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Some Remarks About Reider's Article 'On the Infinitesimal Torelli Theorem for Certain Irregular Surfaces of General Type'

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0. Introduction

In this note I want to show how a careful analysis of Reider's beautiful constructions in [R] leads to an Infinitesimal Torelli theorem for n-dimensional varieties having "enough" 1-forms (see Sect. 3 for precise statements) in dimensions greater than two.

For clarity I have reformulated some essential constructions from Reider's paper in the framework of rank n-vector bundles over n-dimensional varieties in Sects. 1 and 2. In Sect. 3 I apply these to the cotangent bundle. It is instructive to see how Reider's theorem follows from the main result, Theorem 3.3 since it then becomes clear how certain deep properties of surfaces of general type play a crucial rôle: the Miyaoka-Yau-inequality [B-P-V, p. 212] and the Castelnuovo-De Franchis theorem [B-P-V, p. 123].

I also give a theorem valid for threefolds which is very much analogous to Reider's theorem for surfaces. For higher dimensions there is still an inequality which remains to be proven before we have a true generalisation [inequality (7)]. I refer to Sect. 3 for the precise statements.

1. Indecomposable Zero-Cycle Incidence Maps

Let X be a compact complex connected manifold of dimension n and $\mathscr{E}$ a rank n vector bundle on X. We assume that a generic section of $\mathscr{E}$ has zero locus consisting of $m = c_n(\mathscr{E})$ distinct points. So, if

$$I(\mathscr{E}) = \{([e], x) \in \mathbb{P}H^0(\mathscr{E}) \times X; e(x) = 0\}$$

is the incidence variety, the projection induces a generically finite morphism

$$\pi : I(\mathscr{E}) \longrightarrow \mathbb{P}H^0(\mathscr{E}) \ . \qquad \textit{(the zero-cycle incidence-map)} \ .$$

We want to make a geometric assumption which implies that π is *indecomposable*, meaning that π does not factor over two generically finite morphisms of degrees larger than one.

Property I. There exists a pencil P of sections of $\mathscr{E}$ vanishing simultaneously at exactly one point and the variety where two generic members of P are dependent is an irreducible curve.

The following lemma is a trivial, but basic observation:

Lemma 1.1. *If property I holds π is indecomposable.*

Proof. Assume that π is decomposable, e.g. over a Zariski-open subset U of $\mathbb{P}H^0(\mathscr{E}))$ we have a factorization into étale maps:

$$I(\mathscr{E})|U \xrightarrow[\mu]{} T \xrightarrow[\nu]{} U \tag{1}$$

with $\deg \mu \geqq 2$ and $\deg \nu \geqq 2$. Let x be the point in common to the zero-loci $Z(e)$, $e \in P$. The zero-cycle $Z^0(e)$ defined by $\mu^{-1}\mu(([e], x))$ contains $\deg(\mu)$ points, so at least one extra point besides x which has to vary in a curve $C(P)$. Since by definition $C(P)$ belongs to the variety $S(P)$ of points where two generic members of P become dependent – by assumption irreducible – we must have $C(P) = S(P)$. So, if $y \in Z(e) \backslash Z^0(e)$ there must be some $[e'] \in P$, $[e'] \neq [e]$ with $y \in Z(e')$, but then $y = x$. This contradiction shows that $Z(e) = Z^0(e)$. But then we find: $\deg Z(e) = m = \deg \mu$. $\deg \nu \geqq 2$. $\deg \nu = 2$. $\deg Z^0(e)$, which is impossible. So π must be indecomposable. □

The following proposition describes a geometric situation in which Property I holds

Proposition 1.2. *Suppose that $\mathscr{E}$ is spanned by its sections and that the canonical map $\varphi : \mathbb{P}(\mathscr{E}^\vee) \longrightarrow \mathbb{P}H^0(\mathscr{E})^\vee$ (which is well-defined) has degree one onto its image and contracts at most finitely many $(n-1)$-dimensional varieties. Then Property I holds and hence the zero-cycle incidence map is indecomposable.*

Definition. If $\mathscr{E}$ has the properties mentioned in the preceding proposition we call $\mathscr{E}$ *almost very ample.*

Before we start the proof of Proposition 1.2 we first introduce some notation:

$$V := H^0(\mathscr{E})$$

$$\mathscr{S} := \text{universal subbundle on } \operatorname{Gr}(n, V^\vee) = Y \ .$$

Now, since $\mathscr{E}$ is spanned by sections, we have obvious bundle maps

$$\mathscr{E} \,.\, \longrightarrow \mathscr{S} \hookrightarrow Y \times V^\vee$$

The canonical map $\varphi : \mathbb{P}(\mathscr{E}^\vee) \longrightarrow \mathbb{P}V^\vee$ is just the composition of the projectivization of these maps followed by the projection of $Y \times \mathbb{P}V^\vee$ onto the second factor. So we arrive at a commutative diagram defining the *Gauss-map* γ:

$$\begin{tikzcd}
\mathbb{P}(\mathscr{E}^\vee) \arrow[r] \arrow[d] \arrow[rr, bend left, "\varphi"] & \mathbb{P}(\mathscr{S}) \arrow[r] \arrow[d] & \mathbb{P}(V^\vee) \\
X \arrow[r, "\gamma"'] & \operatorname{Gr}(n, V^\vee) &
\end{tikzcd}$$

The degeneracy locus Z^p for linearly independent sections is just $\gamma^{-1}(\Omega)$, where Ω is a Schubert variety associated to the r-tuple $(p,0,0,0\ldots,0)$ (cf. [F, Example 14.3.2]). Now Ω is irreducible of codimension p and hence if φ has generically degree 1, we expect Z^p to be irreducible as well. In fact, given a $(p-1)$-dimensional subvariety of $\mathrm{Gr}(n, V^\vee)$ we can find a Schubert variety Ω avoiding it, so if γ contracts at most a subscheme of dimension p we can move Ω away from the contracted scheme and Z^p remains irreducible. Hence:

Lemma 1.3. *If $\mathscr{E}$ is spanned by its sections, γ has generically degree 1 and contracts at most a p-dimensional subscheme, the locus where $n-p+1$ generic sections become dependent is an irreducible variety of codimension p.* □

Proof of 1.2. Taking $p=n-1$ in Lemma 1.3, we almost have Property I. We still need to see that we can find x so that for a generic pencil P of sections vanishing at x the common zero-locus of sections in P consist of x only. Let P_x be the vector space of sections vanishing at x and let $s \in P_x$ a generic section. Now, since φ is a degree one morphism, for generic x a generic section t belonging to P_x is non-zero at all points where s vanishes. The pencil formed by s and t has the property we want. □

2. Cohomological Properties Implying Decomposability of the Zero-Cycle Incidence-Map

Let X be a compact complex connected manifold, Z an analytic subspace of X (possibly non-reduced or reducible), $\mathscr{F}$ a locally free $\mathscr{O}_X$-module and $z \in H^0(\mathscr{O}_Z)$ having the property

$$\forall f \in H^0(\mathscr{F}) \ , \quad \exists g \in H^0(\mathscr{F}) \quad \text{with} \quad z \cdot f = g|Z \tag{2}$$

Since g is unique up to elements in $H^0(\mathscr{F} \otimes \mathscr{I}_Z)$, the assignment $f \mapsto g$ induces a linear map $H^0(\mathscr{F}) \longrightarrow H^0(\mathscr{F})/H^0(\mathscr{F} \otimes \mathscr{I}_Z)$ and hence a map

$$A(Z,z): H^0(\mathscr{F})/H^0(\mathscr{F} \otimes \mathscr{I}_Z) \longrightarrow H^0(\mathscr{F})/H^0(\mathscr{F} \otimes \mathscr{I}_Z)$$

We apply this to $\mathscr{F} = \det(\mathscr{E})$, where $\mathscr{E}$ is as in Sect. 1, $Z = p_1 + \ldots + p_m$ the zero-locus of a generic section of $\mathscr{E}$. In case the linear system $|\det \mathscr{E}|$ is without base points there is the following interpretation of $H^0(\mathscr{F})/H^0(\mathscr{F} \otimes \mathscr{I}_Z)$.

Lemma 2.1. *Let $\psi: X \longrightarrow \mathbb{P}H^0(\mathscr{F})^\vee)$ be the canonical morphism defined by the linear system $|\mathscr{F}|$, where $\mathscr{F} = \det \mathscr{E}$. The span of $\psi(p_1), \ldots, \psi(p_m)$ is the projectivization of the annihilator of the subspace $H^0(\mathscr{F} \otimes \mathscr{I}_Z)$ of $H^0(\mathscr{F})$, i.e.*

$$\mathrm{Span}\{\psi(p_1), \ldots, \psi(p_m)\} \cong \mathbb{P}[H^0(\mathscr{F})/H^0(\mathscr{F} \otimes \mathscr{I}_Z)]^\vee$$

Moreover each of the lines in $H^0(\mathscr{F})/H^0(\mathscr{F} \otimes \mathscr{I}_Z)^\vee$ corresponding to $\psi(p_j)$ belongs to an eigenspace of the transpose of $A(Z,z)$ and every eigenspace contains at least one such line.

Proof. See the proof of Lemma 2.3 of [R]. □

Next, we want to vary $[e] \in \mathbb{P}H^0(\mathscr{E})$ and formulate a condition which makes it possible to choose $z = z(e) \in H^0(\mathcal{O}_{Z(e)})$ (almost) canonically so that we obtain an eigenspace decomposition of the dual of $H^0(\mathscr{F})/H^0(\mathscr{F} \otimes \mathscr{I}_{Z(e)})$ depending on e only.

Given a section e of $\mathscr{E}$ with zero-scheme $Z = Z(e)$ the evaluation morphism factors as $\mathscr{E}^\vee \longrightarrow \mathscr{I}_Z \longrightarrow \mathcal{O}_X$. In cohomology this leads to a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & H^0(\mathcal{O}_Z)/H^0(\mathcal{O}_X) \arrow[d] \\
H^1(\mathscr{E}^\vee) \arrow[r] \arrow[dr, ".e"'] & H^1(\mathscr{I}_Z) \arrow[d] \\
 & H^1(\mathcal{O}_X) \arrow[d] \\
 & 0
\end{tikzcd} \tag{3}$$

hence we arrive at a well defined map

$$\tau_0(e): \mathrm{Ker}\,(H^1(\mathscr{E}^\vee) \xrightarrow[.e]{} H^1(\mathcal{O}_X)) \longrightarrow H^0(\mathcal{O}_Z)/H^0(\mathcal{O}_X) \tag{4}$$

Property II. $\exists \xi \in H^1(\mathscr{E}^\vee)$ *in the kernel of* e *such that* $\tau_0(e)\xi$ *lifts to a non-constant function* $z(e)$ *satisfying* (2).

The ambiguity in $z(e)$ is only slight: one can arbitrarily add a constant. This does not affect the eigenspace decomposition of $A(Z, z)$, i.e. fixing ξ with property II $A(Z(e), z(e))$ has an eigenspace decomposition

$$H^0(\mathscr{F})/H^0(\mathscr{F} \otimes \mathscr{I}_{Z(e)}) = \bigoplus_{j=1}^{k} U_j([e])$$

depending only on $[e] \in \mathbb{P}H(\mathscr{E})$. If we vary this point in a suitable Zariski open neighbourhood U the k points in the respective Grassmannians of $H^0(\mathscr{F})/H^0(\mathscr{F} \otimes \mathscr{I}_{Z(e)})$ representing these eigenspaces define an étale covering $T \xrightarrow[\mu]{} U$. Since every $x \in Z(e)$ determines a unique eigenspace by Lemma 2.1, we have an obvious map $I(\mathscr{E})|U \xrightarrow[v]{} T$ such that the composition is the zero-cycle incidence map.

Since $\deg v = c_n(\mathscr{E})/\deg \mu$ and $\deg \mu = k$ satisfies the inequalities

$$2 \leqq k \leqq \dim\,(H^0(\det \mathscr{E})/H^0(\det \mathscr{E} \otimes \mathscr{I}_Z))$$

we arrive at the following Lemma:

Lemma 2.2. *The zero-cycle incidence map is decomposable provided*

(i) *property II holds.*

(ii) *the following inequality is satisfied*

$$c_n(\mathscr{E}) \geqq 2(\dim\,(H^0(\det \mathscr{E})/H^0(\det \mathscr{E} \otimes \mathscr{I}_Z)) \tag{5}$$

The following cohomological properties imply Property II

Lemma 2.3. *Assume*

a) *The cup product map* $H^1(\mathcal{O}_X) \longrightarrow \mathrm{Hom}(H^0(\det\mathscr{E}), H^1(\det\mathscr{E}))$ *is injective,*

b) $H^q(\wedge^{q+1}\mathscr{E}^\vee)=0$ *for* $q=1,\dots,n-1$

then any nonzero ξ *in the kernel of the cup product map*

$$\delta: H^1(\mathscr{E}^\vee) \longrightarrow \mathrm{Hom}(H^0(\det\mathscr{E}), H^1(\mathscr{E}^\vee\otimes\det\mathscr{E}))$$

has Property II.

Proof. For brevity introduce

$$T_0(e)=\mathrm{Ker}(H^1(\mathscr{E}^\vee)\longrightarrow H^1(\mathcal{O}))\ ,$$

$$T_1(e)=\mathrm{Ker}(H^1(\mathscr{E}^\vee\otimes\det\mathscr{E})\longrightarrow H^1(\det\mathscr{E}))$$

We have a commutative square

$$\begin{array}{ccc} T_0(e) & \xrightarrow[\delta]{} & \mathrm{Hom}(H^0(\det\mathscr{E}), T_1(e)) \\ \Big\downarrow{\scriptstyle \tau_0(e)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}(1,\tau_1(e))} \\ H^0(\mathcal{O}_Z)/H^0(\mathcal{O}_X) & \longrightarrow & \mathrm{Hom}(H^0(\det\mathscr{E}),\ H^0(\det\mathscr{E}\otimes\mathcal{O}_Z)/H^0(\det\mathscr{E})) \end{array}$$

where the vertical maps come from multiplication and the map $\tau_1(e)$ is defined in the same way as $\tau_0(e)$. From this diagram we infer:

If $0\neq\eta\in\mathrm{Ker}\,\delta\cap T_0(e)$ *and* $\tau_0(e)$ *is injective*

$\xi=\tau_0(e)\eta$ *satisfies the requirements needed in Property II*

So it suffices to show that $\tau_0(e)$ is injective and that $\mathrm{Ker}\,\delta\subset T_0(e)$. To prove the first statement we look at the Koszul resolution of $\mathscr{E}$ defined by e:

$$0\longrightarrow \wedge^n\mathscr{E}^\vee \longrightarrow \wedge^{n-1}\mathscr{E}^\vee \longrightarrow \dots \longrightarrow \wedge^2\mathscr{E}^\vee \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{I}_Z \longrightarrow 0$$

and we observe that b) implies that $H^1(\mathscr{E}^\vee)\longrightarrow H^1(\mathscr{I}_Z)$ is injective, so-recalling the definition of $\tau_0(e)$ [(3) and (4)] we see that $\tau_0(e)$ is injective as well.

The last statement follows from a) and the commutative diagram

$$\begin{array}{ccc} H^1(\mathscr{E}^\vee) & \xrightarrow[\delta]{} & \mathrm{Hom}(H^0(\det\mathscr{E}), H^1(\mathscr{E}^\vee\otimes\det\mathscr{E})) \\ \downarrow{\scriptstyle \cdot e} & & \downarrow{\scriptstyle \cdot e} \\ H^1(\mathcal{O}_X) & \longrightarrow & \mathrm{Hom}(H^0(\det\mathscr{E}), H^1(\det\mathscr{E}))\ . \end{array} \quad \square$$

We introduce some useful *notation*:
if $\mathscr{S}$ is any sheaf on X we let

$$h^j=\dim H^j(X,\mathscr{S})$$

$$\chi_j(\mathscr{S})=\sum_{k=0}^{j}(-1)^k h^k$$

$$\chi_j=h^0 \quad \text{if} \quad j\leqq 0\ .$$

Proposition 2.4. *Inequality* (5) *follows from*

$$c_n(\mathscr{E}) \geqq 2\left[\sum_{j=0}^{n} (-1)^j \chi_{j-2}(\wedge^{n-j}\mathscr{E})\right]. \tag{6}$$

Proof. Break up the Koszul resolution

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{E} \longrightarrow \wedge^2 \mathscr{E} \longrightarrow \ldots \longrightarrow \wedge^{n-1}\mathscr{E} \longrightarrow \mathscr{I}_Z \otimes \wedge^n \mathscr{E} \longrightarrow 0$$

into short exact sequences

$$0 \longrightarrow \mathscr{A}_j \longrightarrow \wedge^j \mathscr{E} \longrightarrow \mathscr{A}_{j+1} \longrightarrow 0\ , \qquad j = 1, \ldots, n-1\ .$$

with $\mathscr{A}_1 = \mathcal{O}_X$ and $\mathscr{A}_n = \mathscr{I}_Z \otimes \wedge^n \mathscr{E}$, and add the inequalities

$$h^0(\mathscr{I}_Z \otimes \wedge^n \mathscr{E}) \geqq h^0(\wedge^{n-1}\mathscr{E}) - h^0(\mathscr{A}_{n-1})$$

$$\ldots\ldots$$

$$0 \geqq (-1)^j [\chi_{j-1}(\mathscr{A}_{n-j}) - \chi_{j-2}(\wedge^{n-j}\mathscr{E}) + \chi_{j-2}(\mathscr{A}_{n-j+1})]$$

We find

$$h^0(\mathscr{I}_Z \otimes \wedge^n \mathscr{E}) \geqq -\sum_{j=1}^{n} (-1)^j \chi_{j-2}(\wedge^{n-j}\mathscr{E}))$$

and so

$$h^0(\wedge^n \mathscr{E}) - h^0(\mathscr{I}_Z \otimes \wedge^n \mathscr{E}) \leqq \sum_{j=0}^{n} (-1)^j \chi_{j-2}(\wedge^{n-j}\mathscr{E}))$$

and (6) implies (5). □

Remark 2.5. For a 2-bundle (6) reduces to

$$c_2(\mathscr{E}) \geqq 2\,[h^0(\wedge^2 \mathscr{E}) - h^0(\mathscr{E}) + 1]$$

and for a 3-bundle to

$$c_3(\mathscr{E}) \geqq 2\,[h^0(\wedge^3 \mathscr{E}) - h^0(\wedge^2 \mathscr{E}) + h^0(\mathscr{E}) - 1 + h^1(\mathcal{O}_X)]$$

Summarising this section we have

Theorem 2.6. *Let $\mathscr{E}$ be a rank n vector bundle on a complex projective variety X of dimension n. Assume that*

a) *The cup product map* $H^1(\mathcal{O}) \longrightarrow \mathrm{Hom}(H^0(\det \mathscr{E}), H^1(\det \mathscr{E}))$ *is injective,*
b) $H^q(\wedge^{q+1}\mathscr{E}^\vee) = 0$ *for* $q = 1, \ldots, n-1$
c) $c_n(\mathscr{E}) \geqq 2\left[\sum_{j=0}^{n} (-1)^j \chi_{j-2}(\wedge^{n-j}\mathscr{E})\right]$
d) *The kernel of the cup product map*

$$\delta : H^1(\mathscr{E}^\vee) \longrightarrow \mathrm{Hom}(H^0(\det \mathscr{E}), H^1(\mathscr{E}^\vee \otimes \det \mathscr{E}))$$

is not injective

then the zero cycle-incidence map is decomposable.

3. Applications to Infinitesimal Torelli-Problems

Let us first observe that the results of Sects. 2 and 3 combined yield:

Theorem 3.1. *Let $\mathscr{E}$ be a rank n vector bundle on a complex projective variety X of dimension n. Assume that*

a) *The cup product map* $H^1(\mathcal{O}_X) \longrightarrow \mathrm{Hom}(H^0(\det \mathscr{E}), H^1(\det \mathscr{E}))$ *is injective,*

b) $H^q(\wedge^{q+1}\mathscr{E}^\vee)=0$ *for* $q=1, \ldots, n-1$

c) $c_n(\mathscr{E}) \geqq 2\left[\sum_{j=0}^{n}(-1)^j \chi_{j-2}(\wedge^{n-j}\mathscr{E})\right]$

d) *$\mathscr{E}$ is almost very ample (cf. Definition in Sect. 1), then the cup-product map*

$$\delta: H^1(\mathscr{E}^\vee) \longrightarrow \mathrm{Hom}(H^0(\det \mathscr{E}), H^1(\mathscr{E}^\vee \otimes \det \mathscr{E})$$

is injective.

Proof. By Lemma 1.1 Property I (a consequence of d) by Proposition 1.2)) guarantees the indecomposability of the zero-cycle incidence map. If δ would not be injective a), b) and c) imply that the zero-cycle incidence map is decomposable by Theorem 2.6. □

We want to apply this theorem to $\mathscr{E} = \Omega^n$. The kernel of the cup product map in this case measures the extend to which the Infinitesimal Torelli theorem with respect to n-forms fails.

Let us investigate the various conditions of the preceding theorem.

Lemma 3.2. *The following properties of holomorphic* 1*-forms imply condition a) in Theorem 3.1:*

1) *Any non zero* 1*-form can be complemented to a set of n linearly independent* 1*-forms,*

2) *Any n-form obtained by wedging n* 1*-forms is zero if and only if these* 1*-forms are linearly dependent.*

Proof. Condition a) follows if there does not exist a holomorphic 1-form $\alpha \neq 0$ such that $\bar{\alpha} \wedge \beta = 0$ for all $n-1$-forms β. So suppose such a form $\alpha = \alpha_1$ exists. Choose $n-1$ additional 1-forms $\alpha_2, \ldots, \alpha_n$ which together with α_1 form a linearly independent set. Since $0 = \bar{\alpha}_1 \wedge \alpha_2 \ldots \wedge \alpha_n$, we have $0 = (\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n) \wedge (\bar{\alpha}_1 \wedge \bar{\alpha}_2 \ldots \wedge \bar{\alpha}_n)$. So $\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n = 0$ and hence $\alpha_1, \ldots, \alpha_n$ must be dependent, contradicting our assumption 2). □

Observing that condition 1) is automatically satisfied if the cotangent bundle is everywhere spanned by 1-forms, we look for conditions guaranteeing 2).

Lemma 3.3 (Castelnuovo-De Franchis). *Suppose that the cotangent bundle is everywhere spanned by its sections. Assume that for all d with $1 \leqq d \leqq n-1$ there does not exist a rational map from X onto a variety Y of dimension d with $h^{0,d}(Y) \geqq d+1$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be n* 1*-forms with $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n = 0$. Then $\omega_1, \omega_2, \ldots, \omega_n$ are linearly dependent.*

Proof. (Compare the proof of Proposition X.9 in [B]) The hypothesis implies that for some rational functions g_i we have e.g. $\omega_1 = g_2\omega_2 + \ldots + g_n\omega_n$. Since $d\omega_j = 0$, we deduce that ω_i depends on $dg_2, \ldots, dg_n$ (over the field of rational functions of X) and so

$$\omega_2 \wedge \ldots \wedge \omega_n = h\, dg_2 \wedge \ldots \wedge dg_n$$

$$\omega_1 \wedge \ldots \wedge \hat{\omega}_i \ldots \wedge \omega_n = \pm g_i h\, dg_2 \wedge \ldots \wedge dg_n$$

Now $d(\omega_2 \wedge \ldots \wedge \omega_n) = 0 = dh \wedge dg_2 \wedge \ldots \wedge dg_n$, hence $h, g_2, \ldots, g_n$ are algebraically dependent, so the rational map

$$X \underset{\varphi}{\dashrightarrow} \mathbb{P}^n \,(p \mapsto (h(p), g_2(p), \ldots, g_n(p))$$

maps onto a variety of dimension $\leqq n-1$, say Z. By a sequence of blowings up $\varepsilon : \hat{X} \longrightarrow X$ the composition $f = \varepsilon \circ \varphi$ is a morphism, which has a Stein-factorization:

$$f : \hat{X} \xrightarrow[g]{} Y \xrightarrow[h]{} Z \subset \mathbb{P}^n$$

If $x_1, x_2, x_3, \ldots, x_n$ are the standard affine coordinates, we have

$$f^*(x_1\, dx_2 \wedge \ldots \wedge dx_n) = \varepsilon^*(\omega_2 \wedge \ldots \wedge \omega_n)$$

$$f^*(x_1 x_i\, dx_1 \wedge \ldots \wedge d\hat{x}_i \ldots \wedge dx_n) = \pm \varepsilon^*(\omega_1 \wedge \ldots \wedge \hat{\omega}_i \ldots \wedge \omega_n)$$

A local computation shows that the forms $h^*(x_1\, dx_2 \wedge \ldots \wedge dx_n)$ and $h^*(x_1 x_i\, dx_1 \wedge \ldots \wedge d\hat{x}_i \ldots \wedge dx_n)$ are regular on Y and – if we assume that Z has dimension $n-1$ – linearly independent, hence $h^{0,n-1}(Y) \geqq n$ contradicting our assumption in case $\dim Z = n-1$. The general case goes similarly. □

Remark. In the case of surfaces, the condition of the Lemma means that the surface does not admit a holomorphic map onto a curve of genus at least 2 and the statement of the lemma is the classical *Castelnuovo-De Franchis-lemma*, [B-P-V, p. 123]).

As a direct consequence of the preceding discussion we have

Theorem 3.4. *Assume*

a) *For all d with $1 \leqq d \leqq n-1$ there does not exist a rational map from X onto a variety Y of dimension d with $h^{0,d}(Y) \geqq d+1$.*

b) $H^{q+1,n-q}(K_X) = 0$ *for* $q = 1, \ldots, n-1$,[1]

c) $c_n(\Omega_X^1) \geqq 2\left[\sum_{j=1}^{n} (-1)^j \chi_{j-2}(\Omega_X^{n-j})\right]$, *or -equivalently*

$$\left.\begin{array}{lll} h^{1,n-1} + \ldots + h^{n-1,1} & \geqq 2\,[h^{1,n-2} + \ldots h^{n-2,1}] & (n \text{ even}) \\ h^{1,n-1} + \ldots + h^{n,0} & \geqq h^{1,n-2} + \ldots h^{n-1,0} & (n \text{ odd}) \end{array}\right\} \tag{7}$$

d) *The cotangent bundle is almost very ample,*

then the infinitesimal Torelli theorem with respect to n-forms holds for X.

Remarks

Condition b) holds if K_X is ample, by Nakano's vanishing theorem. Furthermore, $h^{n,0}(K_X) = H^0(K_X^{\otimes 2}) = 0$ as soon as K_X is "nef and big" as a consequence of the

[1] E.g., if the canonical bundle is ample (see the remarks after 3.4)

Kawamata-Viehweg vanishing theorem [K]. It follows that for surfaces b) is a consequence of d).

Condition c) for surfaces by Remark 2.5 we must have $c_2(X) \geqq 2 \cdot \chi(\mathcal{O}_X)$ whereas in fact $c_2(X) \geqq 3 \cdot \chi(\mathcal{O}_X)$ [B-P-V, p. 212]. Alternatively, (7) in this case simply means $h^{1,1} \geqq 2q$. This can be proved in a more elementary fashion for surface without rational pencils as in [B-P-V, p. 125]. There we have the inequality $h^{1,1} \geqq 2q-1$, but in the Kähler case one can modify the proof a bit by restricting to primitive forms yielding the slightly sharper inequality $h^{1,1} \geqq 2q$.

These remarks imply that our general theorem implies Reider's theorem:

Corollary 3.5. *If X is a surface of general type having the following properties:*
a) *It does not admit a holomorphic map onto a curve of genus* $\geqq 2$,
b) *The cotangent bundle is almost very ample, then the Infinitesimal Torelli holds with respect to 2-forms.* □

For threefolds we have:

Theorem 3.6. *Let X be a threefold having the following properties:*
a) *X has almost very ample cotangent bundle,*
b) $H^{2,2}(K_X)=0$,
c) *X admits no rational map onto a curve of genus $\geqq 2$ or onto a surface with $p_g \geqq 3$, then the infinitesimal Torelli theorem with respect to 3-forms holds for X.*

Proof. In view of the preceding remarks, we only have to show that in our case the inequality (7) holds, which now reads:

$$h^{3,0}+h^{1,2}+h^{2,1} \geqq h^{2,0}+h^{1,1} \tag{8}$$

We use [G-L], Proposition 3.4 applied to a 1-form ω having 0-dimensional zero locus. We conclude:

$$h^{2,1} \geqq h^{2,0}$$

$$h^{1,2} \geqq h^{1,1}-h^{1,0}$$

Now we consider the natural map

$$\wedge^3 H^0(\Omega_X^1) \longrightarrow H^0(\Omega_X^3)$$

Using an argument as in [B-P-V, p. 125] we find

$$h^{3,0} \geqq 3(h^{1,0}-3)+1=3h^{1,0}-8$$

Adding up we find (8) since $h^{1,0} \geqq 4$ because the cotangent bundle is almost very ample. □

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References

[B] Beauville, A.: Complex algebraic surfaces. Cambridge: Cambridge Univ. Press 1983

[B-P-V] Barth, W. Peters, C., Ven, Van de, A.: Compact complex surfaces. Erg. Math. 3. Folge, Vol. 4. Berlin Heidelberg New York: Springer 1984

[F] Fulton, W.: Intersection theory. Erg. Math. 3. Folge, Vol. 2. Berlin Heidelberg New York: Springer 1984

[G-L] Green, M., Lazarsfeld, R.: Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville. Invent. Math. **90**, 389–407 (1988)

[K] Kawamata, Y.: A generalization of the Kodaira-Ramanujam vanishing theorem. Math. Ann. **261**, 43–46 (1982)

[R] Reider, I.: On the Infinitesimal Torelli Theorem for certain irregular surfaces of general type. Math. Ann. **280**, 285–302 (1988)

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