From (1.3) we may conclude that $(\mathrm{Lie}(B) + \mathfrak{g}^{-1})/\mathrm{Lie}(B)$ is an $\mathrm{ad}(B)$-invariant subspace of $\mathrm{Lie}(G)/\mathrm{Lie}(B)$ and defines a holomorphic subbundle $T^h(D)$ of $T(D)$, the *horizontal tangent bundle.*

For computations that follow, we identify a Zariski open neighbourhood of F in $\check{D}$ with a Zariski open subset of the subgroup $N := \exp(\mathfrak{g}^{-1})$ of $G_{\mathbb{C}}$ under the projection $n \to nF$. This is possible since N maps biregularly onto $N \cdot F$. This identification yields an explicit isomorphism:

$$T^h_F(D) \cong \mathfrak{g}^{-1}. \tag{1.5}$$

Let us also observe that the polarization (1.2) induces a hermitian metric on the tangent space $T_F(D)$ given by

$$\Phi(X, Y) = (-1)^p \Psi(X, \bar{Y}), \quad X, Y \in \mathfrak{g}^p \tag{1.6}$$

This metric yields a G-invariant metric on D denoted with the same symbol. The curvature of this metric is computed in [G-S]. We present Deligne's version of the result ([D1, proof of Théorème 5.16]-note that Lie brackets make sence because of the identification (1.5)):

$$\theta(X, \bar{Y})Z = [CX, [\bar{Y}, Z]^-]^- - [\bar{Y}, [CX, Z]^-]^- - [[X, \bar{Y}], Z]^- \quad X, Y, Z \in \mathfrak{g}^-.$$

Here $C = (-1)^p$ on $\mathfrak{g}^p$ and the superscript means projection onto $\mathfrak{g}^-$.

PROPOSITION 1.7. *For* $X, Y, Z \in \mathfrak{g}^{-1}$ *we have*

$$\theta(X, \bar{Y})Z = -[[X, \bar{Y}], Z] + [\bar{Y}, [X, Z]].$$

Proof. Since $\mathfrak{g}^{-p} = \bar{\mathfrak{g}}^p$ we have $\bar{Y} \in \mathfrak{g}^1$ and hence $[\bar{Y}, Z] \in \mathfrak{g}^0$ and so this projects to 0 in $\mathfrak{g}^-$. On $\mathfrak{g}^{-1}$ we have $C = -1$ and since $[X, Z] \in \mathfrak{g}^{-2}$ we have $[\bar{Y}, [CX, Z]^-]^- = -[\bar{Y}, [X, Z]]$. Similarly $[[X, \bar{Y}], Z] \in \mathfrak{g}^{-1}$. □

COROLLARY 1.8. *For* X, $Y \in \mathfrak{g}^{-1}$ *the holomorphic bisectional curvature* $H(X, Y) := \Phi(\theta(X, \bar{X})Y, Y)$ *is equal to*

$$2\Phi([X, Y], [X, Y]) - \Phi([\bar{X}, Y], [\bar{X}, Y])$$

so in particular $H(X, Y) \leqslant 0$ *If* X *and* Y *commute.*

Proof. We use the Definitions (1.2) and (1.6), the ad($\mathfrak{g}$)-skewsymmetry of Ψ and the Jacobi identity in the computation that follows:

$$\begin{aligned} H(X, Y) &= -\Psi(\theta(X, \bar{X})Y, \bar{Y}) = \Psi([[X, \bar{X}], Y]\bar{Y}) - \Psi([\bar{X}, [X, Y]], \bar{Y}) \\ &= -\Psi([[\bar{X}, Y], X], \bar{Y}) - \Psi([[Y, X], \bar{X}], \bar{Y}) + \Psi([[X, Y], \bar{X}]], \bar{Y}) \\ &= 2\Psi([X, Y], [\bar{X}, \bar{Y}]) - \Psi([\bar{X}, Y], [X, \bar{Y}]) \\ &= 2\Phi([X, Y], [X, Y]) - \Phi([\bar{X}, Y], [\bar{X}, Y]). \end{aligned}$$

□

2. Variations of Hodge structure

Let us fix a connected complex analytic manifold M, $o \in M$ and suppose that H is a finite dimensional real vector space endowed with a non-degenerate bilinear form ψ. As before we let G be the group of automorphisms of (H, Ψ). We fix a *monodromy representation*

$$\sigma: \pi_1(M, o) \to G \tag{2.1}$$

and we let H_M be the corresponding local system on M. By definition it underlies a *ψ-polarized variation of weight w* if there is a holomorphic filtration

$$\mathscr{H}_M := H_M \otimes \mathcal{O}_M = \mathscr{F}^0 \supset \mathscr{F}^1 \supset \cdots \supset \mathscr{F}^w \supset 0 \tag{2.2}$$

such that first of all every fibre is a ψ-polarized weight w Hodge structure and secondly if the flat connection on H maps any local holomorphic section of $\mathscr{F}^p$ to $\mathscr{F}^{p-1} \otimes \Omega^1_M$. If we assume that the Hodge numbers of the Hodge structures are $\{h^{p,w-p}\}$, the filtration (2.2) yields a holomorphic map from the universal cover of M to the period domain $D := D\{H, \psi, h^{p,w-p}\}$ and the second condition means that its derivative maps the tangent space at every point into the horizontal subspace at the image point. Such a map therefore is called *horizontal.* In view of (2.1) we obtain a holomorphic map (the *period map*)

$$f: M \to D/\Gamma, \quad \Gamma := \mathrm{Im}(\sigma) \subset G \tag{2.3}$$

which is *locally liftable*, i.e. every $m \in M$ has a neighbourhood U in M such that $f|U$ lifts to a holomorphic map $U \to D$. Moreover every such lift is *horizontal.* In fact giving a variation of Hodge structure underlying H_M with fixed Hodge numbers $\{h^{p,w-p}\}$ is equivalent to giving a locally liftable holomorphic map (2.3) all of whose local lifts are horizontal. For simplicity we call such maps *horizontal.*

A variation of Hodge structure over $M \times N$ (N a connected complex manifold) is called *a family of variations of Hodge structure on M parametrized by N.* In what follows we are only interested in families for which the base N is a polydisk and we tacitly assume therefore that the monodromy from the N-factor is trivial. Equivalently this means giving a holomorphic horizontal map $\boldsymbol{f}: M \times N \to D/\Gamma$ with $\Gamma = \mathrm{Im}\{\pi_1(M \times \{*\}) \text{ in } G\}$.

If for some point $n \in N$, $\boldsymbol{f}|M \times \{n\} = f: M \to D/\Gamma$ we say that $\boldsymbol{f}$ is a *deformation* of f.

If $\{\boldsymbol{f}: M \times N \to D/\Gamma, n \in N\}$ is a deformation of a period map $f: M \to D/\Gamma$ every tangent vector $X \in T_n(N)$ in the usual way defines an *infinitesimal deformation* $\partial_X \in H^0(M, f^*T(D/\Gamma))$. Since $\boldsymbol{f}$ is locally liftable, the bundle $f^*T^h(D/\Gamma)$ exists and we can easily see that $\partial_X \in H^0(M, f^*T^h(D/\Gamma))$. The identification (1.5) induces an identification

$$f^*T^h(D/\Gamma) \cong \mathscr{E}nd^{-1,1}(\mathscr{H}_M). \tag{2.4}$$

If M is quasi-projective and the local monodromy operators are quasi-unipotent, we can form the *quasi canonical extension* (see e.g. the discussion in [P1, 2.1]) $\mathscr{E}nd_{\mathrm{can}}(\mathscr{H}_M)$ of the bundle $\mathscr{E}nd\ (\mathscr{H}_M)$ to $\bar{M}$. The sub bundle $\mathscr{E}nd^{-1,1}(\mathscr{H}_M)$ extends to a coherent subsheaf $\mathscr{E}nd^{-1,1}_{\mathrm{can}}(\mathscr{H}_M)$ of $\mathscr{E}nd_{\mathrm{can}}(\mathscr{H}_M)$.

PROPOSITION 2.5. *If M is quasi-projective and all local monodromy operators are quasi-unipotent, the infinitesimal deformation $\partial_X \in H^0$ $(M, \mathcal{E}nd^{-1,1}\, \mathcal{H}_M)$ extends to a section of $\mathcal{E}nd_{\mathrm{can}}^{-1,1}(\mathcal{H}_M)$.*

Proof. For simplicity we only look at smooth points y of the compactifying divisor Y and small transversal disks at y and we assume that the local monodromy around y is unipotent. So let Δ be a unit disk transversal to Y at y and let $p: \mathfrak{h} \to \Delta^*$, $p(z) = \exp(2\pi i z)$, be the universal cover. The local system $p^*(H_M|\Delta^*)$ is trivialized by a flat frame $\{\mathbf{e}_1, \ldots, \mathbf{e}_p\}$. This twisted sections $\tilde{\mathbf{e}}_j(z) = \exp(z \log T)\, \mathbf{e}_j$ define a holomorphic frame for the bundle $\mathcal{H}_M|\Delta^*$ and the $\mathcal{O}_\Delta$-submodule of $j_*(\mathcal{H}_M|\Delta^*)$ generated by this frame by definition is the quasi canonical extension of $\mathcal{H}_M|\Delta^*$ to Δ. Let $f: \Delta^* \to D/\langle T\rangle$ be the period map for the weight w Hodge structure on H_M/Δ^* and let $\tilde{f}: \mathfrak{h} \to D$ be a lifting of f. By the Nilpotent Orbit Theorem [S, Theorem 4.12] the twisted map $\exp(-z \log(T))\tilde{f}(z)$ descends to a holomorphic map $g: \Delta^* \to \check{D}$ which extends to over the origin to a holomorphic map $\bar{g}$. If we have a 1-parameter deformation f_t of f we obtain 1-parameter deformations g_t for g and also for $\bar{g}$. It then follows that the infinitesimal deformation $(\partial/\partial t)g|_{t=0}$ considered as a section of $\mathcal{E}nd(\mathcal{H}|\Delta^*)$ extends to a section of $\mathcal{E}nd(\mathcal{H}_{\mathrm{can}}|\Delta) = \mathcal{E}nd_{\mathrm{can}}(\mathcal{H}|\Delta)$. □

It follows that infinitesimal deformations are sections of the quasi canonical extension $\mathcal{T}_f$ of the sheaf $f^*T^h(D/\Gamma)$ identified with $\mathcal{E}nd^{-1,1}\mathcal{H}_M$:

$$\mathcal{T}_f := f^*T^h(D/\Gamma)_{\mathrm{can}}.$$

If H_M is defined over $\mathbb{Z}$ the Monodromy Theorem states that indeed all local monodromy operators are quasi-unipotent [S] so that the quasi canonical extensions exist. Moreover the monodromy group Γ acts discontinuously on D so that the quotient D/Γ is an analytic space.

As in [N] we can construct a *maximal* family of horizontal holomorphic maps $M \times T \to D/\Gamma$ deforming f, with tangent space at f isomorphic to $H^0(\bar{M}, \mathcal{T}_f)$.

Continuing as in [N], the set

$$\mathrm{Mor}^h(M, D/\Gamma) = \{\textit{horizontal holomorphic maps } M \to D/\Gamma\}$$

then can be given an analytic structure with as chart at f the intersection of a small ball centered at the origin of $H^0(\bar{M}, \mathcal{T}_f)$ and the analytic subset which is the set of zeroes of the *obstruction map* $H^0(\bar{M}, \mathcal{T}_f) \to H^1(\bar{M}, \mathcal{T}_f)$. Summarizing we have:

COROLLARY 2.6. *Assume that M is a smooth quasi-projective complex variety and H_M is defined over $\mathbb{Z}$. The set* $\mathrm{Mor}^h(M, D/\Gamma)$ of *horizontal holomorphic maps*

$M \to D/\Gamma$ has the structure of a finite dimensional analytic variety (possibly with infinitely many components). Its Zariski-tangent space at f can be identified with the space $H^0(\bar{M}, \mathcal{T}_f)$, where $\mathcal{T}_f := f^*T^h(D/\Gamma)_{\mathrm{can}}$.

If $f\colon M \to D/\Gamma$ is a period map and t a section of $f^*(T(D/\Gamma))$ we can compute the length of the tangent vector $f_*t(m)$, $m \in M$ with respect to the G-invariant metric Φ (see (1.6)).

COROLLARY 2.7. *Let ∂_X be an infinitesimal deformation considered as a section t of $f^*(T^h(D/\Gamma))$. The function $\Phi(f_*t(m), f_*t(m))$ is bounded on M.*

Proof. This is a local assertion near points y of the compactifying divisor Y. It suffices to restrict everything to a unit disk Δ transversal to Y at y. Monodromy acts on $End(H_{\Delta^*})$. We let V be the invariant subsystem. The section t induces a section of $V \otimes \mathcal{O}_{\Delta^*}$ so we restrict our attention to this bundle. By [D2, Proposition 1.13] the variation of Hodge structure on $End(H_{\Delta^*})$ induces one on V and so we can apply Schmid's asymptotic analysis from [S] to t. In particular, by [S, Corollary (6.7′)] if $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is a flat frame for V the functions $\Phi(\mathbf{e}_j, \mathbf{e}_j)$ have bounded length. Moreover, since by Proposition 2.5 the section t extends to give a section of the quasi canonical extension of V to Δ, we can apply the results of [P1, §2.2] to conclude that $\Phi(t, t) \leqslant \mathrm{Const.}\,(\log(1/|z|))^k$, $k \in \mathbb{Z}$ and z a coordinate on Δ. So, if $t = t_1\mathbf{e}_1 + \cdots + t_m\mathbf{e}_m$ we see that the functions $|t_j|$ are of at most logarithmic growth, hence bounded. So $\Phi(t, t)$ itself is bounded. □

3. Rigidity

We start with a general Lemma due to Carlson and Toledo (see [C-T, Proposition 5.2]):

LEMMA 3.1. *Let $g\colon U \to D$ a period map associated to a polarized variation of Hodge struction on a polydisk U. Let $u \in U$, $F = g(u)$, $\mathfrak{a} = g_*(T_uU) \subseteq T^h_FD$. The identification* (1.5) *makes $\mathfrak{a}$ into an abelian subspace of $\mathfrak{g}$.*

Let $f\colon M \to D/\Gamma$ be a period map associated to a polarized variation of Hodge structure. We use the hermitian metric Φ as defined in (1.6) and we let ∇ be the metric connection on the tangent bundle of D/Γ. Recall the identification (2.4) for the pull back under f of the horizontal tangent bundle of D/Γ.

THEOREM 3.2. *Assume that M is quasi-projective and that H_M is defined over $\mathbb{Z}$.*

(i) An infinitesimal deformation $t \in H^0(M, f^*T^h(D/\Gamma))$ is a flat section of $\mathcal{E}nd(\mathcal{H})^{-1,1}$, i.e. t can be considered as a global endomorphism of the local system H_M of Hodge type $(-1,1)$.

(ii) If t is as in (i), $m \in M$, $\tilde{f}$ a local lift of f around m the tangent vector $\tilde{f}_*(t(m)) \in T_{\tilde{f}(m)}D$ commutes with all $\bar{X}$ where X is a tangent vector in $\mathfrak{t}_m := \tilde{f}_*T_m(M) \subseteq \mathfrak{g}$ and the bar denotes complex conjugation in $\mathfrak{g}$.

Proof. Let $f: M \times N \to D/\Gamma$ be a deformation of the period map f and t the corresponding infinitesimal deformation vector. Let $\tilde{f}$ be a local lift of f around $m \times \{*\}$ extending $\tilde{f}$ and let $\mathfrak{a}_m = \tilde{f}_* T_{(m \times \{*\})}(M \times N)$ while we have $\mathfrak{t}_m = \tilde{f}_* T_m M$. We let $U \in T_m(M)$, $X = \tilde{f}_* U \in \mathfrak{t}_m \subseteq \mathfrak{a}_m$ and $Y = \tilde{f}_*(t(m)) \in \mathfrak{a}_m$. By Lemma 3.1 we have $[X, Y] = 0$ and so by Corollary (1.8) the holomorphic bisectional curvature $H(X, Y)$ is non-positive. Since

$$\partial\bar{\partial}\tilde{f}^*\Phi(t, t)(U, \bar{U}) = \tilde{f}^*\Phi(\nabla t(U), \nabla t(U)) - H(X, Y)$$

it follows that $\tilde{f}^*\Phi(t(m), t(m))$ is a plurisubharmonic function. By Corollary 2.7 this function is bounded and so, by the maximum principle, it must be constant. Now (i) follows.

For (ii) we observe that t being flat and horizontal, we have $H(X, Y) = 0$ for all $X \in \mathfrak{t}_m$, hence by Corollary 1.8 $[\bar{X}, Y] = 0$ for all $X \in \mathfrak{t}_m$. □

REMARK. The first part of the theorem generalizes a computation of Faltings [F, §4].

We are now going to give a geometric condition on period maps which implies rigidity. We start with the following.

DEFINITION 3.3. A period $f: M \to D/\Gamma$ is said to be *regularly tangent* if for some $m \in M$ and some local holomorphic lift g of f near m the zero endomorphism is the only endomorphism of H_M (the local system on M underlying the variation of Hodge structure giving f) commuting with the subspace of $\mathfrak{g}^1$ complex conjugate to $\mathfrak{t}_m := g_* T_m M \subseteq \mathfrak{g}^{-1}$.

An immediate consequence of Theorem 3.2 (ii) and Definition 3.3 is the following:

THEOREM 3.4. *Suppose that M is a smooth complex quasi-projective variety and H_M a local system of $\mathbb{Z}$-modules carrying a polarized variation of Hodge structures. This variation is rigid if* $(\mathrm{End}\, H_M \otimes \mathbb{C})^{-1,1} = 0$ *or if the period map associated to the variation of Hodge structure is regularly tangent.*

3.5. Examples of regularly tangent maps.

(i) If $X \in \mathfrak{g}^{-1}$ the adjoint is $\bar{X}$. The endomorphism $\mathrm{ad}_{[X, \bar{X}]}$ preserves the Hodge decomposition. We call X regular if it is invertible on $\mathfrak{g}^{-1}$, the $(-1, 1)$-part of $\mathfrak{g}$.

Claim. If a tangent vector to M is regular in $\mathfrak{g}^{-1}$ the tangent map is regularly tangent.

Proof. $[\bar{X}, Y] = 0$ and $[X, Y] = 0$ together imply $[[X, \bar{X}], Y] = \mathrm{ad}_{[X, \bar{X}]} Y = 0$ and hence $Y = 0$. □

(ii) Let $q: \mathfrak{g}^{-1} \to \bigoplus_{p=0}^{w-1} \mathrm{Hom}(H^{p,w-p}, H^{p-1,w+1-p})$ be the natural projection and introduce the *infinitesimal tangent map*

$$\kappa := q|\mathfrak{t}_m \to \bigoplus_{p=0}^{w-1} \mathrm{Hom}(H^{w-p,p}, H^{w-p-1,p+1})$$

with components κ_p, which induce $\mathbb{C}$-linear maps

$$\tau_p : \mathfrak{t}_m \otimes H^{p,w-p} \to H^{p-1,w+1-p}.$$

Suppose that τ_p is *surjective* for $p = 0, \ldots, w-1$. If $Y \in \mathfrak{g}^{-1}$ and $Y_p = \kappa_p(Y)$, for all $h_p \in H^{p,w-p}$ we can write $Y_p h_p = X_p h'_p$, $h'_p \in H^{p,w-p}$ and if $X \in \mathfrak{t}_m$ is such that $\kappa_p(X) = X_p$ one has $[\bar{X}, Y] = 0$. Starting at $p = 0$, one inductively finds that $\bar{X}_p X_p h'_p = 0$ and hence that $h'_p = 0$ and so $Y_p = 0$. Concluding we have shown: *If the infinitesimal tangent map κ induces surjective maps τ_p, the period map is rigid.*

REMARK. If f is regularly tangent smooth deformations of f are trivial, but since the deformation space need not be reduced at f we cannot conclude that $\mathrm{End}^{-1,1}\, H_M \otimes \mathbb{C} = 0$ in this case. The examples following the next proposition yield instances where f regularly tangent does imply $\mathrm{End}^{-1,1}\, H_M \otimes \mathbb{C} = 0$.

PROPOSITION 3.6. *Assume that M is quasi-projective and that H_M is defined over $\mathbb{Z}$. Suppose that for one (and hence all) $m \in M$ the subspace $\mathfrak{g}^{-1}(F_m)$ of $\mathfrak{g}$ commutes with the subspace $\mathrm{End}^{-1,1}(\mathscr{H}_M)$ consisting of the flat endomorphisms of $\mathscr{H}_M$ of type $(-1, 1)$. Then the variety $\mathrm{Mor}^h(M, D/\Gamma)$ is smooth with tangent space $\mathrm{End}^{-1,1}(\mathscr{H}_M)$.*

Proof. Because of Corollary 2.6, we only need to prove that the space $\mathrm{Mor}^h(M, D/\Gamma)$ is smooth at f. So fix some $Y \in H^0(\bar{M}, \mathscr{T}_f)$ considered as a flat global endomorphism of the local system $H_M \otimes \mathbb{C}$. Let $\{F^p\}$ be the Hodge-flag defined by f. As in the proof of [F, Theorem 2] we look at the deformed flag given by

$$\{F_t^p := \exp(tY)F^p\} \in \check{D}, \quad t \in \mathbb{C} \text{ small.}$$

For small t this flag actually belongs to D. The corresponding deformation of f is horizontal, since for all $X \in \mathfrak{g}^{-1}$ we have

$$X(F_t^p) = X \exp(tY)F^p = \exp(tY)XF^p \subset \exp(tY)F^{p-1} = F_t^{p-1}.$$

The infinitesimal deformation corresponding to F_t by construction is Y. □

EXAMPLES. If $[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}] = 0$, the assumption of the preceding theorem holds.

Examples include:
- Weight one Hodge structures (see [F, Theorem 2]),
- Weight two Hodge structures with $h^{2,0} = 1$, e.g. those that arise for families of K3-surfaces.

In order to be able to deduce rigidity for families of projective varieties over a fixed quasi-compact base some form of the Torelli–theorem must hold for the class $\mathscr{C}$ of manifolds one considers:

Any deformation within the class $\mathscr{C}$ is locally trivial in the complex analytic sense if and only if any local lift of the period map (with respect to k-forms) is constant. (3.7)

Examples of varieties for which (3.7) *holds*

(i) $\mathscr{C} = \{$abelian varieties of genus g and fixed polarization$\}$.
(ii) $\mathscr{C} = \{$curves of genus $g\}$.
(iii) $\mathscr{C} = \{$K3-surfaces with polarization of a fixed degree$\}$.
(iv) $\mathscr{C} = \{$complete intersection of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n\}$. In order to have local Torelli here we need ([F1]):
- when $m = 1$ and $n - m = 1$ we should have $k \neq 3$,
- when $m = 2$ and $n - m$ even we should have $(d_1, d_2) \neq (2, 2)$.

In the first three examples $\mathrm{Mor}^h(M, D/\Gamma)$ is smooth (see the Examples after Proposition 3.6) and so f being regularly tangent implies the vanishing of $\mathrm{End}^{-1,1} H_M \otimes \mathbb{C}$ (cf. the Remark preceding Proposition 3.6).

If a deformation of a family over a fixed base induces a trivial deformation of variation of Hodge structure, the Hodge structure in each point of the base remains fixed and if (3.7) holds, the fibre at this point does not change and hence the family itself must be rigid. So we arrive at the following theorem:

THEOREM 3.8. *Consider a class $\mathscr{C}$ of algebraic manifolds satisfying the local Torelli property* (3.7) *and fix a quasi-projective manifold M. Let X be a smooth quasi-projective variety and $X \to M$ be a proper holomorphic map with fibres in $\mathscr{C}$. Such a fibration with M fixed and fibres in $\mathscr{C}$ is rigid provided the global endomorphism of the local system of the primitive* kth *cohomology groups of the fibres of type $(-1, 1)$ is zero. This is the case if the period map for the family over M is regularly tangent.*

Examples of rigid families

1. Any family of abelian varieties or K3-surfaces over a smooth quasi-projective base for which the period map is regularly tangent.

2. A family of genus g curves over a quasi-projective M having the property that the Kodaira-Spencer map $T_m M \to H^1(\theta_C)$ (C the fibre over $m \in M$) is surjective.

In fact the cup product

$$H'(\theta_C) \otimes H^0(\omega_C) \to H^1(\mathcal{O}_M)$$

is surjective since the dual map

$$H^0(\omega_M) \to \mathrm{Hom}(H^0(\omega_M), H^0(\omega_M^{\otimes 2}))$$

obviously is injective (it comes from multiplication of 1-forms). Now we apply 3.5(ii).

3. Any family of degree d smooth hypersurfaces in $\mathbb{P}^{n+1}$ with surjective Kodaira Spencer mapping is rigid (with the exception of $(d, n) = (3, 2)$). In fact then local Torelli holds and also the cup products (X is a degree d hypersurface given by $F = 0$):

$$H^1(\theta_X) \otimes H^{n-k,k}(X) \to H^{n-k-1,k+1}(X)$$

can be identified with product maps

$$R_F^d \times R_F^{t(k)} \to R_F^{t(k+1)}$$

between suitable graded pieces of the jacobian ring R_F of X (See e.g. [P-S, §13]). In particular they are all surjective and we can apply 3.5(ii). Compare this with [C-D]: for most (d, n) the families are maximal, so in particular rigid. However the following values of (d, n) yield instances of non-maximal but rigid families:

$$(d, n) = (3, 4), (4, 4), (3, 5) \text{ or if } n = 1, d \geqslant 5.$$

Section 4. Finiteness

As before we assume that the vector space H carries a lattice $H_{\mathbb{Z}}$ of maximal rank, ψ is defined over $\mathbb{Z}$, M is a smooth quasi-projective complex variety and H_M is defined over $\mathbb{Z}$. The complexified system is semisimple by [S, §7]:

$$H_M \otimes \mathbb{C} = \bigoplus_i H_{M,i} \otimes V_i, \quad H_{M,i} \text{ irreducible.}$$

Deligne's arguments in [D2] imply that the variation of Hodge structure on H_M

is reflected in a decomposition $V_i = \oplus_p V_i^p$ (complex weight 0 Hodge structure) for all V_i, unique up to a possible shift $\{h_i^{p,-p}\} \to \{h_i^{p-1(i),-p+1(i)}\}$ for the Hodge indices $h_i^{p,-p} = \dim V_i^p$ on V_i. Since the Hodge numbers $\{h^{p,w-p}\}$ are fixed, the Hodge numbers of these weight zero Hodge structures are also fixed up to a finite number of possible shifts for the Hodge indices $h_i^{p,-p}$.

Complex weight 0 Hodge structures with given Hodge numbers on V_i are parameterized by a flag manifold, so the variations of Hodge structure on H_M polarised by ψ and with Hodge numbers $\{h^{p,w-p}\}$ (zero for $p > 0$ or $p < 0$) are parameterized by a disjoint union of a finite number of products of homogeneous domains, open subsets of unions of products of the flag manifolds corresponding to points yielding real polarisable variations on $H_{\mathbb{Z}}$ as opposed to complex variations. It is a *generalized period domain* and we denote it by

$$P = P(H_M, \{h^{p,w-p}\}).$$

Note that P need not be connected and that different connected components may very well have different dimensions.

To every polarizable variation $(H_M, \{F^p\})$ of Hodge structure on H_M with $\{h^{p,w-p}\}$ as Hodge numbers, there is associated a unique point $\tau(H_M, \{F^p\})$ in P and to every deformation of such a Hodge structure with fixed monodromy with connected base N there is associated a holomorphic map

$$\tau: N \to \{\text{Connected component of } P\}.$$

Theorem 3.2(i) implies that this map is *horizontal* in the obvious sense. Since in general one cannot expect that $\mathrm{End}^{p,-p} H_M = 0$ for $|p| > 1$ not all holomorphic τ will be horizontal. So if τ is rigid the relevant connected component of P need not be a point and a Arakelov-type finiteness statement cannot be expected. One way of forcing this brutally is by demanding $\mathrm{End}\, H_M = \mathrm{End}^{0,0}\, H_M$ and this then yields Simpson's finiteness statement:

THEOREM (Simpson [Si]). *There are only finitely many polarisable variations of Hodge structure on a fixed local system H_M for which* $\mathrm{End}\, H_M \otimes \mathbb{C} = \mathrm{End}^{0,0}\, H_M$.

REMARK. Simpson formulates his theorem only for M a curve, but his proof is also valid for any quasi-projective smooth complex M.

Next, we vary the local system as well (keeping the underlying $\mathbb{Z}$-module $H_{\mathbb{Z}}$ as well as ψ fixed). By [D2, Théorème 0.1] up to isomorphy, there are at most finitely many local systems H_M on M with $(H_{\mathbb{Z}}, \psi)$ as fibre which carry a polarisable variation of Hodge structure. In other words, $\mathrm{Mor}^h(D/G_{\mathbb{Z}})$ is a disjoint union of $G_{\mathbb{Z}}$-orbits of $\mathrm{Mor}^h(D/\Gamma_i)$, i in some finite set. Then Simpson's theorem coupled with Theorem 3.8, implies the following finiteness statement:

THEOREM 4.1. *Suppose that M is a smooth complex quasi-projective variety. Fix a euclidean lattice (H, ψ). There are at most finitely many isomorphy classes of variations of ψ-polarized Hodge structure such that the monodromy has values in* $\mathrm{Aut}(H_{\mathbb{Z}}, \psi)$ *and such that* $\mathrm{End}\, H_M \otimes \mathbb{C} = \mathrm{End}^{0,0}\, H_M$. *Also there are at most finitely many such isomorphy-classes of variation of Hodge structure for which the period map is regularly tangent and for which in addition* $\mathrm{End}\, H_M{}^{-p,p} = 0$ *if* $|p| > 1$.

Situations where this last condition is automatically satisfied include the following two cases (see the Examples after Proposition 3.3):
- the weight one case,
- the weight two case where $h^{2,0} = 1$. This, together with the Torelli theorem for abelian varieties, resp. K3-surfaces leads to:

THEOREM 4.2. *Let M be a smooth quasi-complex projective variety*

(i) *There are at most finitely many non-isomorphic families of abelian varieties over M for which* $\mathrm{End}\, R^1 f_* \mathbb{C}$ *has pure Hodge type* $(0, 0)$. *In particular finiteness holds for those families for which the period map is regularly tangent. If* $\mathrm{End}\, R^1 f_* \mathbb{C}$ *has not pure Hodge type* $(0, 0)$ *there are infinitely many non-isomorphic families of abelian varieties over M.*

(ii) *There are at most finitely many non-isomorphic families of* K3*-surfaces over M for which* $\mathrm{End}\, R^2 f_* \mathbb{C}$ *has pure Hodge type* $(0, 0)$. *In particular finiteness holds for those families for which the period map is regularly tangent. If* $\mathrm{End}\, R^2 f_* \mathbb{C}$ *has not pure Hodge type* $(0, 0)$ *there are infinitely many non-isomorphic families of* K3*-surfaces over M.*

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