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A Criterion for Flatness of Hodge Bundles over Curves and Geometric Applications

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Introduction

The following conjecture plays a central rôle in the birational classification of complex projective varieties.

Conjecture $C_{n,m}$. *Let X, resp. Y be an n-dimensional, resp. m-dimensional nonsingular complex projective variety and let $f: X \to Y$ be a surjective morphism, whose general fibre F is connected. Then sub-additivity for the Kodaira-dimensions holds:*

$$\kappa(X) \geqq \kappa(Y) + \kappa(F).$$

For the definition of the Kodaira-dimension as well as for a discussion of the conjecture $C_{n,m}$ I refer to the Bourbaki-talk by Esnault [7]. Here, I only want to say that $C_{n,n-1}$ and $C_{n,1}$ are true, and that for applications to Albanese mappings a stronger version of $C_{n,m}$ is needed, which in [7] is called "Conjecture $C^+_{n,m}$". If $m=1$, it merely says that in addition to the validity of $C_{n,1}$, in the special case when Y is an elliptic curve and $\kappa(X) = \kappa(F) = 0$, there should exist a surjective morphism $Y' \to Y$ such that the fibre product $X \times_Y Y'$ is birational to a product $Y' \times F$. For $n=2$, $m=1$ this last requirement follows from a result of Arakelov [1, Theorem 1.1] which implies that $\deg(f_*\omega_{X/Y}) \geqq 0$ and equality holds if and only if f is "isotrivial", i.e. after a finite covering of Y, the fibre product is birational to a product. (Here $\omega_{X/Y}$ is the relative dualizing sheaf, see Sect. 5.1).

Kawamata's proof of $C_{n,1}$ as presented in [13] depends on a result of Fujita [11] about line bundle-quotients of $f_*\omega_{X/Y}(\dim Y = 1)$. Very roughly, Fujita's result says that Arakelov's inequality generalizes to the case when $\dim X$ is arbitrary (but still $\dim Y = 1$). Here, I want to look more closely to what happens if $\deg(f_*\omega_{X/Y}) = 0$. The natural framework for this question is Griffith's theory of variations of Hodge structure. Our main result is the following.

Main Theorem (= Theorem 4.1). *Let C be a smooth projective curve and $C_0 \subset C$ a Zariski-open subset on which one has a (real) variation of Hodge structure $\mathscr{V} = \mathscr{F}^0 \supset \ldots \supset \ \supset \mathscr{F}^m \supset 0$ of weight m. Let $'\mathscr{F}^j$ be the canonical extension of $\mathscr{F}^j$ and let*

$\mathscr{K}('\mathscr{V})=\bigotimes_{j=1}^{m}(\det '\mathscr{F}^j)$. *We have* (i) $\deg('\mathscr{F}^m)\geqq 0$ *and equality holds if and only if* $\mathscr{F}^m$ *is a flat subbundle of* $\mathscr{V}$ *and the local monodromies are unipotent on* $\mathscr{F}^m$, (ii) $\deg\mathscr{K}('\mathscr{V})\geqq 0$ *and equality holds if and only if all the Hodge bundles* $\mathscr{F}^j$ *are flat subbundles of* $\mathscr{V}$ *and all local monodromies are unipotent.*

This theorem can directly be translated in the "geometric situation", where the variation of Hodge structure comes from the usual Hodge filtration on the primitive cohomologygroups $H_0^n(X_t, \mathbb{C})$ ($n=\dim X_t$) of the smooth fibres X_t of a surjective holomorphic map $f: X\to C$ with X a compact Kähler manifold; the Zariski-open subset C_0 is precisely the set of regular values of f. This translation is Proposition 5.1.1. Loosely speaking it says that $\deg(f_*\omega_{X/C})>0$ if and only if the variation of Hodge-structure given by $H_0^n(X_t, \mathbb{C})\supset H^0(\omega_{X_t})$ is non-trivial, i.e. if and only if the periods of the holomorphic n-forms of the fibres X_t are *not* constant. Obviously, to draw any geometric conclusion in case these periods are constant, one needs to know that constant periods imply that the fibres are isomorphic (see 5.1.2 for a precise statement). So, if this property holds, $\deg(f_*\omega_{X/C})=0$ implies that $f|f^{-1}(C_0)$ is a fibre bundle in the complex-analytic sense (Theorem 5.1.3). To extend the bundle over the punctures, I make use of certain results of Fujiki about the properness of the relative Chow-scheme (see Sect. 5.2). The final geometric result is stated as Theorem 5.3.1, which is indeed a generalization of Arakelov's result:

Theorem. *Let X be a compact connected Kähler-manifold, C a smooth curve and $f: X\to C$ a proper surjective holomorphic map with connected fibres. Then* $d=\deg f_*\omega_{X/C}\geqq 0$. *If* $d=0$ *and if moreover the Torelli-property* (5.1.2) *holds, f is a fibre bundle over C_0 = set of non-critical values of f. There exists a finite cover $\sigma: D\to C$, branched at most in critical values of f and one further point, such that the fibre product $X\times_C D$ is bimeromorphic to a fibre bundle over D with typical fibre F. If moreover the automorphisms of F act faithfully on $H^{n,0}(F)$, we can take* $\sigma=\mathrm{id}$, *i.e. f itself is bimeromorphically a fibre bundle.*

As a consequence, I prove $C_{n,1}^{+}$ under the assumption that the Torelli-property holds for the generic fibre. It is probably possible to remove the Torelli-property, using Viehweg's techniques from [24].

I want to make two remarks concerning the above theorem. First of all it is stated in the setting of Kähler-geometry and as such appears to be new. Secondly, if X is *not* assumed to be Kähler, the inequality $d\geqq 0$ does not hold as demonstrated by an example of Blanchard [2], as was kindly pointed out to me by K. Ueno.

Finally, some remarks about the content of this paper. Whereas the main theorem in Sect. 4 and its geometric consequences in Sect. 5 have been mentioned already, some justification should be given for the inclusion of Sects. 1–3, which contain results, more or less known to the expert. There are two reasons to include this material, apart from the desire to be as self-contained as possible. The first one is, that the results I need are rather hard to locate in the literature. For example the asymptotic behaviour of the Hodge metric near a puncture is only barely outlined in [25]. The second reason is that it is quite crucial to know this behaviour in detail, since one must first of all show that $\int_{C_0}\partial\bar\partial\log h^j$ converges [h^j is the Hodge metric on $\det(\mathscr{F}^j)$] and secondly one has to interpret the residues of h^j near the punctures in case this integral vanishes (see Proposition 2.2.1 and the proof of the Main Theorem in Sect. 4).

1. Variations of Hodge Structure

1.1. Basic Definitions

We start with the following data.

(i) A real vector space V of finite dimension,
(ii) A non-negative integer m,
(iii) A non-degenerate real-valued bilinear form Q on V such that $Q(v,w)=(-1)^m Q(w,v)$ for all $v,w\in V$,
(iv) A connected complex manifold B,
(v) A representation of the fundamental group on V:

$$\varrho:\pi_1(B)\to \mathrm{Aut}(V,Q):=G_{\mathbb{R}} \quad (\textit{monodromy-representation}).$$

These data determine a locally constant sheaf V_B on B in the usual way: the fundamental group $\pi_1(B)$ acts as covering transformations of the universal cover $\tilde{B}\to B$ of B and V_B is the quotient of the constant sheaf $V\times\tilde{B}$ on $\tilde{B}$ by the action of $\pi_1(B)$ given by

$$\gamma\cdot(v,z)=(\varrho(\gamma)v,\gamma^{-1}z), \qquad \gamma\in\pi_1(B), \qquad (v,z)\in V\times\tilde{B}.$$

The vectorbundle $V_B\bigotimes_{\mathbb{C}}\mathcal{O}_B=\mathscr{V}$ admits the canonical flat connection

$$\nabla=1\otimes d \qquad \mathscr{V}\to\mathscr{V}\otimes\Omega^1_B\,.$$

It is characterized by the property that its sheaf of flat sections is exactly the locally constant sheaf $V_B\otimes\mathbb{C}$.

The bilinear form Q defines a flat bilinear form on $\mathscr{V}$ with real values on V_B. We shall denote it also by Q.

Now we come to the main definition.

By a (real) variation of Hodge structure of weight m on B underlying $(\mathscr{V},Q)$ we mean a filtration of $\mathscr{V}$ by holomorphic subbundles $\mathscr{F}^p$ (the Hodge bundles)

$$\mathscr{V}=\mathscr{F}^0\supset\mathscr{F}^1\supset\ldots\supset F^m\supset\mathscr{F}^{m+1}=0$$

which satisfies the *horizontality condition*

$$\nabla\mathscr{F}^p\subset\Omega^1_B\otimes\mathscr{F}^{p-1}$$

and the following *polarization conditions*

(i) There is a Q-orthogonal direct sum decomposition

$$\mathscr{V}=\bigoplus_{p+q=n}\mathscr{H}^{p,q} \qquad \mathscr{H}^{p,q}=\mathscr{F}^p\cap\overline{\mathscr{F}^q} \quad \text{(Hodge-decomposition)},$$

where the bar denotes complex conjugation with respect to the real structure coming from the one on $V\otimes\mathbb{C}$.

(ii) If $\pi_{p,q}: \mathscr{V} \to \mathscr{H}^{p,q}$ is the projection with respect to the Hodge-decomposition and if $C = \sum_{p+q=m} i^{p-q} \pi_{p,q}$ the hermitian form h on $\mathscr{V}$ defined by

$$h(v, w) = Q(Cv, \bar{w}) \qquad \text{(Hodge metric)}$$

is positive definite.

The following situation, referred to as "*geometric situation*", gives a standard example. Let X, B be connected Kähler manifolds and

$$f: X \to B$$

a smooth, surjective and proper holomorphic map. Let ω be a Kähler class of X and consider on each fibre $X_b = f^{-1}(b)$ the primitive cohomology groups $H_0^m(X_b, \mathbb{C})$ with respect to $\omega|X_b$. Since f is locally trivial in the C^∞-sense, fixing a base point $* \in B$ there is a representation of $\pi_1(B, *)$ on $H^m(X_*, \mathbb{C})$ as well as on $H_0^m(X_*, \mathbb{C})$. So the preceding construction gives a flat vector bundle $(\mathscr{H}^m, \nabla)$, whose fiber over $b \in B$ is $H_0^m(X_b, \mathbb{C})$. It carries the usual Hodge filtration, which fits together to a filtration $\mathscr{H}^m = \mathscr{F}^0 \supset \ldots \supset \mathscr{F}^m \supset 0$. It is a basic result, due to Griffiths, that these subbundles $\mathscr{F}^p$ are in fact *holomorphic* subbundles and that they satisfy the horizontality condition above. The polarization conditions follow from the usual Hodge-Riemann bilinear relations, since we work with primitive cohomology. We refer to [5] for proofs of these statements and for references to the original papers.

1.2. The Period Map

A given variation of Hodge structures $(\mathscr{V}, Q, \mathscr{F})$ induces on each fiber $\mathscr{F}_b$ of $\mathscr{V}(b \in B)$ a polarized (real) Hodge structure, whose Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} \mathscr{H}_b^{p,q}$ are constant on B. This leads us to consider polarized Hodge structures on V with given Hodge numbers $h^{p,q}$. To describe this set, we use the language of Hodge-frames. Given a decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=m} H^{p,q}, \qquad H^{p,q} = \overline{H^{q,p}}, \qquad \dim_{\mathbb{C}} H^{p,q} = h^{p,q}$$

an hermitian form h can be defined as in 1.1. and we only consider those decompositions that are h-orthogonal and for which h is positive definite. A Hodge-frame adapted to such a decomposition, by definition is an $(m+1)$-tuple $\{\mathbf{e}^0, \ldots, \mathbf{e}^m\}$ where each $\mathbf{e}^j$ is a set

$$\mathbf{e}^j = \{e_1^j, \ldots, e_{k_j}^j) \qquad k_j = h^{j,m-j}$$

of vectors giving an orthonormal basis for $H^{j,m-j}$ and where $e_k^j = \bar{e}_k^{m-j}$. The manifold of Hodge frames can be identified with the real Liegroup $G_{\mathbb{R}}$. So, if we fix a reference Hodge structure we obtain a representation of the set of polarized Hodge structures on V of weight m and with given Hodge numbers $h^{p,q}$ as a homogeneous manifold

$$D = G_{\mathbb{R}}/K,$$

where K is the (compact) isotropy group of the reference Hodge structure. The manifold D is an open submanifold in its compact dual

$$\check{D} = G_{\mathbb{C}}/\check{K},$$

where $G_{\mathbb{C}} = \mathrm{Aut}(V_{\mathbb{C}}, Q)$ and $\check{K}$ is the isotropy group in $G_{\mathbb{C}}$ of the reference Hodge structure. The manifold $\check{D}$ is the closed subvariety inside the flag manifold of flags $(V_{\mathbb{C}} = F^0 \supset F^1 \supset \ldots \supset F^m \supset 0;\ \dim_{\mathbb{C}} F^j = \sum_{i \geqq j} h^{i,m-i})$ which only satisfy the first polarization condition in 1.1. The points of D correspond to flags that in addition satisfy the second polarization condition.

Returning to the case of a variation of Hodge structure on B, we can transport the Hodge filtration on $\mathscr{V}$ to a holomorphically varying filtration on the induced bundle on $\tilde{B}$, the universal cover of B. This bundle is trivial, so we can identify it with $V_{\mathbb{C}} \times \tilde{B}$ and over each point $z \in \tilde{B}$ we have a Hodge structure on $V_{\mathbb{C}}$, hence a unique point in D. This yields a holomorphic map

$$\tilde{\Phi}: \tilde{B} \to D \quad \text{("period map").}$$

In case V carries an integral lattice $V_{\mathbb{Z}}$ and Q is defined over $\mathbb{Z}$ (e.g. if in the geometric situation the fibres are curves, or if f is a projective morphism) the monodromy-group Γ (i.e. the image of the monodromy-representation) belongs to $G_{\mathbb{Z}} = \mathrm{Aut}(V_{\mathbb{Z}}, Q)$ and Γ acts properly discontinuously on D and so $\Gamma \backslash D$ carries the structure of a normal analytic space [3]. The map $\tilde{\Phi}$ descends to a holomorphic map

$$\Phi: B \to \Gamma \backslash D \quad \text{(period map).}$$

1.3. Positivity of the Chern Forms of Hodge Bundles

We specify the situation of the preceding sections to the case where B is a 1-dimensional connected complex manifold. We recall Griffiths' curvature estimates from [12] in this case. We note firstly that the connection ∇ induces an $\mathcal{O}_B$-linear map $\mathscr{F}^j \to (\mathscr{V}/\mathscr{F}^j) \otimes \Omega^1_B$, the second fundamental form of $\mathscr{F}^j$ in $\mathscr{V}$. If $\mathrm{Gr}^j = \mathscr{F}^j/\mathscr{F}^{j+1}$, the horizontality of ∇ implies that the second fundamental form induces $\mathcal{O}_B$-linear maps

$$\sigma^j: \mathrm{Gr}^j \to \mathrm{Gr}^{j-1} \otimes \Omega^1_B$$

or – equivalently – holomorphic (1,0)-forms τ^j with values in $\mathrm{Hom}(\mathrm{Gr}^j, \mathrm{Gr}^{j-1})$. There are natural isomorphisms of C^∞-vectorbundles $\mathrm{Gr}^j \xrightarrow{\sim} \mathscr{H}^{j,m-j}$ and we use these to transport the Hodge metric to Gr^j. We let $(\sigma^j)^*$ be the adjoint of σ^j with respect to the Hodge metric and let $(\tau^j)^*$ its associated (0,1)-form with values in $\mathrm{Hom}(\mathrm{Gr}^{j-1}, \mathrm{Gr}^j)$. The curvature form θ^j of the metric connection on Gr^j is given by

$$\theta^j = -(\tau^j)^* \wedge \tau^j - \tau^{j+1} \wedge (\tau^{j+1})^*$$

(compare [20, Lemma 7.18]). The associated real (1,1)-form

$$c(\mathrm{Gr}^j) = (i/2\pi) \quad \mathrm{trace}\, \theta^j$$

is the Chern form of Hodge metric on Gr^j. If we choose a local Hodge frame field for Gr^j, i.e. a Hodge frame depending in a C^∞ manner on a local parameter t on B and express τ^j in the frame as a matrix of (1,0)-forms

$$\tau^j = (\tau^j_{k,l}\, dt)$$

an easy calculation shows that $c(\mathrm{Gr}^j)$ locally is given by the real (1,1)-form

$$(i/2\pi)\left\{\sum_{k,l} |\tau^j_{k,l}|^2 - \sum_{r,s} |\tau^{j+1}_{r,s}|^2\right\} dt \wedge d\bar{t}.$$

In particular, we see that $c(\mathscr{F}^m)$ is positive semi-definite and is identically zero precisely when the bundle map σ^m is identically zero. This is the case if and only if ∇ preserves $\mathscr{F}^m$, i.e. if and only if $\mathscr{F}^m$ is a flat sub-bundle of $\mathscr{V}$. Summarizing, we have

(1.3.1) **Proposition.** *Given a variation of Hodge structures* $\mathscr{V} = \mathscr{F}^0 \supset \ldots \supset \mathscr{F}^m \supset 0$ *of weight* m *over a* 1*-dimensional complex manifold* B. *Then the Chern form* $c(\mathscr{F}^m)$ *is positive semi-definite and it vanishes if and only if* $\mathscr{F}^m$ *is a flat subbundle of* $\mathscr{V}$.

To keep track of the information in the remaining Hodge bundles, we must keep in mind the alternation of signs in the preceding formula for $c(\mathrm{Gr}^j)$. Following Griffiths [12, p. 147] we introduce the *canonical bundle of a variation of Hodge structure* $(\mathscr{V}, \mathscr{F}^0)$

$$\mathscr{K}(\mathscr{V}) = \bigotimes_{j=1}^{m} (\det \mathrm{Gr}^j)^{\otimes j} = \bigotimes_{j=1}^{m} (\det \mathscr{F}^j).$$

Since the metric on $\mathscr{K}(\mathscr{V})$ induced by the Hodge metric has curvature $\omega(\mathscr{V}) = \sum_{j=1}^{m} j\,\mathrm{Trace}(\theta^j)$, the associated Chern form $i/2\pi\ \omega(\mathscr{V})$ locally is expressed as $(i/2\pi)\sum_j \sum_{k,l} |\tau^j_{k,l}|^2 dt \wedge d\bar{t}$, which is semi-positive and as before we can prove (cf. [12, Proposition 7.15]).

(1.3.2) **Proposition.** *Given a variation of Hodge structure* $\mathscr{V} = \mathscr{F}^0 \supset \mathscr{F}^1 \supset \ldots \supset \mathscr{F}^m \supset 0$ *of weight* m *over a* 1*-dimensional complex manifold* B. *Then the Chern form for the Hodge metric on the bundle* $\mathscr{K}(\mathscr{V})$ *is positive semi-definite and it vanishes* if and only if *all Hodge bundles* $\mathscr{F}^j$ *are flat subbundles of* $\mathscr{V}$.

(1.3.3) *Remark.* If $\mathscr{V}$ is defined over $\mathbb{Z}$, one has a period map $\Phi: B \to \Gamma\backslash D$ and flatness of all bundles $\mathscr{F}^j$ is equivalent with Φ being constant. In the general case $\tilde{\Phi}: \tilde{B} \to D$ is a well-defined holomorphic mapping and flatness of all of the bundles $\mathscr{F}^j$ means that $\tilde{\Phi}$ is constant.

2. Asymptotic Analysis of the Hodge Bundles and Metrics in the Curve Case

2.1. The Quasi-Canonical Extension

Let C be a smooth complex projective curve and let C_0 be the complement in C of finitely many points. For every $p \in C\backslash C_0$, a small loop in C_0 winding positively about p determines an element in $\pi_1(C_0)$, which we denote by γ_p. Assume now that data as in 1.1 are given (with $B=C_0$) and such that the local monodromy-operators

$$T_p=\varrho(\gamma_p), \quad p\in C\backslash C_0$$

are *quasi-unipotent*, i.e. $(T_p^\nu-1)^{M+1}=0$ for some non-negative integers ν and M. This is the case in the geometric situation: the projective case is treated in [20, cf. Theorem 6.1], and [15]. For the general case see [21, Sect. 2] or [4, Sect. 7].

We define $(\zeta_\alpha=e^{2\pi i\alpha})$

$$V_\alpha = \{v\in V_{\mathbb{C}}; (T_p-\zeta_\alpha 1)^{M+1}v=0\}$$

$$T_\alpha = \zeta_{-\alpha}T_p|V_\alpha$$

$$N_\alpha=\log T_\alpha=\sum_{l=1}^{M}(-1)^l(1/l)(T_\alpha-1)^l.$$

Then $V_\alpha=(0)$ if $\nu\alpha\notin\mathbb{Z}$ and we have a decomposition

(2.1.1) $$V_{\mathbb{C}}=\bigoplus_{0\leqq\alpha<1}V_\alpha.$$

To construct the quasi-canonical extension [6, p. 94] of the bundle $\mathscr{V}$, we restrict $\mathscr{V}$ to a unit coordinate disc (Δ, t) centered at $p\in C\backslash C_0$ and we form the universal cover of Δ^*:

$$\tau: H\to\Delta^*, \quad \tau(z)=e^{2\pi i t}.$$

Sections of $\mathcal{O}_{\Delta^*}(\mathscr{V})$ correspond to holomorphic maps $s: H\to V_{\mathbb{C}}$ satisfying $s(z+1)=T_p s(z)$. If $v\in V_\alpha$ we obtain a section $\tilde{v}\in\Gamma(\mathcal{O}_{\Delta^*}(\mathscr{V}))$ by setting

(2.1.2) $$\tilde{v}(z)=\exp(2\pi i\alpha z+zN_\alpha)\cdot v,$$

and if $\{v_1,\ldots,v_n\}$ is a basis of $V_{\mathbb{C}}$ adapted to the splitting (2.1.1), the sections $\{\tilde{v}_1,\ldots,\tilde{v}_n\}$ trivialize $\mathcal{O}_{\Delta^*}(\mathscr{V})$. If $j: C_0\to C$ is the inclusion, the $\mathcal{O}_C$-submodule of $j_*\mathscr{V}$ with basis $\{\tilde{v}_1,\ldots,\tilde{v}_n\}$ defines an extension of $\mathscr{V}$ to all of C as a locally free sheaf. This sheaf is independent of the choice of basis and of the choice of the t-coordinate and is denoted by $'\mathscr{V}$. It carries a connection $'\nabla$ with logarithmic residues, in fact, we have

$$'\nabla\tilde{v}=(2\pi i\alpha+N_\alpha)\tilde{v}\otimes(dt/t)$$

and the residue at 0 [w.r. to a basis adapted to (2.1.1)] is the matrix $\oplus\left(\alpha+\frac{1}{2\pi i}N_\alpha\right)$. For later reference we introduce the rational number

(2.1.3) $$\alpha_p=\mathrm{Trace}(\mathrm{Res}_p{'\nabla})=\sum_{0\leqq\alpha<1}\alpha\dim_{\mathbb{C}}V_\alpha.$$

Sometimes it is useful to pull back $\mathcal{O}_{\Delta^*}(\mathscr{V})$ via the ν-sheeted cover

$$\sigma_\nu\Delta_u^*\to\Delta^* \quad (\sigma_\nu(u)=u^\nu).$$

In $\sigma_\nu^*(\mathscr{V}))$ we have T_p^ν as unipotent monodromy, so repeating the preceding construction with T_p^ν, using now $N=\log T_p^\nu=\sum_{\mu=0}^{\nu+1}\nu N_\mu$, we find a holomorphic section $\tilde{v}'(u)=\exp(1/2\pi i\, N \log u)\, V$, for $\sigma_v^*(\mathcal{O}_{\Delta^*}(\mathscr{V}))$ and we have

(2.1.4) $$\sigma_v^*(\tilde{v})=u^{v\alpha}\tilde{v}'.$$

Let us finally consider a variation of Hodge structure $\{\mathscr{F}^0\}$ or C_0 underlying $\mathscr{V}$. The canonical extension of the Hodge bundle $\mathscr{F}^k$ is defined as follows. We define isomorphisms ϕ_z of $V_{\mathbb{C}}$ by letting $\phi_z(v)=\tilde{v}(z)(v\in V_\alpha)$. The subspaces $\phi_z^{-1}\mathscr{F}^k(z)$ are T_p-invariant and give a holomorphically varying subspace $'\mathscr{F}^k(t)$ of $'\mathscr{V}(t)(t=\tau(z))$. The corresponding section of the appropriate Graßmann bundle is algebraic with respect to the intrinsic algebraic structure on $'\mathscr{V}$ (compare [20, pp. 234–236]), hence extends over the origin. This gives $'\mathscr{F}^k(0)\subset{}'\mathscr{V}(0)$, and doing this at each puncture yields $'\mathscr{F}^k$, the canonical extension of $\mathscr{F}^k$. From the construction it follows that $'\mathscr{F}^k$ is a locally free subsheaf of $'\mathscr{V}$ and that $v\in{}'\mathscr{F}^k(0)$ if and only if there exists a germ of a holomorphic section w of $'\mathscr{V}$ near 0 such that $\tilde{v}+tw$ is a holomorphic section of $\mathscr{F}^k$ in Δ^*.

2.2. Asymptotic Analysis of the Hodge Metric

In view of later applications, we need bounds for the Hodge metric and its logarithmic derivatives when a point on the curve C approaches a puncture $C\backslash C_0$. Schmid's asymptotic analysis in [20] can be modified slightly to give an asymptotic expansion for Hodge-length from which these bounds are derived. We follow the outline given by Zucker in [25].

As before, we let (Δ, t) be a coordinate disc centered at a puncture $p\in C\backslash C_0$, and we let $\tau: H\to\Delta^*$ be the universal covering of Δ^*. The theory of the period map (cf. Sect. 1.2) gives a holomorphic map

$$\tilde{\Phi}: H\to D$$

In order to simplify the argument, we *assume that T_p is unipotent*. Later on we shall make the necessary changes in the quasi-unipotent case. The map $\tilde{\Psi}: H\to D$ defined by $\tilde{\Psi}(z)=\exp(-zN)\tilde{\Phi}(z)\,(N=\log T_p)$ satisfies $\tilde{\Psi}(z+1)=\tilde{\Psi}(z)$, so it descends to $\Psi:\Delta^*\to D$. By Schmid's nilpotent orbit theorem [20, 4.9] this map extends to a holomorphic map $\Psi:\Delta\to D$ and the nilpotent orbit $\exp(zN)\circ\Psi(0)$ is strongly asymptotic to $\tilde{\Phi}(z)$. Let us paraphrase the argument on [loc. cit., pp. 252–253] leading to a bound on the Hodge metric, to show that actually a nice asympotic expansion holds. Let $0\in D$ be some reference Hodge filtration. One compares the two Hodge filtrations $\exp(zN)\circ\Psi(0)$ and $\tilde{\Phi}(z)$ by transporting them isometrically back to 0 and by measuring in the Hodge metric there. Let us start with some real-analytic trivialization of the bundle of Hodge frames in some neighbourhood U of 0. If $\{\mathrm{v}^j(0)\}$ is a Hodge frame for the reference Hodge filtration, we thus have a real-analytic map

$$\tilde{s}: U\to G_{\mathbb{R}}$$

with $\tilde{s}(0)=1$ and such that $\{\tilde{s}(u)\mathrm{v}^j(0)\}$ is a Hodge frame for $u\in U$. The SL_2-orbit theorem [loc. cit, Theorem 5.13] yields a real-analytic map

$$n: H\to G_{\mathbb{R}}$$

with $\exp(zN)\circ\Psi(0)=n(z)\circ 0$ (in Schmid's notation $n(z)=m(z)(s(z))^{-1}$, [loc. cit., p. 253]). As mentioned before, the nilpotent orbit theorem implies that the element $s(z)\in G_{\mathbb{R}}$ defined by

$$s(z)=\tilde{s}\circ(n(z)^{-1}\circ\tilde{\Phi}(z)),\qquad \operatorname{Im} z\gg 0$$

is strongly asymptotic to 1 in any $G_{\mathbb{R}}$-invariant norm. The inverse of the isometry

$$m(z)=s(z)n(z)\in G_{\mathbb{R}}$$

is now used to transport $\tilde{\Psi}(z)$ back to 0.

The nilpotent map N determines a filtration (m is the weight of the Hodge filtration) $0\subset W_0\subset W_1\subset\ldots\subset W_{2m-1}\subset W_{2m}=V_{\mathbb{C}}$ [loc. cit., p. 247] and if $v\in W_k$, Schmid obtains an expansion ($z=x+iy$)

$$m(z)^{-1}v=s(z)^{-1}\{y^{(k-m)/2}\exp(-xN)v+y^{(k-m-1)/2}\exp(-xN)v_{m+1}$$
$$+\text{lower order terms}\}$$

So for $\tilde{v}(z)=\exp(zN)v$ we obtain an expression [26, (4.6)]:

$$m(z)^{-1}\tilde{v}=s(z)^{-1}\{y^{(k-m)/2}\exp(iyN)v+\text{lower order terms}\}$$
$$=s(z)^{-1}\left\{y^{(k-m)/2}\left(\sum_{j=0}^{M}\frac{i^j}{j!}N^j v\right)+\text{lower order terms}\right\}$$

and for the Hodge metric we find ($v,w\in W_k$):

$$h(\tilde{v}(z),\tilde{w}(z))_{\tilde{\Phi}(z)}=h(m(z)^{-1}\tilde{v}(z),m(z)^{-1}\tilde{w}(z))_0$$
$$=f(e^{2\pi iz})\{y^{k-m}a_k(v,w)+y^{k-m-1}a_{k+1}(v,w)+\ldots\},$$

where $g(z)=f(e^{2\pi iz})$ is a real-analytic function, defined for $\operatorname{Im} z\gg 0$ and with $g(\infty)=1$, and where $a_{k+j}(v,w)$ ($j\geqq 0$) are real constants depending on v and w alone. Because of the construction of the canonical extensions $'\mathcal{F}^p$ as given in Sect. 2.1, these expansions are also valid for sections of $'\mathcal{F}^p$ near the punctures (see [26, (5.3)]). Similarly, one finds bounds for sections $s(t)$ generating the determinant bundle $\det{}'\mathcal{F}^p$ near a puncture (or the canonical bundle $\mathcal{K}('\mathcal{V})$). The expansions are of the form

$$h(s(t),s(t))=f(t)\{(-\log|t|)^{k-m}h_k+(-\log|t|)^{k-m-1}h_{k+1}+\ldots\}$$

for some real-analytic $f(t)$ with $f(0)=1$ and, k an integer between 0 and $2m$.

The general case follows easily, using (2.1.4). One finds that the Hodge-norm for a section $s(t)$ generating $\det{}'\mathcal{F}^j$ near a puncture has an asymptotic expansion of the form

$$h(s(t),s(t))=|t|^{-2\alpha_p^j}g(t)\{(\log(1/|t|))^{\beta_p}+\text{lower order terms}\},$$

where $g(t)$ is real analytic, $g(0)\neq 0$, $\alpha_p^j\in\mathbb{Q}$, $\alpha_p^j\geqq 0$, and $\beta_p\in\mathbb{Z}$, $-m\leqq\beta_p\leqq m$. In the special case when $\mathcal{F}^j$ is a flat subbundle of $\mathcal{V}$, the residue $\alpha_p(\mathcal{F}^j)=\operatorname{Trace}(\operatorname{Res}_p{}'V|\mathcal{F}^j)$ is well-defined [see (2.1.3) for the case $j=0$] and the number α_p^j coincides with this residue. Summarizing, we have

(2.2.1) **Proposition.** *If* $s(t)$ *is a generating section of the line bundle* $\det{}'\mathcal{F}^j$ *in a coordinate* (Δ,t) *centered at* $p\in C\backslash C_0$, *we have the following expansion for its*

Hodge-norm

$$h(s(t), s(t)) = |t|^{-2\alpha_p^j} g(t) \{(\log(1/|t|))^{\beta_p} + \textit{lower order terms}\},$$

where $\alpha_p^j \in \mathbb{Q}, \alpha_p^j \geqq 0, \beta_p \in \mathbb{Z}, -m \leqq \beta_p \leqq m$ and wheere $g(t)$ is analytic with $g(0) \neq 0$. If $\mathscr{F}^j$ is a flat subbundle of $\mathscr{V}$ the number α_p^j is equal to the residue $\alpha_p(\mathscr{F}^j)$ and vanishes if and only if T_p has unipotent action on the $\mathbb{C}$-subvector space of $V_{\mathbb{C}}$ which gives the flat subbundle $\mathscr{F}^j$.

The last clause follows, since $\mathrm{Trace}(\mathrm{Res}_p{}'V|\mathscr{F}^j) = \Sigma\alpha$, where the summation is over those α for which $V_\alpha \cap \mathscr{F}^j \neq (0)$ (repeated $\dim_{\mathbb{C}} V_\alpha \cap \mathscr{F}^j$ times), so it vanishes if and only if the $\mathbb{C}$-subvector space of $V_{\mathbb{C}}$ giving $\mathscr{F}^j$ belongs entirely to V_0, i.e. if and only if T_p acts in a unipotent manner on this subspace.

(2.2.2) *Remark.* An analogous statement holds for the linebundle $\mathscr{K}('\mathscr{V}) = \bigotimes_j (\det \mathrm{Gr}^j)^{\otimes j} = \bigotimes_j (\det \mathscr{F}^j)$. In this case flatness of all the $\mathscr{F}^j$'s implies that the Hodge length of a generating section of $\mathscr{K}('\mathscr{V})$ near a puncture has an expansion

$$h(s(t), s(t)) = |t|^{-2\alpha_p} g(t) \{(\log(1/|t|)^{\beta} p + \ldots\},$$

where α_p is the number $\mathrm{Trace}(\mathrm{Res}_p V)$ [see (2.1.3)], so it vanishes if and only if T_p is unipotent.

3. Singular Hermitian Metrics on Line Bundles Over Curves

As in Sect. 2 we let C be smooth projective curve and C_0 the complement of finitely many points of C. We let $\mathscr{L}$ be a holomorphic line bundle on C. A hermitian metric h on $\mathscr{L}|C_0$ is said to be *good* at $p \in C \backslash C_0$ (cf. [18] Sect. 1) if for some coordinate neighbourhood (Δ, t) centered at p and for a generating holomorphic section s of $\mathscr{L}|\Delta$ the following bounds are valid

$$C_1(\log(1/|t|))^\beta \leqq h(s(t), s(t)) \leqq C_2(\log(1/|t|))^\beta,$$

(3.1) $$C_1, C_2 > 0, \beta \text{ a non-negative integer}$$

$$|\partial/\partial t \log h(s(t), s(t)| \leqq C_3 |t|^{-1} (\log(1/|t|)^{-1}, \quad C_3 > 0$$

$$|\partial^2/\partial t\bar{\partial} t \log h(s(t), s(t)| \leqq C_4 |t|^{-2} (\log(1/|t|)^{-2}, \quad C_4 > 0.$$

This notion is independent of the choice of s and of (Δ, t), and since it easily seen that

$$\Gamma(\Delta, \mathscr{L}) = \{s \in \Gamma(\Delta^*, \mathscr{L});\ h(s(t), s(t)) \leqq C(\log 1/|t|)^\beta \text{ for } C > 0,\ \beta \text{ depending on } s\}$$

it follows that if there is to exist a metric h on $\mathscr{L}|C_0$ good at all points $p \in C \backslash C_0$, the extension $\mathscr{L}$ of $\mathscr{L}|C_0$ cannot be arbitrary. It is uniquely characterized by the above growth condition. For instance, if $\mathscr{L}|C_0$ is the determinant of a Hodge bundle $\mathscr{F}^j$ as in Sect. 2, the line bundle $\mathscr{L}$ must be the determinant of the canonical extension. Mumford proves [loc. cit.] that for a metric h of $\mathscr{L}$ which is good at each point

$p \in C \backslash C_0$, the Chern form $\frac{1}{2\pi i}\partial\bar{\partial}\log h$ is integrable [this is implied by the third bound in (3.1)] and its integral computes $c_1(\mathscr{L})$ [this is implied by the second and the third bound in (3.1)].

(3.2) *Example.* Suppose, one has an asymptotic expansion $h(s(t), s(t)) = f(t)\{\log(1/|t|)^\beta + \text{lower order terms}\}$ with $\beta \in \mathbb{Z}$ and $f(t)$ real analytic, $f(0) \neq 0$, I claim that the bounds (3.1) hold. The first bound is clear, while the formulas for the logarithmic derivative

$$\partial/\partial t(\log(\log 1/|t|)^\beta) = t^{-1}(\log 1/|t|)^{-1}$$

and

$$\partial^2/\partial t\bar{\partial} t(\log(\log(1/|t|)^\beta) = t^{-2}(\log 1/|t|)^{-2}$$

show that the other two bounds hold as well.

If we consider the asymptotic expansion of the Hodge norm of a generating section of the line bundle $\det{}'\mathscr{F}^j$ near a puncture (Proposition 2.2.1) we see that the Hodge metric h is nearly good. In fact $|t|^{2\alpha_p^j} h(s(t), s(t))$ has an expansion as in the previous example and this implies that for any smoothing h_ε of the metric h, which coincides with h on the annulus $2/3\varepsilon < |t| < 4/3\varepsilon$ the limit

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|t| = \varepsilon} \bar{\partial}\log h_\varepsilon(s(t), s(t))$$

exists and is equal to α_p^j. This motivates the following notion: we say that a *hermitian metric* h for $\mathscr{L}|C_0$ *has residue* α_p at p, if $|t|^{2\alpha_p} h(s(t), s(t)$ satisfies the bounds (3.1), so that

$$\alpha_p = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|t| = \varepsilon} \bar{\partial}\log h_\varepsilon(s(t), s(t)), \tag{3.3}$$

where h_ε is a smoothing of h, chosen as above. Now we have (cf. [13, Lemma 5]).

(3.4) **Proposition.** *If $\mathscr{L}$ is a holomorphic line bundle on C and h a hermitian metric on $\mathscr{L}|C_0$ with residue α_p at $p \in C \backslash C_0$. Then the (1, 1) form $\partial\bar{\partial}\log h$ is integrable on C_0 and*

$$c_1(\mathscr{L}) = \left(\frac{1}{2\pi i}\right) \int_{C_0} \partial\bar{\partial}\log h + \sum_{p \in C \backslash C_0} \alpha_p .$$

Proof. Let h_ε be a global smoothing of h which behaves as the preceding smoothing near the punctures. Then

$$\begin{aligned} c_1(\mathscr{L}) &= \frac{1}{2\pi i} \int_C \partial\bar{\partial}\log h_\varepsilon \\ &= \frac{1}{2\pi i}\left[\int_{C_0} \partial\bar{\partial}\log h + \sum_{p \in C \backslash C_0} \lim_{\varepsilon \to 0} \int_{|t| = \varepsilon} \bar{\partial}\log h_\varepsilon(s(t), s(t)\right]. \end{aligned}$$

Since $\partial\bar{\partial}\log h$ and $\partial\bar{\partial}\log(|t|^{2\alpha_p}h)$ are equal on Δ^*, and since the estimates (3.1) assure convergence of $\int_{\Delta^*}\partial\bar{\partial}\log(|t|^{2\alpha_p}h)$, the first term in this expression converges. By (3.3) the limit on the right equals α_p, thereby completing the proof. □

(3.5) *Remark.* Comparing this result with [13, Lemma 5] we see that in our case integrability of the integral is guaranteed by the very definition of the notion of "residue". In applications (see Sect. 4) it therefore suffices to establish certain asymptotic expansions.

4. Proof of the Main Theorem

(4.1) **Theorem.** *Let C be a smooth projective curve and $C_0\subset C$ a Zariski-open subset on which one has a (real) variation of Hodge structure $\mathscr{V}=\mathscr{F}^0\supset\ldots\supset\mathscr{F}^m\supset 0$ of weight m. Let $'\mathscr{F}^j$ be the canonical extension of $\mathscr{F}^j$ and let $\mathscr{K}('\mathscr{V})=\bigotimes_{j=1}^{m}(\det{}'\mathscr{F}^j)$. We have*

(i) $\deg('\mathscr{F}^m)\geqq 0$ *and equality holds if and only if $\mathscr{F}^m$ is a flat subbundle of $\mathscr{V}$ and the local monodromies are unipotent on $\mathscr{F}^m$.*

(ii) $\deg\mathscr{K}('\mathscr{V})\geqq 0$ *and equality holds if and only if all the Hodge bundles $\mathscr{F}^j$ are flat subbundles of $\mathscr{V}$ and all local monodromies are unipotent.*

Proof. The degree of a holomorphic line bundle $\mathscr{L}$ on C is $c_1(\mathscr{L})$. The Hodge metric h^j on $\det(\mathscr{F}^j)$ has a certain asymptotic behaviour, expressed by Proposition 2.2.1. According to the discussion after Example 3.2, it follows that it has residue α_p^j at p, with $\alpha_p^j\geqq 0$. By Proposition 3.4 we have

$$c_1('\mathscr{F}^j)=c_1(\det{}'\mathscr{F}^j)=\frac{1}{2\pi i}\int_{C_0}\partial\bar{\partial}\log h^j+\sum_p\alpha_p^j.$$

Now $\frac{1}{2\pi i}\partial\bar{\partial}\log h^m=i/2\pi\,\mathrm{trace}\,\theta^m$, where θ^m is the curvature of the metric connection on $\mathscr{F}^m$, so by Proposition 1.3.1 its integral over C_0 is non-negative and vanishes if and only if $\mathscr{F}^m$ is a flat subbundle of $\mathscr{V}$. So $c_1('\mathscr{F}^m)\geqq 0$ and equality holds if and only if $\mathscr{F}^m$ is a flat subbundle of $\mathscr{V}$ and all the α_p^m vanish. Again by Proposition 2.2.1 this occurs if and only if $\mathscr{F}^m$ is flat and T_p acts in a unipotent manner on $\mathscr{F}^m$ near each puncture $p\in C\backslash C_0$. This concludes the proof of (i). The proof of (ii) goes along similar lines, using semi-positivity of the Chern from of $\mathscr{K}(\mathscr{V})$ (see Proposition 1.3.2) and the meaning of the numbers α_p in this case (see Remark 2.2.2). □

(4.2) *Remark.* Suppose that the automorphism group of the fibre of a certain flag of the $\mathscr{F}^j$'s is finite (e.g. if the variation is defined over $\mathbb{Z}$). Then, if it consists of flat bundles on which the local monodromies T_p act in a unipotent manner, one must have $T_p=id$ and also the canonical extension of the flag must be flat over C.

(4.3) *Remark.* If the Hodge structures at every point of C_0 are irreducible over $\mathbb{R}$, then flatness of $\mathscr{F}^m$ implies that $\mathscr{F}^m=\mathscr{F}^{m-1}= \quad =\mathscr{F}^0=\mathscr{V}$ for obvious reasons: if $\mathscr{F}^m$ is flat it splits off as a direct summand, say $\mathscr{V}=\mathscr{F}^m\oplus\mathscr{V}_1$ and $\mathscr{V}_1$ inherits from

$\{\mathscr{F}^{\cdot}\}$ a Hodge structure of weight m, so $\mathscr{V}_1 = 0$, This situation often arises in geometry, so in that case it does not make any difference to look at $\mathscr{F}^m$ or at $\mathscr{K}(\mathscr{V})$.

5. Applications to Geometry

5.1. The Main Theorem in the Geometric Situation

Recall the geometric situation, described in Sect. 1.1, where now $B = C_0$, a Zariski open subset of a smooth projective curve C. More precisely, we have a compact connected Kähler manifold X and a surjective holomorphic map $f: X \to C$. We let C_0 be the set of critical values of f, so that f restricts to a smooth family of connected Kähler manifolds over C_0. The primitive cohomology groups $H_0^m(X_t, \mathbb{C})$, $t \in C_0$ of the fibres X_t fit nicely together to a flat vector bundle $(\mathscr{H}^m, \nabla)$ and the Hodge filtrations fit together to give a variation of Hodge structure of weight m on C_0 underlying $\mathscr{H}^m$. The Hodge bundles have canonical extensions to all of C, and can be described in geometric terms (cf. [21', Theorem 2.11]). In case $m = n = \dim_{\mathbb{C}} X_t$ there is a connection with the *relative dualizing sheaf:*

$$\omega_{X|C} = \mathscr{K}_X \otimes f^* \mathscr{K}_C^{\vee}.$$

Over C_0 we have $f_*(\omega_{X|C}|C_0) = \mathscr{F}^n$ and Kawamata has shown [13, Lemma 1]:

$$f_*(\omega_{X|C}) = {}'\mathscr{F}^n.$$

The proof of this equation is highly non-trivial; it makes essential use of Schmid's asymptotic expansion as presented in Sect. 2.2. For details we refer to Kawamata's paper [13]. Cf. also [25, pp. 51–52].

Consequently, the main theorem translates as follows:

(5.1.1) **Proposition.** *The degree of $f_*(\omega_{X|C})$ is non-negative and vanishes if and only if $\mathscr{F}^n = f_*(\omega_{X|C}|C_0)$ is a flat subbundle of $\mathscr{H}^n = \bigcup_{b \in C_0} H_0^n(X_b, \mathbb{C})$ and all local monodromies $T_p (p \in C \setminus C_0)$ act in a unipotent manner on $\mathscr{F}^n$.*

We want to translate the flatness of $\mathscr{F}^n$ into properties of the smooth fibration $f|f^{-1}(C_0)$. Obviously, one needs a Torelli-type theorem for the fibres. To be more precise, if $\tilde{C}_0$ is the universal cover of C_0 and if D is as in Sect. 1.2 (but with $F^0 = F^1 = \quad = F^{n-1}$, since we only consider variations of F^n) the holomorphic "period map" $\Phi: \tilde{C}_0 \to D$ should be constant precisely, when the pulled back family has constant fibres and hence is trivial by a result of Grauert and Fischer [8]. Going back to C_0, the Hodge bundle $\mathscr{F}^n$ should be flat precisely when f is an analytically local trivial family over C_0. This will be certainly the case, if the generic smooth fibre X_b has the following property:

(5.1.2) A germ of a deformation of X_t is trivial if and only if the period map associated to the variation of the n-th Hodge bundle ($n = \dim_{\mathbb{C}} X_t$) is constant.

This property is a sort of weak *Torelli property* for X_t and in this form holds many cases, e.g. if $\mathscr{K}_{X_t} = \mathscr{O}_{X_t}$, for curves, for complete intersections in projective space [17].

(5.1.3) **Theorem.** *Let $f: X \to C$ as before. Then* $\deg(f_* \omega_{X|C}) \geqq 0$. *Assume that the Torelli property* (5.1.2) *holds for a generic smooth fibre of f. Then* $\deg(f_* \omega_{X|C}) = 0$ *if and only if the following two properties hold.*

(i) *The smooth fibres of f form a locally trivial fibre bundle (in the complex analytic sence)*

(ii) *The local monodromy around a singular fibre is the identity on* $H^{n,0}(X_b)$, *b near a critical value of f.*

Proof. The inequality follows from Proposition 5.1.1. By the same proposition, if equality holds $\mathscr{F}^n$ is a flat subbundle of $\mathscr{H}^n$ and the infinitesimal Torelli property garantuees (i). For (ii), we observe that the group of isometries of $H_0^n(X_b)$ coming from automorphisms of X_b respecting the Kähler class is finite by [16, Proposition 2.2]. So if $f|f^{-1}(C_0)$ is a fibre bundle and if the local monodromies are unipotent, they must be the identity. Since this last hypothesis is verified by Theorem 4.1, we are done. □

The next question is whether the fibre bundle $f|f^{-1}(C_0)$ is *trivial* near the punctures. To answer this, we introduce the operation of *extracting* an *N-th root.* We let Y be a Kähler manifold., $\pi: Y \to \Delta$ a proper holomorphic map with connected fibres, such that $0 \in \Delta$ is the only critical value of π. So π is smooth over Δ^* with compact connected Kähler manifolds as fibres. We shall refer to this situation as *a Kähler degeneration.* If $\sigma_N: \Delta_u \to \Delta = \Delta_t$ is the map given by $\sigma_N(u) = u^N = t$, we can form the fibre product $Y \times_{\Delta} \Delta_u$ and its normalisation Y_N maps to Δ_u. In general Y_N has singularities over the origin and we choose some resolution $\tilde{Y}_N$ of Y_N and the resulting Kähler degeneration $\pi_N: \tilde{Y}_N \to \Delta_u$ is called a *degeneration obtained after extracting an N-th root of* π.

If $Y_0 = \pi^{-1}(0)$ is a divisor $\sum_{i=1}^{k} n_i D_i$, taking N any multiple of l.c.m. $(n_1, \ldots, n_k)$ we find a fibre $\pi_N^{-1}(0)$ having at least k components with multiplicity 1 (the proper transforms of D_i) and hence π_N has a section (perhaps after shrinking Δ_u somewhat).

Let us apply this to the situation of the preceding theorem. So, near a puncture f is a fibre bundle with typical fibre F, hence is associated to a representation $\varrho: \pi_1(\Delta^*) \to \operatorname{Aut} F$. Moreover $\varrho(\gamma)$ (γ a generator of $\pi_1(\Delta^*)$) acts trivially on $H^{n,0}(F)$, so by [16, Proposition 2.2], some power $\varrho(\gamma^M)$ belongs to $\operatorname{Aut}^0 F$. Since the Torelli-property implies that F cannot be ruled, by [16, Theorem 4.9] it follows that $T = \operatorname{Aut}^0 F$ is a complex torus. After extracting an M-th rooth the fibre bundle has a T as its group and by the preceding remark, after further extraction of roots and shrinking of Δ^*, the associated principal bundle has a section, hence is trivial. So also the F-bundle is trivial. It follows that near a puncture, after extracting a root, the bundle is trivial. So we have proved.

(5.1.4) **Lemma.** *In the situation of Theorem 5.1.3 assuming* $\deg(f_*, \omega_{X|C}) = 0$, *locally in a suitable neighbourhood* $(\Delta, 0)$ *of a critical value* $0 \in C$ *of f, after extracting a suitable N-th root, the fibration* $f|f^{-1}(\Delta^*)$ *is trivial.*

A further question is whether one can extend the fibre bundle $f|f^{-1}(C_0)$ over the punctures. This we treat in the next subsection.

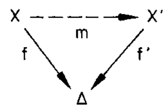
5.2. Degenerations of Trivial Families

The crucial auxiliary result is the following.

(5.2.1) **Lemma.** *Let $\pi: Y \to \Delta$ be a Kähler degeneration and let S^* be a subvariety of $Y^* = \pi^{-1}(\Delta^*)$ which is proper over Δ^*, there exist points $t_0 \in \Delta^*$ arbitrarily close to 0 such that, perhaps after shrinking Δ and extracting an N-th root, the subvariety $S^*_{t_0}$ extends to a cycle S on Y, which is flat and proper over Δ.*

Proof. We employ the relative Chow-scheme $\mu: \mathscr{C}(Y|\Delta) \to \Delta$ introduced by Fujiki [9]. Since Y is Kähler and π is proper, by [10, Theorem 5.2], $\mathscr{C}(Y|\Delta)$ has at most countably many components, all of which are proper over Δ. Arbitrarily near 0 there are points $t_0 \in \Delta^*$ such that the connected component $\mathscr{C}_0$ of $\mathscr{C}(X|\Delta)$ containing the cycle $S^*_{t_0}$ does not entirely map to t_0 (here we use countability). Since $\mu_0 = \mu|\mathscr{C}_0 \to \Delta$ is proper, after shrinking Δ and taking an N-th root, the holomorphic map μ_0 has a section. This section yields the desired S flat over Δ. □

(5.2.2) **Corollary.** *Let $f: X \to \Delta$ and $f': X' \to \Delta$ be two Kähler degenerations, isomorphic as families over Δ^*. Then, perhaps after shrinking and extracting an N-th root, there is a commutative diagram*



with m a bimeromorphic map inducing biholomorphic maps between the fibres over Δ^*.

Proof. One can apply the previous lemma with Y a resolution of singularities of $X \underset{\Delta}{\times} X'$ and S^* the graph of an isomorphism $X^* \to (X')^*$. So, after shrinking and extracting a root, S^* extends to a cycle $S \subset X \underset{\Delta}{\times} X'$ such that $S_{t_0} = S^*_{t_0}$. Since $S^*_{t_0}$ gives an isomorphism and since the set $\{t \in \Delta,\ S_t$ does *not* give an isomorphism$\}$ is an analytic subset of Δ, after shrinking Δ still further, all cycles $S_t, t \in \Delta^*$ give isomorphisms. The cycle S then gives the desired meromorphic correspondence. □

This corollary in particular applies when a given Kähler degeneration over Δ is *trivial* over Δ^*.

(5.2.3) **Corollary.** *Let $f: X \to \Delta$ be a Kähler degeneration, such that $f|f^{-1}(\Delta^*)$ is biholomorphically a product. Then, after extracting an N-th root and shrinking Δ, the new degeneration is bimeromorphically a product.*

5.3. A Generalization of a Result of Arakelov

Let us combine Theorem 5.1.3, Lemma 5.1.4, and Corollary 5.2.3. The last two results describe the situation locally and since we can always find a finite cover $\sigma: D \to C$ branched at most in the critical points of f and one further point such that near every critical point σ is totally ramified with given order, we arrive at the following result, which is a generalization of [1, Theorem 1.1]. (Also compare [19, Proposition 5].)

(5.3.1) **Theorem.** *Let X be a compact connected $(n+1)$-dimensional Kähler manifold, $f: X \to C$ a holomorphic map onto a curve and $C \setminus C_0$ the set of critical values. We have*

$$d = \deg f_* \omega_{X|C} \geqq 0.$$

If $d=0$ and the generic smooth fibre has the Torelli-property (5.1.2) *f is a fibre bundle over C_0; after a finite covering $\sigma: D \to C$ branched at most in critical values of f and one further point, the fibre product $X \underset{C}{\times} D$ is bimeromorphic to a fibre bundle over D.* □

(5.3.2) **Corollary.** *If the generic fibre F of f has the Torelli-property, e.g. if $\omega_F \cong \mathcal{O}_F$, the conjecture $C^+_{n,1}$ (cf. the Introduction) holds.*

Proof. As mentioned in the Introduction, the conjecture $C_{n,1}$ has been proved by Kawamata [13]. So, to prove $C^+_{n,1}$, it suffices to see that if C is an elliptic curve the assumption $\kappa(X)=0$ implies $d=0$, because then we can apply Theorem 5.3.1. Suppose that on the contrary $d>0$. Then by Riemann-Roch $\dim_{\mathbb{C}} H^0((f_*\omega_{X|C})^{\otimes \nu})$ grows like $a\nu (a \geqq 1)$ for ν large. On the other hand there is a natural injection $H^0(f_*(\omega_{X|C})^{\otimes \nu}) \to H^0(f_*(\omega_{X|C}^{\otimes \nu}))$ (it comes from the injection $f^* f_* \omega_{X|C} \to \omega_{X|C}$), so $\omega_{X|C} = \mathcal{K}_X$ has positive Kodaira-dimension contradicting $\kappa(X)=0$. □

Let us give an example to show that if in Theorem 5.3.1 we have $d=0$ we really have to take a finite cover before we arrive at a situation where the bundle is bimeromorphically locally trivial.

(5.3.3) *Example.* Let E be an elliptic curve and F a curve of genus $g \geqq 2$ having a nontrivial automorphism ϕ of prime order p, acting as a translation on E. The quotient X of $E \times F$ by the cyclic group generated by ϕ is smooth and if $C = F/\{\phi^k\}$ we have a holomorphic map $f: X \to C$ induced by the projection $E \times F \to F$. Away from the fixed points of ϕ on C the fibration is locally a product, but f itself is *not* a product bundle. The Hodge bundle $\mathcal{F}^1$ however is flat and the canonical extension is a flat g-bundle on C, becoming trivial when lifted to F.

The point of the previous example is that the structure groups of the bundles $f|f^{-1}(C_0)$ and $\mathcal{F}^1$ differ, because of the existence of non-trivial automorphisms acting trivially on $H^{1,0}(X_C)$ (namely translations of $X_C \cong E$). In the final section we shall investigate what happens if this is not the case.

5.4. A Special Case

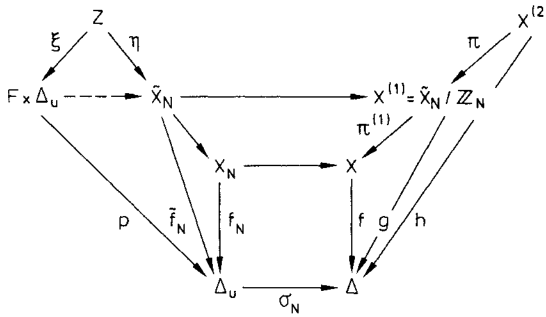
(5.4.1) **Proposition.** *Let $f: X \to \Delta$ be a Kähler degeneration. Assume*

(i) *The family $f|f^{-1}(\Delta^*)$ is a fibre bundle in the analytic sense with typical fibre F of dimension n.*

(ii) *The local monodromy is the identity on $H^{n,0}(F)$.*

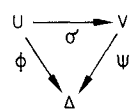
(iii) *The group of automorphisms of F act faithfully on $H^{n,0}(F)$. Then X is bimeromorphically equivalent to $F \times \Delta$.*

Proof. The three assumptions imply that $f|f^{-1}(\Delta^*)$ is a trivial fibre bundle in the analytic sense, so we can apply Corollary 5.2.3. We only have to show that we can take $N=1$. Let us consider the operation of extracting an N-th root in more detail. We have a commutative diagram



Here X_N is the normalization of $X \underset{\Delta}{\times} \Delta_u$. It admits a natural $\mathbb{Z}_N$-action covering the action of $\mathbb{Z}_N$ on Δ_u given by multiplication with N-th roots of unity. The manifold $\tilde{X}_N$ is a $\mathbb{Z}_N$-equivariant resolution of singularities (compare [22, Theorem 2.12]) and $\pi: X^{(2)} \to X^{(1)} = \tilde{X}_N/\mathbb{Z}_N$ is a resolution of the finitely many quotient-singularities of $X^{(1)}$. Finally ξ and η are proper modifications establishing a bimeromorphic equivalence between $\tilde{X}_N$ and $Y \times \Delta_u$. We shall make repeated use of the following

Lemma. *If one has a commutative diagram*



with U and V complex manifolds, σ a proper modification and ϕ and ψ proper maps, then the maps $R^k\phi_*\mathcal{O}_V \to R^k\psi_*\mathcal{O}_U$ induced by σ are isomorphisms ($h=0,1,\ldots$).

This follows from the fact that the natural maps $H^k(\psi^{-1}(W), \mathcal{O}_V) \to H^k(\phi^{-1}(W), \mathcal{O}_U)$ induced by σ are isomorphisms for all open subsets W of Δ.

If we apply this lemma to the left hand triangle we find $H^{n,0}(F)\otimes\mathcal{O}_{\Delta_u} \xrightarrow{\sim} (R^n f_{\tilde{N}})_*\mathcal{O}_{\tilde{X}_N} \xrightarrow{\sim} (R^n f_N)_*\mathcal{O}_{X_N}$. The group $\mathbb{Z}_N$ acts on the stalk at 0 of $(R^n f_{\tilde{N}})_*\mathcal{O}_{\tilde{X}_N}$ and the quotient map $\tilde{X}_N \to \tilde{X}_N/\mathbb{Z}_N = X^{(2)}$ induces an injection $(\sigma_N^*(R^n f_x \mathcal{O}_X))_0 \to ((R^n f_N)_*\mathcal{O}_{X_N})_0$ onto the $\mathbb{Z}_N$-invariants. On the other hand, since X^1 has at most cyclic quotient singularities, $R^m\pi_*\mathcal{O}_{X^{(2)}}=0$ for $m \geqq 1$ [23] and one obtains an isomorphism $R^n g_*\mathcal{O}_{X^{(1)}} \xrightarrow{\sim} R^n h_*\mathcal{O}_{X^{(2)}}$. Since $\pi^{(1)}\circ\pi$ is bimeromorphic, the preceding lemma says that $R^n f_*\mathcal{O}_X$ and $R^n h_*\mathcal{O}_{X^{(2)}}$ are isomorphic. Combining everything we have an isomorphism

$$\sigma_N^*((R^n f_*\mathcal{O}_X))_0 \xrightarrow{\sim} H^{n,0}(F)^{\mathbb{Z}_N} \otimes (\mathcal{O}_{\Delta_u})_0$$

By relatively duality (see [14, Corollary (24)]) $R^n f_* \mathcal{O}_X$ is dual to $f_* \omega_{X|\Delta}$. This last sheaf is the canonical extension of the free sheaf $(f_* \omega_{X|\Delta})|\Delta^*$, hence is itself free. So $R^n f_* \mathcal{O}_X$ is free and the stalk of $\sigma_N^*(R^n f_* \mathcal{O}_X)$ at 0 is $H^{n,0}(F) \otimes (\mathcal{O}_{\Delta_u})_0$. The previous equation shows that $H^{n,0}(F) = H^{n,0}(F)^{\mathbb{Z}_N}$, hence $N=1$ since $\mathbb{Z}_N$ acts faithfully on $H^{n,0}(F)$ by assumption. □

(5.4.2) **Corollary.** *If in the situation of Theorem 5.3.1 with $d=0$ in addition to the validity of the Torelli-property for the generic fibre F, one also knows that* Aut *F acts faithfully on $H^{n,0}(F)$, the manifold X itself is bimeromorphic to a fibre bundle over C and it becomes trivial after a finite unramified covering.*

Proof. Only the last statement requires proof. It follows directly from the fact that the group of the fibre bundle must be finite by [16, Proposition 2.2], since it preserves a Kähler class and since moreover $\mathrm{Aut}^0 F = \{1\}$ as a consequence of the assumption. □

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