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Inventiones mathematicae

Automorphisms of Enriques Surfaces

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0. Introduction

The aim of this note is to compute the group $\mathrm{Aut}(Y)$ of (biholomorphic) automorphisms for the general Enriques surface Y. The basic tool is the global Torelli theorem for projective K3-surfaces as it was given by Piatetski-Shapiro and Shafarevich [11] and refined by Burns and Rapaport [2]. The essential result is that – in contrast to the case of curves – $\mathrm{Aut}(Y)$ is big for general Y and small for special Y.

In this paper we consider the complex case only. Recall that an Enriques surface Y is a (projective) complex surface with universal double cover a K3-surface. One knows that $H^2(Y,\mathbb{Z})=\mathbb{Z}^{10}\oplus\mathbb{Z}_2$ and that the cup-product provides $H^2(Y,\mathbb{Z})/\mathrm{torsion}=\mathbb{Z}^{10}$ with the structure of a lattice M of signature (1, 9).

Theorem. *For a generic Enriques surface Y the representation of* $\mathrm{Aut}(Y)$ *on $H^2(X,\mathbb{Z})$ defines an isomorphism of* $\mathrm{Aut}(Y)$ *with the 2-congruence subgroup of $O^\uparrow(M)$, where $O^\uparrow(M)$ is the group of isometries of M not interchanging the two positive half-cones in $M\otimes_{\mathbb{Z}}\mathbb{R}$, or in other words, the reflection group of the lattice M.*

Here the notion "generic" needs some explanation. Horikawa [7, 8] defined a quasi-projective period domain D^0/Γ for Enriques surfaces. The assertion in the theorem holds for all surfaces Y with period point $\tau(Y)\in D^0/\Gamma$ in the complement of countably many images of 9-dimensional analytic varieties (recall $\dim D^0/\Gamma=10$). It was pointed out to us by Dolgachev that the theorem also follows from results of Nikulin [10], although it is not stated there explicitely.

For special Y the automorphism group can be quite different. We describe a 2-dimensional family of surfaces Y where $\mathrm{Aut}(Y)$ in general is $\mathbb{Z}_2\times D_\infty$, but for special cases $\mathbb{Z}_4\times D_\infty$ or D_4. Here $D_4(D_\infty)$ is the dihedral group $\mathbb{Z}_2\ltimes\mathbb{Z}_4$ $(\mathbb{Z}_2\ltimes\mathbb{Z})$. The example of surfaces with finite group $\mathrm{Aut}(Y)$ was communicated to us by Dolgachev.

We apply the knowledge of $\mathrm{Aut}(Y)$ for generic Y to count the number of inequivalent realisations of Y as elliptic fibre space over $\mathbb{P}_1$, as double cover of a quadri-nodal complete intersection of two quadrics in $\mathbb{P}_4$ (double plane realisation), as sextic surface in $\mathbb{P}_3$ passing doubly through the edges of a tetrahedron (Enriques-realisation), or as smooth surface in $\mathbb{P}_5$ of degree 10 (deformations of Reye-congruences). There are

$527=$	$17\cdot 31$	realisations as elliptic fibration
$67456=$	$2^7\cdot 17\cdot 31$	double plane realisations
$5396480=$	$2^{11}\cdot 5\cdot 17\cdot 31$	Enriques-realisations
$25903104=$	$2^{13}\cdot 3\cdot 17\cdot 31$	realisations as 10th degree surface in $\mathbb{P}_5$.

We owe much to stimulating discussions on this subject with many other geometers, in particular to I. Dolgachev.

1. Some Lattices and Their Isometries

1.1. **Preliminaries.** A *lattice* is a free $\mathbb{Z}$-module of finite rank endowed with an integral quadratic form. $L\perp M$ denotes the *orthogonal direct sum* of two lattices L and M. $L^\vee=\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ is the *dual $\mathbb{Z}$-module* (the canonical quadratic form on $L^\vee$ in general is not integral). The symmetric bilinear form on a lattice L, associated with the quadratic form, usually is denoted by $\langle\ ,\ \rangle$.

This form defines the *correllation* morphism

$$\varphi_L: L\to L^\vee, \qquad x\to\langle x, -\rangle.$$

If L is *nondegenerate*, φ_L is injective, and we may identify L with the submodule $\varphi_L(L)\subset L^\vee$. Then $L^\vee/L$ is a finite abelian group, trivial precisely when L is *unimodular*.

A submodule $M\subset L$ is called *primitive*, if L/M is free of torsion. In this case every integral functional on M extends to L, i.e., the restriction $L^\vee\to M^\vee$ is surjective.

If L is nondegenerate and M is primitive and nondegenerate, the composition $L\xrightarrow{\sim}L^\vee\to M^\vee\to M^\vee/M$ is surjective. It sends $x\in L$ to the $\varphi(M)$-equivalence class of $\langle x,-\rangle|M$. So its kernel is $M\perp M^\perp$ and we obtain an isomorphism $i_M: L/(M\perp M^\perp)\to M^\vee/M$. Interchanging the rôles of M and $M^\perp$ we obtain $i_{M^\perp}: L/(M\perp M^\perp)\xrightarrow{\sim}(M^\perp)^\vee/M^\perp$. Then we put $j_M=i_{M^\perp}\circ i_M^{-1}$: $M^\vee/M\xrightarrow{\sim}(M^\perp)^\vee/M^\perp$.

(1.1) **Lemma** [9, Prop. 1.1]. *Let L be a unimodular lattice, $M\subset L$ a nondegenerate primitive sublattice, and α: $M\to M$, β: $M^\perp\to M^\perp$ isometries. Then the isometry (α,β) of $M\perp M^\perp$ extends to L if and only if the automorphisms $\tilde\alpha$ on $M^\vee/M$ induced by α and $\tilde\beta$ on $(M^\perp)^\vee/M^\perp$ induced by β satisfy $j_M\circ\tilde\alpha=\tilde\beta\circ j_M$.*

(1.2) **Corollary.** *Let $M\subset L$ and $\alpha: M\to M$ be as above. If α extends to an isometry of L restricting to $\pm\mathrm{id}$ on $M^\perp$, then this extension is unique. Such an extension exists if and only if $\alpha^\vee: M^\vee\to M^\vee$ induces $\pm\mathrm{id}$ on $M^\vee/M$.*

We shall use the following *notation*: For $n\in\mathbb{N}$, by nL we denote the sublattice $\{n\cdot x: x\in L\}$ of L, whereas $L(n)$ is the $\mathbb{Z}$-module L endowed with the quadratic form $x \to nx^2$. A *root* in the lattice L is an element w of square $w^2 = -2$. Any root w defines the *reflection* $s_w \in O(L)$, $s_w(x) = x + \langle x, w \rangle w$. Given a lattice L we put $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$, $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$, $L_{\mathbb{C}} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and for any homomorphism $g: L \to L$ we denote by $g_{\mathbb{Q}}$, $g_{\mathbb{R}}$, resp. $g_{\mathbb{C}}$ the natural extension of g to these vector spaces.

1.2. **Application.** For applying 1.2 to the Picard lattice of K3 and Enriques surfaces we fix the following notation.

$$\mathbb{H} = \mathbb{Z}e_1 + \mathbb{Z}e_2 \qquad \text{with } e_1^2 = e_2^2 = 0,\ e_1 e_2 = 1 \text{ (hyperbolic plane)},$$
$$\mathbb{E} = E_8(-1) \qquad \text{(root lattice for Dynkin diagram } E_8\text{)},$$
$$L = \mathbb{H} \perp \mathbb{H} \perp \mathbb{H} \perp \mathbb{E} \perp \mathbb{E},$$
$$M = \mathbb{H} \perp \mathbb{E},$$
$$s: L \to L \text{ the involution } (h_1, h_2, h_3, e_1, e_2) \mapsto (-h_1, h_3, h_2, e_2, e_1),$$
$$L^+ = \{x \in L: s(x) = x\} = \{(0, h, h, e, e) \in L: h \in \mathbb{H}, e \in \mathbb{E}\},$$
$$L^- = \{x \in L: s(x) = -x\} = \{(h_1, h, -h, e, -e) \in L: h_1, h \in \mathbb{H}, e \in \mathbb{E}\}.$$

There are obvious isometries

$$\varepsilon^+: L^+ \to M(2) \qquad (0, h, h, e, e) \mapsto (h, e),$$
$$\varepsilon^-: L^- \to \mathbb{H} \perp M(2), \quad (h_1, h, -h, e, -e) \mapsto (h_1, h, e).$$

In particular this shows

$$(L^\pm)^\vee / L^\pm = M/2M = (\mathbb{Z}_2)^{10}.$$

L^+ and L^- are primitive nondegenerate sublattices of L, one the orthogonal complement of the other. Using that all elements in $(L^\pm)^\vee/L^\pm$ are induced by inner products with elements $(0, h, 0, e, 0)$ one traces the isomorphism $j: (L^-)^\vee/L^- \to (L^+)^\vee/L^+$ and finds: j is induced by the obvious isometry

$$\psi: L^- \to \mathbb{H} \perp L^+, \qquad (h_1, h, -h, e, -e) \mapsto (h_1, h, h, e, e).$$

We also put

$$\Gamma = \{g \in O(L): gs = sg\}.$$

For any $g \in \Gamma$ we have $g: L^\pm \to L^\pm$, and there are obvious restrictions $r^\pm: \Gamma \to O(L^\pm)$.

(1.3) **Lemma.** *For $g \in O(L^\pm)$ the following properties are equivalent*:

a) *there is a (unique) extension $\gamma \in \Gamma$ of g with $r^\mp(\gamma) = \mathrm{id}$.*

b) *g belongs to the 2-congruence subgroup of $O(L^\pm)$.*

The proof follows from Corollary (1.2), because g induces the identity on $(L^\pm)^\vee/L^\pm$ if and only if it belongs to the 2-congruence subgroup.

The quadratic form on M has signature (1, 9). So the set $\{x \in M_{\mathbb{R}}: x^2 > 0\}$ consists of two disjoint cones $\mathscr{C}_M$ and $-\mathscr{C}_M$. We put $O^\uparrow(M) = \{g \in O(M): g\mathscr{C}_M = \mathscr{C}_M\}$.

Then $O(M)$ is the direct product $O^\uparrow(M) \times \{\pm \mathrm{id}\}$.

1.3. **On the Root Lattice $\mathbb{E}$.** In this section we collect a few properties of $\mathbb{E}$ which are needed later. We use the description [1, p. 268] of $\mathbb{E}$. So $\mathbb{E}\subset\mathbb{R}^8$ (with the negative of the usual inner product) is the set of vectors $(x^1,\ldots,x^8)$ where either all x^i are integers or all x^i are half-integers, and $\sum x^i\in\mathbb{Z}$ is even. The 240 roots are $(0\ldots0,\ \pm1_i,0\ldots0,\ \pm1_j,0\ldots0)$, $1/2(\pm1,\ldots,\pm1)$.

(1.4) **Lemma.** *There are exactly* 135 *equivalence classes* mod $2\mathbb{E}$ *of vectors* $x\in\mathbb{E}$ *with* $x^2=-4$.

Proof. An integral vector $x\in\mathbb{E}$ with $x^2=-4$ is up to permutation of the coordinates of the form $\pm(2,0,\ldots,0)$ or $(\pm1,\pm1,\pm1,\pm1,0,0,0,0)$. Since $(0\ldots0,\ \pm2_i,0\ldots0,\ \pm2_j,0\ldots0)\in2\mathbb{E}$, all vectors $(0\ldots0,\pm2,0\ldots0)$ are equivalent mod $2\mathbb{E}$. Of the second type there are $2^4\cdot\binom{8}{4}$ vectors and mod $2\mathbb{E}$ each of them is equivalent with 2^4 ones.

So there are $\binom{8}{4}=70$ inequivalent ones. Any half-integral vector is up to permutation of coordinates of the form

$$\pm\tfrac{1}{2}(3,-1,\ldots,-1),\qquad \pm\tfrac{1}{2}(3,1,1,-1,\ldots,-1),$$
$$\pm\tfrac{1}{2}(3,1,1,1,1,-1,-1,-1),\ \pm\tfrac{1}{2}(3,1,1,1,1,1,1,-1).$$

Here all vectors of the first and of the last type are equivalent mod $2\mathbb{E}$. Vectors of the first, second, and third type are inequivalent mod $2\mathbb{E}$. There are $2.8\cdot\binom{7}{2}$ vectors of the second type, each equivalent with 2.6 of them. So there are $\frac{1}{2}7.8=28$ inequivalent ones mod $2\mathbb{E}$. Of the third type there are $2.8\cdot\binom{7}{3}$ vectors, each of them equivalent with 16 ones, so 35 inequivalent ones. Altogether we have $1+70+1+28+35=135$. □

(1.5) **Corollary.** *Choosing* 135 *representatives of the equivalence classes above, one from each of the* 120 *pairs* $\pm w$ *of roots, and* 0, *one obtains a system of representatives of* $\mathbb{E}$ mod $2\mathbb{E}$.

Proof. We only have to show that $w_1-w_2\in2\mathbb{E}$ for two roots w_1,w_2 implies $w_2=\pm w_1$. But if $w_1-w_2\in2\mathbb{E}$, then $(w_1-w_2)^2=-4-2w_1w_2$ is divisible by 8. Since $|w_1w_2|\leqq2$ this implies $w_1w_2=\pm2$, i.e., $w_2=\pm w_1$. □

We denote by $W=W(E_8)$ the *Weyl group*. Since the Dynkin diagram of E_8 admits no symmetries, W coincides with $O(\mathbb{E})$, see [1, p. 270]. W contains in particular

- all permutations of coordinates x^i
- simultaneous changes $x^i,x^j\mapsto -x^i,-x^j$ of the signs of two coordinates.

(1.6) **Lemma.** *W operates transitively on the set of all ordered 8-tuples of roots $w_1,\ldots,w_8\in\mathbb{E}$ satisfying $\langle w_i,w_j\rangle=-1$ whenever $i\neq j$.*

Proof. $W(E_8)$ operates transitively on the roots, so we may assume

$$w_1=\tfrac{1}{2}(1,\ldots,1).$$

If w_i, $i\geqq2$, is integral, then

$$w_i=(0\ldots0,1,0\ldots0,1,0\ldots0).$$

If w_i is not integral, say $w_i = \frac{1}{2}(1, 1, 1, 1, 1, 1, -1, -1)$ we use the reflection s_w with $w = \frac{1}{2}(-1, 1, 1, 1, 1, -1, -1, -1) \perp w_1$ and transform w_i into an integral root. After permuting coordinates we have

$$w_2 = (1, 1, 0 \ldots 0).$$

Since $w_2 \perp w$, by the same argument we may assume w_3 integral. Then after permutation

$$w_3 = (1, 0, 1, 0 \ldots 0).$$

Again $w_3 \perp w$ and we may arrange it that w_4 is integral, i.e.,

$$w_4 = (1, 0, 0, \ldots 1 \ldots 0) \quad \text{or} \quad w_4 = (0, 1, 1, 0 \ldots 0).$$

In the second case we transform w_4 under s_u with

$$u = \tfrac{1}{2}(1, -1, -1, -1, -1, 1, 1, 1) \perp w_1, w_2, w_3 \quad \text{into } \tfrac{1}{2}(1, 1, 1, -1, -1, 1, 1, 1)$$

and then with

$$u = \tfrac{1}{2}(1, -1, -1, 1, 1, 1, -1, -1) \perp w_1, w_2, w_3 \quad \text{into } (1, 0, 0, 0, 0, 1, 0, 0).$$

So after permutation

$$w_4 = (1, 0, 0, 1, 0, 0, 0, 0).$$

Still $w_4 \perp w$, hence we may assume w_5 integral, and after permutation

$$w_5 = (1, 0, 0, 0, 1, 0, 0, 0) \perp w$$
$$w_6 = (1, 0, 0, 0, 0, 1, 0, 0).$$

So, after permutation, we have

$$w_7 = (1, 0, 0, 0, 0, 0, 1, 0) \quad \text{or} \quad \tfrac{1}{2}(1, 1, 1, 1, 1, 1, -1, -1).$$

In the first case necessarily $w_8 = (1, 0, \ldots, 0, 1)$ and in the second case there is no w_8 at all. □

1.4. **Reduction Modulo 2.** In this section we examine the reduction morphism $O(M) \to \mathrm{Aut}(M/2M) = GL(10, \mathbb{F}_2)$. By $\bar{x} \in M/2M$ we denote the class represented by $x \in M$. On $M/2M$ we have the bilinear form

$$\langle \bar{x}, \bar{y} \rangle = \langle x, y \rangle \bmod 2.$$

Since the form on M is even, by

$$q(\bar{x}) = \tfrac{1}{2} x^2 \bmod 2$$

one defines a nondegenerate quadratic form q on $M/2M$, i.e., a form satisfying

$$q(\bar{x} + \bar{y}) = q(\bar{x}) + q(\bar{y}) + \langle \bar{x}, \bar{y} \rangle.$$

On each $\mathbb{F}_2$-vector space of even dimension $2k$ there are - up to conjugation - exactly two such forms, q^+ and q^-, differing by their number $\nu=2^{k-1}(2^k \pm 1)$ of zeros.

Using Lemma (1.4) we count the zeros of $q|\mathbb{E}/2\mathbb{E}$ and find

$$\nu_{\mathbb{E}}=2^8-120=136=2^3(2^4+1).$$

We observe that the elements $\bar{x}\in\mathbb{E}/2\mathbb{E}$ with $q(\bar{x})=1$ are precisely the images of the roots.

Now $\mathbb{H}/2\mathbb{H}=\mathbb{F}_2^2$ and q has 3 zeros on it. The zeros of q on $M/2M$ are exactly the pairs $(\bar{h},\bar{x})$, $h\in\mathbb{H}$, $x\in\mathbb{E}$ satisfying $q(\bar{h})=(\bar{x})$. Their number is

$$3\cdot\nu_E+256-\nu_E=3\cdot 136+120=528=2^4(2^5+1).$$

Hence q and $q|\mathbb{E}/2\mathbb{E}$ is the corresponding form q^+. Its group of automorphisms is denoted by $O^+(2k,\mathbb{F}_2)$. E.g. in the book [5] one finds (Chap. III, § 10)

i) $|O^+(2k,\mathbb{F}_2)|=2^{1+k(k-1)}\cdot(2^k-1)\cdot\sum_{j=1}^{k-1}(2^{2j}-1)$,

ii) the group $O^+(2k,\mathbb{F}_2)$ is generated by transvections if $k\neq 2$; these are maps $\bar{x}\mapsto\bar{x}+\langle\bar{x},\bar{a}\rangle\bar{a}$, $q^+(\bar{a})=1$.

iii) the group $O^+(2k,\mathbb{F}_2)$ contains a normal subgroup of index 2 consisting of all products of an even number of transvections. For $k\geqq 3$ this group is simple. In our cases, $k=4$ and 5, we find

$$|O(\mathbb{E}/2\mathbb{E},q)|=2^{13}\cdot 3^5\cdot 5^2\cdot 7,$$
$$|O(M/2M,q)|=2^{21}\cdot 3^5\cdot 5^2\cdot 7\cdot 17\cdot 31.$$

Now any root $w\in M$ reduces in $M/2M$ to an element $\bar{w}$ with $q(\bar{w})=1$, and the reflection s_w reduces to the transvection defined by $\bar{w}$. Conversely, if $\bar{a}\in M/2M$ with $q(\bar{a})=1$, then $\bar{a}=(\bar{h},\bar{x})$, $h\in\mathbb{H}$, $x\in\mathbb{E}$, such that one of the following holds:

- either $q(\bar{h})=1$ and $q(\bar{x})=0$, i.e. $h\in\mathbb{H}$ is modulo $2\mathbb{H}$ equivalent with h_1+h_2, $(h_1+h_2)^2=2$, and $x\in\mathbb{E}$ to an element of square -4 (cf. 1.3). So $\bar{a}$ is the image of a root in M.
- or $q(\bar{h})=0$ and $q(\bar{x})=1$, i.e., $h\in\mathbb{H}$ is equivalent to $0, h_1$, or h_2 and $\bar{x}\in\mathbb{E}$ is equivalent to a root. In this case too, $\bar{a}$ is the image of a root in M.

This proves that all transvections are reductions mod 2 of reflections s_w and the reduction maps

$$W(E_8)\to O(\mathbb{E}/2\mathbb{E},q) \qquad O(M)\to O(M/2M,q)$$

are surjective. Since $O(M)=O^{\uparrow}(M)\times\{\pm\mathrm{id}\}$, even $O^{\uparrow}(M)\to O(M/2M,q)$ is surjective. Recalling that

$$|W(E_8)|=2^{14}\cdot 3^5\cdot 5^2\cdot 7$$

we find the following well known

(1.7) **Proposition.** a) *The 2-congruence subgroup of $W(E_8)$ is just* $\{\pm\mathrm{id}\}$ (cf. [1] Exercise in Chap. 6, § 4).

b) *The 2-congruence subgroup in $O(M)$ has index* $2^{21}\cdot 3^5\cdot 5^2\cdot 7\cdot 17\cdot 31$.

1.5. **An Auxiliary Result.** Denote by K the lattice $\mathbb{H}\perp\mathbb{H}(2)$ and fix a basis $h_1, h_2\in\mathbb{H}$, $k_1, k_2\in\mathbb{H}(2)$ with $h_1^2=h_2^2=k_1^2=k_2^2=0$, $\langle h_1, h_2\rangle=1$, $\langle k_1, k_2\rangle=2$. Let $G\subset O(K)$ be the subgroup acting trivially on $K^\vee/K$. It contains all $g\in O(K)$ such that $g(k_i)-k_i\in 2K$ for $i=1,2$. In 4.5 we shall apply the following fact. (We are indebted to Y. Namikawa for pointing out to us an error in the first version of this lemma.)

(1.8) **Lemma.** *All vectors $x\in K$ of square $x^2=-4$, which are of the form*

$$x=2s_1 h_1+2s_2 h_2+t_1 k_1+t_2 k_2, \qquad s_i, t_i\in\mathbb{Z},$$

are under G conjugate with k_1+k_2.

Proof. Given x as above we put

$$x_1=2s_1 h_1+t_1 k_1, \quad x_2=-2s_2 h_2-t_2 k_2,$$
$$e_1=t_2 h_1-s_2 k_1, \quad e_2=-t_1 h_2+s_2 k_2.$$

They satisfy

$$x_1^2=x_2^2=e_1^2=e_2^2=\langle x_i, e_j\rangle=0$$

and since

$$x^2=4(2s_1 s_2+t_1 t_2)=-4$$

we have additionally

$$\langle x_1, x_2\rangle=2, \quad \langle e_1, e_2\rangle=1.$$

So e_1, e_2, x_1, x_2 form a basis of K with e_1, e_2 generating a sublattice $\mathbb{H}$ and x_1, x_2 generating a sublattice $\mathbb{H}(2)$. Then there is some $g\in O(K)$ mapping

$$e_1\mapsto h_1, \quad e_2\mapsto h_2, \quad x_1\mapsto k_1, \quad x_2\mapsto k_2,$$
$$x=x_1+x_2\mapsto k_1+k_2.$$

Since $t_1\cdot t_2=1-2s_1 s_2$ is odd, the vectors

$$x_1-k_1=2s_1 h_1+(t_1-1)k_1 \quad x_2-k_2=-2s_2 h_2-(t_2+1)k_2$$

belong to $2K$ and $g\in G$.

2. Periods of Enriques Surfaces

2.1. **Notation.** Let X be any complex projective surface. The cup-product form $\langle\ ,\ \rangle$ on $H^2(X,\mathbb{R})$ restricts to the subspace $H^{1,1}(X,\mathbb{R})=H^2(X,\mathbb{R})\cap H^{1,1}(X)$ as a form of signature $(1, h^{1,1}(X)-1)$. The set $\{x\in H^{1,1}(X,\mathbb{R}): \langle x, x\rangle>0\}$ consists of two disjoint connected cones. For two elements in the same connected component the cup-product is positive, while it is negative for two elements in different components. So only one of the cones, say $\mathscr{C}_X$, *contains classes of ample divisors.*

The inclusion $\mathbb{Z}\to\mathbb{R}$ induces a map $H^2(X,\mathbb{Z})\to H^2(X,\mathbb{R})$. Its image $H^2(X,\mathbb{Z})$/torsion is denoted by $H^2(X,\mathbb{Z})_f$. Its elements are called the *integral* *points* of $H^2(X,\mathbb{R})$. The cup-product provides $H^2(X,\mathbb{Z})_f$ with a quadratic form. The sublattice

$$S_X = H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})_f$$

is called the *algebraic lattice*. Its elements are precisely the cohomology classes d of divisors D on X. $T_X = S_X^{\perp} \subset H^2(X,\mathbb{Z})_f$ is called the *transcendental lattice*. A curve $D \subset X$ is called *nodal* or (-2)-*curve*, if it is smooth rational with $D^2 = -2$. A *nodal class* is the class $d \in S_X$ of such a curve. We put

$$\mathscr{C}_X^+ = \{x \in \mathscr{C}_X : \langle x, d\rangle > 0 \text{ for all nodal classes } d\}.$$

(2.1) **Lemma.** *If X is a* K3 *or Enriques surface, the set $\mathscr{C}_X^+ \cap H^2(X,\mathbb{Z})_f$ of integral points in $\mathscr{C}_X^+$ consists precisely of the classes of ample divisors.*

Proof. By the Nakai-Moishezon criterion a divisor D with $D^2 > 0$ is ample if and only if $D \cdot E > 0$ for all irreducible curves $E \subset X$. But for such a curve the adjunction formula shows $E^2 = -2$ or $E^2 \geqq 0$. In the second case its class e belongs to the closure of $\mathscr{C}_X$ and hence $\langle x, e\rangle > 0$ for all $x \in \mathscr{C}_X$. It follows that an integral point of $\mathscr{C}_X^+$ is the class of an ample divisor and conversely. □

Therefore, in the case of a K3 or Enriques surface X, $\mathscr{C}_X^+$ is called the *ample cone*.

In the remainder of this section X is a K3-surface.

(2.2) **Lemma.** *Let Δ^+ denote the set of all classes $d \in H^2(X,\mathbb{Z})$ of effective divisors satisfying $d^2 = -2$. Then*

$$\mathscr{C}_X^+ = \{x \in \mathscr{C}_X : \langle x, d\rangle > 0 \text{ for all } d \in \Delta^+\}.$$

Proof. Let $\mathscr{C}'$ denote the cone on the right-hand side. Since Δ^+ contains all nodal classes, obviously $\mathscr{C}' \subset \mathscr{C}_X^+$. Conversely, if $d \in \Delta^+$ and $\langle x, d\rangle \leqq 0$ for some $x \in \mathscr{C}_X^+$, then also $\langle y, d\rangle < 0$ for some integral point $y \in \mathscr{C}_X^+$. This contradicts (2.1). So $\langle x, d\rangle > 0$ for all $d \in \Delta^+$, $x \in \mathscr{C}_X^+$ and this shows $\mathscr{C}_X^+ \subset \mathscr{C}'$. □

(2.3) **Lemma.** *For an isometry g of $H^2(X,\mathbb{Z})_f$ the following properties are equivalent.*

i) $g_{\mathbb{R}} \mathscr{C}_X^+ \subset \mathscr{C}_X^+$.
ii) *g maps each class of an ample divisor to the class of an ample divisor.*
iii) *g maps the class of one ample divisor to the class of an ample divisor.*
iv) *$g_{\mathbb{R}} \mathscr{C}_X = \mathscr{C}_X$ and $g\Delta^+ = \Delta^+$.*

Proof. i) ⇒ ii) follows from (2.1), ii) ⇒ iii) is trivial. If iii) holds then of course $g\mathscr{C}_X = \mathscr{C}_X$. If $d \in \Delta^+$, then $(gd)^2 = -2$, so by Riemann-Roch either gd or $-gd$ is effective. But let $a \in H^2(X,\mathbb{Z})$ be an ample class with ga ample again. Then $\langle gd, ga\rangle = \langle d, a\rangle$ is positive and $-gd$ cannot be effective. This proves iv). The step iv) ⇒ i) follows from (2.2). □

We denote by $O^{\uparrow}(X) \subset O(H^2(X,\mathbb{Z}))$ the subgroup of isometries g with properties i)–iv).

2.2. **The Universal Covering of an Enriques Surface.** Let Y be an Enriques surface and $\pi: X \to Y$ its universal (double) covering. Let $\sigma \in \operatorname{Aut}(X)$ be the covering involution. According to Horikawa [7, Theorem 5.1] there is an identification $H^2(X,\mathbb{Z}) \xrightarrow{\sim} L$ such that σ^* acts on $H^2(X,\mathbb{Z})$ as the involution s from 1.2. The map $\pi^*: H^2(Y,\mathbb{Z})_f \to H^2(X,\mathbb{Z})$ is an isomorphism onto $L^+ \subset L$. In particular there is an isometry $H^2(Y,\mathbb{Z})_f \xrightarrow{\sim} M$. Such an identification $H^2(X,\mathbb{Z}) \to L$ is called a *marking of the Enriques surface Y*. Let $H^{1,1}(X,\mathbb{R})^\sigma$ denote the vector subspace of σ^*-invariants.

(2.4) **Lemma.** *The map $\pi^*: H^{1,1}(Y,\mathbb{R}) \to H^{1,1}(X,\mathbb{R})$ maps $\mathscr{C}_Y^+$ bijectively onto $\mathscr{C}_X^+ \cap H^{1,1}(X,\mathbb{R})^\sigma$. The integral points in $\mathscr{C}_Y^+$ correspond* 1 *to* 1 *under π^* to the classes of ample σ-invariant divisors on X.*

Proof. To test whether $x \in \mathscr{C}_X$ belongs to $\mathscr{C}_X^+$ we have to check $\langle x,d\rangle > 0$ for nodal classes d. If $\sigma^* x = x$ then $\langle x,d\rangle = \frac{1}{2}\langle x, d+\sigma^* d\rangle$, and if $\langle d,\sigma^* d\rangle > 0$, then $(d+\sigma^* d)^2 = -4+2\langle d,\sigma^* d\rangle \geqq 0$, because this number is divisible by 4. Hence $d+\sigma^* d \in \bar{\mathscr{C}}_X$ and $\langle x,d\rangle > 0$. So we have to check $\langle x,d\rangle > 0$ only for nodal classes d with $\langle d,\sigma^* d\rangle = 0$. If now $D \subset X$ is the (-2)-curve representing d, then $\langle d,\sigma^* d\rangle = 0$ if and only if $D \cap \sigma^* D = \emptyset$, i.e., if and only if $\pi(D) = \pi(\sigma D)$ is a (-2)-curve on Y. Since every σ^*-invariant $x \in \mathscr{C}_X$ is of the form $\pi^* y$, $y \in \mathscr{C}_Y$, it follows that $H^{1,1}(X,\mathbb{R})^\sigma \cap \mathscr{C}_X^+ = \pi^* \mathscr{C}_Y^+$. If $c \in \mathscr{C}_Y^+$ is an integral point, then $\pi^* c$ is the class of a σ-invariant divisor. Since we have proven $\pi^* c \in \mathscr{C}_X^+$, from Lemma (2.1) we obtain the ampleness of this divisor. □

2.3. **The Torelli Theorem for K3-Surfaces.** In this section we state the global Torelli theorem [11, p. 534], [2, Cor. 32] in the form we need it.

So let X be a projective K3-surface and ω_X a nonzero holomorphic 2-form on X. This ω_X spans $H^{0,2}(X)$ and is unique up to scalars. Using the Hodge decomposition we view $H^{0,2}(X)$ as a subspace of $H^2(X,\mathbb{C})$ and ω_X as a class in $H^2(X,\mathbb{C})$. Obviously

$$\langle \omega_X, \omega_X\rangle = 0, \qquad \langle \omega_X, \bar{\omega}_X\rangle > 0.$$

For $\mathrm{Re}\,\omega_X$ and $\mathrm{Im}\,\omega_X \in H^2(X,\mathbb{R})$ these relations are equivalent with

$$\langle \mathrm{Re}\,\omega_X, \mathrm{Re}\,\omega_X\rangle = \langle \mathrm{Im}\,\omega_X, \mathrm{Im}\,\omega_X\rangle > 0,$$
$$\langle \mathrm{Re}\,\omega_X, \mathrm{Im}\,\omega_X\rangle = 0.$$

So $\mathrm{Re}\,\omega_X$ and $\mathrm{Im}\,\omega_X$ span in $H^2(X,\mathbb{R})$ a two-dimensional subspace, on which the cup-product is positive definite.

Since $H^{1,1}(X) = \mathrm{Re}\,\omega_X^\perp \cap \mathrm{Im}\,\omega_X^\perp$, we have

$$S_X = H^2(X,\mathbb{Z}) \cap \mathrm{Re}\,\omega_X^\perp \cap \mathrm{Im}\,\omega_X^\perp.$$

If $\rho = \operatorname{rank} S_X$ is the *Picard number*, we have signature $S_X = (1, \rho-1)$, signature $T_X = (2, 20-\rho)$.

(2.5) **Theorem** (Global Torelli). *Let g be an isometry of $H^2(X,\mathbb{Z})$. Then g is induced by a unique automorphism of X if and only if $g \in O^\uparrow(X)$ and $g_{\mathbb{C}}\omega_X = \lambda\omega_X$ for some $\lambda \in \mathbb{C}$.*

(2.6) **Corollary.** *The representation of* $\mathrm{Aut}(X)$ *on $H^2(X,\mathbb{Z})$ is faithful and identifies* $\mathrm{Aut}(X)$ *with a subgroup of $O^\uparrow(X)$.*

2.4. **Periods of Enriques Surfaces.** We recall Horikawa's results [7, 8] on the moduli space of Enriques surfaces. Let $Y = X/\sigma$ be an Enriques surface and $\varphi: H^2(X, \mathbb{Z}) \to L$ a marking of Y. Since on Y there are no holomorphic 2-forms, we have $\sigma^* \omega_X = -\omega_X$. So $\varphi_{\mathbb{C}}(\omega_X)$ defines a point $\tau(Y, \varphi)$ in the period domain

$$D := \{\mathbb{C} \cdot \omega \in \mathbb{P}(L_{\mathbb{C}}^-) \colon \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0\}.$$

This D is the union of two copies of a bounded symmetric domain of type IV and dimension 10. The group Γ (or rather $r^-(\Gamma)$) acts on D in a properly discontinuous way. It contains an involution interchanging the two connected components of D [7, Lemma 8.1]. Since $r^-(\Gamma)$ is an arithmetic group [7], by Baily-Borel the analytic space D/Γ carries the structure of a quasi-projective variety. Since two markings for Y differ by an element in Γ, the Γ-equivalence class $\tau(Y) \in D/\Gamma$ of $\tau(Y, \varphi)$ is independent of the choice of φ. This point $\tau(Y)$ is called the *period point* of Y.

Horikawa proves:

(i) $\tau(Y_1) = \tau(Y_2)$ if and only if Y_1 is isomorphic with Y_2.

(ii) The points $\tau(Y, \varphi)$ belong to

$$D^0 = D \setminus \bigcup_{d \text{ root in } L^-} d^\perp,$$

where $d^\perp = \{\mathbb{C} \cdot \omega \in \mathbb{P}(L_{\mathbb{C}}^-) \colon \langle \omega, d \rangle = 0\}$.

(iii) All points $\tau \in D^0$ are of the form $\tau(Y, \varphi)$ for some marked Enriques surface Y, φ.

(2.7) **Lemma.** *The union of all hyperplanes $d^\perp$, $d \in L^-$ a root, is locally finite in D. Hence $D \cap \bigcup_d d^\perp$ is an analytic set in D.*

Proof. If the union is not locally finite, there are infinitely many distinct roots $d_\nu \in L^-$ and points $\omega_\nu \in D \cap d_\nu^\perp$ such that $\omega = \lim_{\nu \to \infty} \omega_\nu \in D$. Since ω_ν converges to ω, the hyperplanes $(\operatorname{Re} \omega_\nu)^\perp$ and $(\operatorname{Im} \omega_\nu)^\perp$ as points in $\mathbb{P}((L_{\mathbb{R}}^-)^\vee)$ converge to $(\operatorname{Re} \omega)^\perp$, resp. $(\operatorname{Im} \omega)^\perp$. The cup-product on L^- has signature (2, 10) with $\operatorname{Re} \omega$, $\operatorname{Im} \omega$ spanning a plane, on which this form is positive definite. So the cup-product is negative definite on $(\operatorname{Re} \omega)^\perp \cap (\operatorname{Im} \omega)^\perp$. In particular the vectors in $(\operatorname{Re} \omega)^\perp \cap (\operatorname{Im} \omega)^\perp$ of square -2 form a compact sphere, and there is a compact neighborhood of this sphere containing all vectors of square -2 in $(\operatorname{Re} \omega_\nu)^\perp \cap (\operatorname{Im} \omega_\nu)^\perp$ for all $\nu \in \mathbb{N}$. All the infinitely many roots d_ν would belong to this compact set, a contradiction. □

The analytic set $D \cap \bigcup d^\perp$ in D is Γ-invariant. So its image in D/Γ is analytic too. The Baily-Borel compactification $\overline{D/\Gamma}$ of D/Γ is obtained by attaching a curve [11, §4, Lemma 1]. By the extension theorem of Remmert-Stein [12, Satz 13] the analytic hypersurface in D/Γ extends to a hypersurface in the projective variety $\overline{D/\Gamma}$. It follows that Horikawa's period domain D^0/Γ is quasi-projective [8, Thm. 3.1].

2.5. **Nodal Curves on Enriques Surfaces.** If $C \subset Y$ is a nodal curve, then $\pi^* C \subset X$ decomposes as $B + \sigma B$ with a nodal curve B on X satisfying $B \cdot \sigma(B) = 0$.

Conversely, given nodal curves $B, \sigma(B)$ on X with $B \cdot \sigma(B)=0$, there is a nodal curve $C=\pi(B) \subset X$ such that $B+\sigma(B)=\sigma^* C$. So there is a 1 to 1 correspondence between nodal classes $c \in H^2(Y, \mathbb{Z})_f$ and pairs $b, \sigma^*(b)$ of nodal classes on X satisfying $\langle b, \sigma^*(b)\rangle=0$. Fix a marking $H^2(X, \mathbb{Z})=L$ as above. On L^- the linear forms $\langle b, -\rangle$ and $\langle sb, -\rangle$ differ only by the sign. For each root $c \in H^2(Y, \mathbb{Z})_f=M$ with $\pi^* c=b+\sigma(b)$ put $D_c=D \cap b^\perp$. Since D is not contained in any hyperplane and $b \notin L^+$, we have $D_c \neq D$. Since there are only countably many roots in L^+, the set

$$D_{\mathrm{gen}}=D^0 \setminus \bigcup_{c \text{ root in } L^+} D_c$$

is still dense in D^0 and D_{gen}/Γ is dense in D^0/Γ. The period point $\tau(Y) \in D^0/\Gamma$ is contained in the image of $\bigcup D_c$ if and only if Y contains a (-2)-curve. So, if we understand by a "generic" Enriques surface Y a surface with $\tau(Y) \in D_{\mathrm{gen}}/\Gamma$, then we have shown:

(2.8) **Proposition.** *The generic Enriques surface contains no* (-2)*-curve.*

Now for given Y we define the following sublattices $L_1, L_2, L_3, L_4 \subset L$: Let $M' \subset M$ be the smallest primitive sublattice containing all nodal classes and $L_1 = \pi^* M' \subset L^+$. Let $L_2 \subset L^-$ be the smallest primitive sublattice containing all the classes $d-s(d)$, where $d+s(d)=\pi^* c$, $c \in M$ a nodal class. We put $L_3 = L_2^\perp \cap S_X \cap L^-$ and $L_4=T_X$. Finally we let $N \subset L^-$ be the smallest primitive sublattice containing L_3 and L_4. Since the form on $S_X \cap L^-$ is negative definite we have

$$L_{\mathbb{R}}^-=L_2 \otimes \mathbb{R} \perp \underbrace{L_3 \otimes \mathbb{R} \perp L_4 \otimes \mathbb{R}}_{N_{\mathbb{R}}}.$$

Notice that the sublattice $N \subset L^-$ determines L_2, L_2 determines M' and hence L_1. We call N the *nodal type* of the marked surface Y, φ. Proposition (2.8) means of course $L_1=L_2=0$ for generic Y.

2.6. **Generic Enriques Surfaces of Fixed Nodal Type.** We fix a primitive sublattice $N \subset L^-$ and consider all marked Enriques surfaces Y, φ of fixed nodal type N. Their period points $\tau(Y, \varphi)$ belong to $D^0 \cap \mathbb{P}(N_{\mathbb{C}})$. If there is at least one surface of nodal type N, then $D^0 \cap \mathbb{P}(N_{\mathbb{C}})$ is a non-empty open set in a quadric of $\mathbb{P}(N_{\mathbb{C}})$. Put $n=\operatorname{rank} N$. If $n \geqq 3$, the union of countably many hyperplanes of $\mathbb{P}(N_{\mathbb{C}})$ intersects $D^0 \cap \mathbb{P}(N_{\mathbb{C}})$ in a set with dense complement. We apply this simultaneously to two different kinds of hyperplanes.

a) The hyperplanes $d^\perp \cap N_{\mathbb{C}}$ where $d \in L$, $d \notin N^\perp$, is a nodal class satisfying $\langle d, s(d)\rangle=0$. The period points in the complement of these hyperplanes belong to surfaces of nodal type precisely equal to N (and not smaller).

b) The hyperplanes of $\mathbb{P}(N_{\mathbb{C}})$ defined over some algebraic number field $k \subset \mathbb{C}$. In particular we take k the extension of $\mathbb{Q}$ obtained by adjoining all primitive l-th roots of unity with Euler function $\varphi(l) \leqq n$. Since $n \leqq 12$, only the following values of l occur: $l=1, \ldots, 16, 18, 20, \ldots, 32, 36, 42$.

(2.9) **Lemma.** *Let* Y, φ *be a marked Enriques surface of nodal type* N *with period point* $\tau(Y, \varphi) \in D^0 \cap \mathbb{P}(N_{\mathbb{C}})$ *not contained in any proper linear subspace of*

$\mathbb{P}(N_{\mathbb{C}})$ defined over k. Let $g \in \mathrm{Aut}(X)$ be an automorphism commuting with the covering involution σ. Then $g^*|N = \pm \mathrm{id}_N$.

Proof. Since g commutes with σ it induces an automorphism of Y, so g^* leaves invariant all sublattices L_i, $N \subset Y$. Let ω_X be a nonzero holomorphic 2-form on X. Then $g^*\omega_X = \lambda\omega_X$ with $\lambda \in \mathbb{C}$. Since $\langle g^*\omega_X, \overline{g^*\omega_X}\rangle = \langle \omega_X, \bar{\omega}_X \rangle$, obviously $|\lambda| = 1$. But since X is projective, from [13, p. 178/179] it follows that λ is a root of unity. Since λ is an eigenvalue for $g^*|N_{\mathbb{C}}$ and $n \leqq 12$, we have $\deg[\mathbb{Q}(\lambda):\mathbb{Q}] \leqq 12$. So $\lambda^l = 1$ with $\varphi(l) \leqq 12$ and $\lambda \in k$. g^* is defined over $\mathbb{Q}$, so the λ-eigenspace for $g^*|N \otimes \mathbb{C}$ is defined over k. But then the assumption on $\tau(Y, \varphi)$ implies that this eigenspace is all of $N_{\mathbb{C}}$, i.e., $g^*|N_{\mathbb{C}} = \lambda \cdot \mathrm{id}$. This is possible only if $\lambda = \pm 1$. □

If $n = \mathrm{rank}\, N = 2$ we may embed N in the euclidean plane $\mathbb{R}^2$ such that the form on N is induced by the usual inner product on $\mathbb{R}^2$. The two isotropic subspaces in $N_{\mathbb{C}} = \mathbb{C}^2$ are generated by the vectors $(1, \pm i)$. An orientation reversing orthogonal transformation of $\mathbb{R}^2$ would interchange these two subspaces. So any isometry g of N leaving both lines $\mathbb{C} \cdot (1, \pm i)$ invariant is of the form $g = \pm \mathrm{id}$, unless N is isometric with the period lattice in $\mathbb{C} = \mathbb{R}^2$ of an elliptic curve with $\mathbb{Z}_4$ or $\mathbb{Z}_6$-symmetry. In this case g may be of order 3, 4, or 6. So with these exceptions the analog of Lemma (2.9) for $n = 2$ also holds.

3. Generic Enriques Surfaces

3.1. **Automorphisms of Enriques Surfaces.** As above, fix a marking $H^2(X, \mathbb{Z}) = L$, $H^2(Y, \mathbb{Z})_f = M$ of the Enriques surface Y. Recall that $\Gamma \subset O(L)$ is the subgroup of isometries commuting with s and $O^{\uparrow}(X) \subset O(L)$ the subgroup of isometries g for which $g_{\mathbb{R}}$ leaves invariant $\mathscr{C}_X^+ \subset L_{\mathbb{R}}$. Let $O^{\uparrow}(Y)$ denote the group of isometries $g \in O(M)$ with $g_{\mathbb{R}}$ leaving invariant $\mathscr{C}_Y^+ \subset M_{\mathbb{R}}$. Each automorphism $\alpha \in \mathrm{Aut}(Y)$ lifts (in two ways) to an automorphism $\tilde{\alpha} \in \mathrm{Aut}(X)$ commuting with s. So if we put

$$\mathrm{Aut}(X, \sigma) = \{g \in \mathrm{Aut}(X) : g\sigma = \sigma g\}$$

then $\mathrm{Aut}(Y) = \mathrm{Aut}(X, \sigma)/\{\mathrm{id}, \sigma\}$. Any $g \in \mathrm{Aut}(X)$ commuting with s leaves invariant the sublattices $L_1, \ldots, L_4 \subset L$ defined in 2.5. So we have canonical maps r_i: $\mathrm{Aut}(X, \sigma) \to O(L_i)$, $i = 1, \ldots, 4$, and an embedding (r^+, r_2, r_3, r_4): $\mathrm{Aut}(X, \sigma) \to O(L^+) \times O(L_2) \times O(L_3) \times O(L_4)$ with $r^- = (r_2, r_3, r_4)$.

(3.1) **Proposition.** *Under r^+ the kernel of r^-: $\mathrm{Aut}(X, \sigma) \to O(L^-)$ is identified with the 2-congruence subgroup of $O^{\uparrow}(Y)$.*

Proof. By Lemma (1.3) the kernel of $r^- = \Gamma \to O(L^-)$ is under r^+ identified with the 2-congruence subgroup of $O(L^+)$. It is clear that $r^+(\gamma)_{\mathbb{R}}$ leaves invariant the ample cone $\mathscr{C}_Y^+$ for every $\gamma \in \mathrm{Aut}(X, \sigma)$. So we only have to show the converse: if $\gamma \in \Gamma$ with $r^-(\gamma) = \pm \mathrm{id}$ and $r^+(\gamma)_{\mathbb{R}}$ leaving invariant $\mathscr{C}_Y^+$, then γ is induced by an automorphism of X. But take some arbitrary ample divisor C on Y with class $c \in C_Y^+ \cap H^2(Y, \mathbb{Z})_f$. Then $r^+(\gamma)c$ is the class of an ample divisor again. The class $\pi^* c \in L^+$ is the class of the ample divisor $\pi^* C$ and $\gamma(\pi^* c) = \pi^* r^+(\gamma) c$ too.

So $\gamma \in O^{\uparrow}(X)$ by Lemma (2.3). Since $r^{-}(\gamma) = \pm \mathrm{id}$, the extension $\gamma_{\mathbb{C}}$ leaves invariant the subspace $\mathbb{C}\omega_X \subset L_{\mathbb{C}}^{-}$. The assertion follows from Theorem (2.5). □

(3.2) **Proposition.** *The image of* r^{-}: $\mathrm{Aut}(X, \sigma) \to O(L^{-})$ *is a finite group.*

Proof. The form on L^{-} has signature (2, 10). For each $\gamma \in \Gamma(Y)$ we have $\gamma_{\mathbb{C}}\omega_X = \lambda\omega_X$, $\lambda \in \mathbb{C}$. If $\omega_X = u + iv$, $u, v \in L_{\mathbb{R}}^{-}$, this shows that $\gamma_{\mathbb{R}}$ leaves invariant the 2-dimensional real subspace $\mathbb{R}u \oplus \mathbb{R}v \subset L_{\mathbb{R}}^{-}$, on which the form is positive definite. On $(\mathbb{R}u \oplus \mathbb{R}v)^{\perp} \subset L_{\mathbb{R}}^{-}$ the form is negative definite. So $r^{-}\gamma$ belongs to the compact group $O(\mathbb{R}u \oplus \mathbb{R}v) \times O((\mathbb{R}u \oplus \mathbb{R}v)^{\perp})$. □

Combining propositions (3.1) and (3.2) we find

(3.3) **Theorem.** *For every Enriques surface Y the automorphism group* $\mathrm{Aut}(Y)$ *contains the 2-congruence subgroup of* $O^{\uparrow}(Y)$ *as normal subgroup of finite index.*

Next we consider generic Enriques surfaces and obtain the main result of this paper.

(3.4) **Theorem.** *Let* Y, φ *be a marked Enriques surface with period point* $\tau(Y, \varphi) \in D^0$ *not contained in any proper linear subspace of* $\mathbb{P}(L_{\mathbb{C}}^{-})$ *defined over the number field k from Lemma* (2.9). *Then the representation of* $\mathrm{Aut}(Y)$ *on* $H^2(Y, \mathbb{Z})_f = M$ *identifies* $\mathrm{Aut}(Y)$ *with the 2-congruence subgroup of* $O^{\uparrow}(M)$.

Proof. The assumption implies $\tau(Y, \varphi) \in D_{\mathrm{gen}}$. So $O^{\uparrow}(Y) = O^{\uparrow}(M)$. Also Lemma (2.9) shows that $r^{-}(\mathrm{Aut}(X, \sigma)) = \pm \mathrm{id}$, and the assertion follows from Proposition (3.1). □

Now if a generic surface Y is deformed into less generic ones, the following phenomena, working against each other, can happen.
- Y acquires nodal curves, so $O^{\uparrow}(Y)$ and $\mathrm{Aut}(Y)$ probably too, become smaller.
- Y acquires nodal curves and/or S_X becomes bigger, ω_X becomes more special, hence $r^{-}(\mathrm{Aut}(X, \sigma))$ and $\mathrm{Aut}(Y)$ probably too grow bigger.

We do not know, whether one can control these effects. In particular we do not know the Enriques surface with the "biggest" or the "smallest" automorphism group.

3.2. **Computation of Some Stabilizer Groups.** Denote by $G \subset O^{\uparrow}(M)$ the 2-congruence subgroup. It is the purpose of this section, to compute the stabilizer subgroups $G_c \subset G^{\uparrow}(M)_c$ for certain elements $c \in M$. An element $c \in M$ will be called

primitive, if $\mathbb{Z}c \subset M$ is a primitive sublattice

0-*class*, if $c^2 = 0$

forward pointing, if $c \in \bar{\mathscr{C}}_Y^{+}$ (here and in the sequel fix an isomorphism $M = H^2(Y, \mathbb{Z})_f$)

fwp, if c is forward pointing and primitive.

In particular we consider elements

$$c = e_1 + \ldots + e_n$$

where $e_1, \ldots, e_n$ are 0-classes satisfying

(3.5) $e_i \cdot e_j = 1$ for $i \neq j$.

(3.6) **Lemma.** i) *If* $n \geqq 2$, *then* (3.5) *implies that all* e_i *are primitive.*

ii) *If* $n \geqq 2$ *and one* e_i *is fwp, then so are all.*

iii) *If* $c = e_1 + \ldots + e_n = e'_1 + \ldots + e'_n$ *with* $e_1, \ldots, e_n$ *and* $e'_1, \ldots, e'_n$ *fwp 0-classes satisfying* (3.5), *then up to a permutation we have* $e_1 = e'_1, \ldots, e_n = e'_n$.

Proof. The assertions i) and ii) being obvious, we prove iii). We compute $c \cdot e'_1$ in two ways:

$$(e_1 + \ldots + e_n) e'_1 = \sum_1^n e_i e'_1 \quad \text{and} \quad (e'_1 + \ldots + e'_n) \cdot e'_1 = n - 1.$$

Since e_i and e'_1 both are *fwp*, we have $e_i \cdot e'_1 \geqq 0$ and $e_i \cdot e'_1 = 0$ only if $e_i = e'_1$. This shows $e_i = e'_1$ for some i. The assertion follows by induction on n. □

(3.7) **Proposition.** *For* $n = 1, 2, 3$ *or* 10 *the group* $O^{\uparrow}(M)$ *operates transitively on the set of ordered n-tuples* $e_1, \ldots, e_n \in M$ *consisting of fwp 0-classes satisfying* (3.5).

Proof. $n = 1$. Let $e_1 \in M$ be an arbitrary *fwp* 0-class. Since M is unimodular, there is some $c \in M$ with $e_1 \cdot c = 1$. Put $k = \frac{1}{2} c^2$ and $e_2 = c - k e_1$. Then $e_2^2 = 0$ and $e_1 \cdot e_2 = 1$. So e_1, e_2 are *fwp* generators of a sublattice $\mathbb{H} \subset M$. Since e_1 and e_2 may be permuted by some $g \in O^{\uparrow}(M)$, the assertion will follow from the case $n = 2$.

$n = 2$. It suffices to show that $O^{\uparrow}(\mathrm{M})$ operates transitively on the set of sublattices $\mathbb{H} \subset M$. But when an embedding $\mathbb{H} \subset M$ is given, we have $M = \mathbb{H} \perp \mathbb{H}^{\perp}$ with $\mathbb{H}^{\perp}$ unimodular, even, and negative definite, hence $\mathbb{H}^{\perp} = \mathbb{E}$. Then there is some $g \in O^{\uparrow}(M)$ mapping this decomposition $M = \mathbb{H} \perp \mathbb{E}$ into the standard one.

$n = 3$. From the case $n = 2$ it follows that we may assume e_1, e_2 to be the standard generators of $\mathbb{H}$ in the standard decomposition $M = \mathbb{H} \perp \mathbb{E}$. Then $e_3 = e_1 + e_2 + w$ with $w \in \mathbb{E}$ some root. The assertion follows from the well-known fact that the Weyl group $W(E_8)$ operates transitively on the roots of E_8.

$n = 10$. We take $e_1, e_2 \in \mathbb{H} \subset M$ as in the case $n = 3$ and for $i = 3, \ldots, 10$ we have $e_i = e_1 + e_2 + w_i$ with roots $w_i \in \mathbb{E}$ satisfying

$$w_i \cdot w_j = -1 \qquad \text{whenever } i \neq j.$$

The assertion follows from Lemma (1.6). □

Now we consider sums $c = e_1 + \ldots + e_n$ of *fwp* 0-classes $e_i \in M$ satisfying (3.5). For $n = 1, 2, 3$, and 10 we saw that $O^{\uparrow}(M)$ operates transitively on such c. It follows from Lemma (1.1) and Lemma (3.6) iii) that

$$O^{\uparrow}(M)_c = \mathfrak{S}_n \times O(e_1^{\perp} \cap \ldots \cap e_n^{\perp}),$$

$\mathfrak{S}_n$ the permutation group of degree n. We are interested in the number $N(n)$ = number of G-orbits of elements c.

Clearly the stabilizer subgroup G_c is $G \cap O^{\uparrow}(M)_c$, and since G is a normal subgroup in $O^{\uparrow}(M)$, all G-orbits are equivalent under $O^{\uparrow}(M)$. So the set of G-orbits is a homogeneous space under $O^{\uparrow}(M)/G = O^{+}(10, \mathbb{F}_2)$ and

$$N(n) = |O^{+}(10, \mathbb{F}_2)| / [O^{\uparrow}(M)_c : G_c].$$

The results are given in the following table

n	$O^{\uparrow}(M)_c$	G_c	$[O^{\uparrow}(M)_c : G_c]$	$N(n)$
1	$\mathbb{E} \times W_8$	$2\mathbb{E} \times \{\pm 1\}$	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7$	$17 \cdot 31 = 527$
2	$\mathfrak{S}_2 \times W_8$	$\{\pm 1\}$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 17 \cdot 31 = 67456$
3	$\mathfrak{S}_3 \times W_7$	1	$2^{11} \cdot 3^5 \cdot 5 \cdot 7$	$2^{10} \cdot 5 \cdot 17 \cdot 31 = 2698240$
10	$\mathfrak{S}_{10}$	1	$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$	$2^{13} \cdot 3 \cdot 17 \cdot 31 = 12951552$

Proofs. Recall (Sect. 1.4) that $|O^{+}(10, \mathbb{F}_2)| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$. To compute the stabilizer groups we use the standard decomposition $M = \mathbb{H} \perp \mathbb{E}$ with $\mathbb{H} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Any $g \in O^{\uparrow}(M)$ has a matrix decomposition

$$g = \left(\begin{array}{c|c} g_H & g_{EH} \\ \hline g_{HE} & g_E \end{array}\right) \qquad \begin{array}{ll} g_H: \mathbb{H} \to \mathbb{H}, & g_E: \mathbb{E} \to \mathbb{E} \\ g_{EH}: \mathbb{E} \to \mathbb{H}, & g_{HE}: \mathbb{H} \to \mathbb{E}. \end{array}$$

$n = 1$. Assume $ge_1 = e_1$. Then

$$g = \left(\begin{array}{cc|c} 1 & t & f \\ 0 & 1 & 0 \\ \hline 0 & y & g_E \end{array}\right) \qquad \begin{array}{ll} t \in \mathbb{Z}, & y \in \mathbb{E} \\ f: \mathbb{E} \to \mathbb{Z} & \end{array}$$

and orthogonality of g is equivalent with

$$t = -\tfrac{1}{2} y^2, \qquad g_E \in W_8, \qquad f = -\langle g_E y, \ \rangle.$$

So $O^{\uparrow}(M)_{e_1}$ is $\mathbb{E} \times W_8$, the extended Weyl group, under the identification

$$\mathbb{E} \times W \ni (y, h) \mapsto \left(\begin{array}{cc|c} 1 & -\tfrac{1}{2} y^2 & -\langle hy, \ \rangle \\ 0 & 1 & 0 \\ \hline 0 & y & h \end{array}\right).$$

It is known (Sect. 1.4) that the 2-congruence subgroup of W_8 is just $\{\pm 1\}$. So $G_c = 2\mathbb{E} \times \{\pm 1\}$.

$n = 2$. If $c = e_1 + e_2$, we have $O^{\uparrow}(M)_c = \mathfrak{S}_2 \times W_8$ with $\mathfrak{S}_2$ permuting e_1 and e_2. Since a nontrivial permutation of e_i's cannot belong to G, we have $G_c = \{\pm 1\}$, the 2-congruence subgroup of W_8.

$n = 3$. If $c = e_1 + e_2 + e_3$ and $e_3 = e_1 + e_2 + w$ as above with some root $w \in \mathbb{E}$, then $\mathbb{Z}e_1^{\perp} \cap \mathbb{Z}e_2^{\perp} \cap \mathbb{Z}e_3^{\perp}$ is $w^{\perp} \subset \mathbb{E}$, the orthogonal complement of a root. This $w^{\perp}$ is isomorphic with the lattice $(-E_7)$. So $O^{\uparrow}(M)_c = \mathfrak{S}_3 \times W_7$ with $\mathfrak{S}_3$ permuting e_1, e_2, e_3 and W_7 the Weyl group of E_7. In particular G_c is trivial.

$n = 10$. Obvious. $\square$

3.3. Representations of Generic Enriques Surfaces

a) *Elliptic Pencils.* In this section let Y be an Enriques surface, general in the sense that Theorem (3.4) applies. In particular, there are no (-2)-curves on Y.

We fix an isomorphism $H^2(Y,\mathbb{Z})_f = M$. It is classical that (because of the absence of (-2)-curves) each *fwp* 0-class $e \in M$ is effective. There are exactly two curves E and E' representing e. They are either nonsingular elliptic or irreducible rational with a node or a cusp, and adjoint in the sense that $K_Y = \mathcal{O}_Y(E - E')$. The linear system $|2E|$ is an elliptic pencil without base points. It provides Y with the structure of an elliptic fibre space over $\mathbb{P}_1$. This fibration contains exactly two multiple fibres, namely $2E$ and $2E'$. (The divisors E and E' are called the half-pencils of $|2E|$.) Each elliptic fibration over $\mathbb{P}_1$ is defined by such a linear system $|2E|$ with $e \in M$ some *fwp* 0-class. We shall not distinguish between two elliptic fibrations $Y \to \mathbb{P}_1$ differing by an automorphism of $\mathbb{P}_1$. So two elliptic fibre spaces $Y \to \mathbb{P}_1$ are isomorphic (as fibre spaces, modulo $\mathrm{Aut}(\mathbb{P}_1)$) if and only if they differ by an automorphism of X. So the different representations of X as elliptic fibrations over $\mathbb{P}_1$ correspond 1 to 1 with G-orbits of *fwp* 0-classes $e \in M$. From 6.5 we conclude

(3.8) **Theorem.** *For a general Enriques surface Y there are exactly* $17 \cdot 31$ *non-isomorphic elliptic fibre spaces over $\mathbb{P}_1$ with total space Y.*

b) *Double Plane Representations.* Now we consider pairs $e_1, e_2 \in M$ of *fwp* 0-classes with $e_1 \cdot e_2 = 1$. We let E_i, E_i' be the curves representing e_i, we put $D = 2(E_1^{(\prime)} + E_2^{(\prime)})$ and consider the linear system $|D|$. It is known [3, Theorem 6.1] that this linear system defines a 2 to 1 map of Y onto a "quartic Del Pezzo surface Q of Segre symbol (11)(11)1", i.e., a surface Q in $\mathbb{P}_4$ projectively equivalent to the complete intersection $z_0^2 = z_1 z_2 = z_3 z_4$ of two rank-3 quadrics. The map is ramified over the four nodes of Q and a complete intersection curve $B = Q \cap Z$ with some quadric $Z \subset \mathbb{P}_4$ not passing through any of the four nodes of Q. The absence of (-2)-curves on Y forces B to be nonsingular. The double cover $Y \to Q$ is related to Horikawa's representation of Y [7, 8] through the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathbb{P}_1 \times \mathbb{P}_1 \arrow[r] & Q \end{tikzcd}$$

where $\mathbb{P}_1 \times \mathbb{P}_1 \to Q$ is a double cover ramified over the four nodes of Q.

Up to an automorphism of $\mathbb{P}_4$ the map defined by $|D|$ is uniquely determined by the two classes $e_1, e_2 \in M$. We consider two double plane representations of Y as equivalent, if the pairs (Q, B) defining them differ by an automorphism of $\mathbb{P}_4$, i.e., if the classes D differ by an automorphism of Y. From Sect. 3.2 we obtain:

(3.9) **Theorem.** *For a general Enriques surface there are exactly* $2^7 \cdot 17 \cdot 31$ *inequivalent double plane representations.*

c) *Enriques Representations.* Let $e_1, e_2, e_3 \in M$ be a triplet of *fwp* 0-classes satisfying $e_i \cdot e_j = 1$ for $i \neq j$, let E_i, E_i' be the curves representing e_i, and put $D = E_1$

$+E_2+E_3$ (defined uniquely by $d=e_1+e_2+e_3$ up to the ambiguity between D and $D'=D+K_y$). It is known [3, Theorem 7.4] that $|D|$ defines a birational map of Y onto a sextic surface in $\mathbb{P}_3$ passing doubly through the edges of a tetrahedron. The image surfaces are projectively equivalent if and only if the linear systems $|D|$ differ by an automorphism g of Y. Using the double-plane representation it is easy to see that for general Y (i.e., general choice of the branch curve $B \subset Q$) there is no automorphism $g \in G$ leaving e_1, e_2 invariant and interchanging E_1 and E'_1. This shows that in general the systems $|D|$ and $|D'|$ have projectively inequivalent images. From Sect. 3.2 we conclude:

(3.10) **Theorem.** *For a general Enriques surface there are exactly* $2^{11} \cdot 5 \cdot 17 \cdot 31$ *inequivalent Enriques representations.*

d) *Representations as Surfaces of Degree* 10 *in* $\mathbb{P}_5$. Let $e_1, \ldots, e_{10}$ be *fwp* 0-classes satisfying $e_1 \cdot e_j = 1$ for $i \neq j$ and let E_i, E'_i be the curves representing e_i. We consider the linear system $|D|$ with class

$$d = \tfrac{1}{3} \sum_{1}^{10} e_i.$$

(Notice that because of the explicit form of the e_i given in Sect. 3.2 and Lemma (1.6) one easily checks that $\sum_{1}^{10} e_i$ in M is divisible by 3.) It is known [4, 3.2.1 iii) and 3.3.2] that there are (special) Enriques surfaces carrying such e_i with $|D|$ defining an embedding of Y in $\mathbb{P}_5$ of degree 10 ("Reye-congruences"). So for general Y, the system $|D|$ will also define such an embedding $Y \to \mathbb{P}_5$. As above one proves that $|D|$ and $|D'|$ define projectively inequivalent embeddings. So we conclude from Sect. 3.2

(3.11) **Theorem.** *For a general Enriques surface there are exactly* $2^{14} \cdot 3 \cdot 17 \cdot 31$ *embeddings in* $\mathbb{P}_5$*, defined by linear systems* $|D| = |\frac{1}{3} \sum E_i|$ *as above, as* 10*th degree surfaces which are projectively inequivalent.*

4. Examples of Enriques Surfaces with Small Automorphism Group

In this section we use the double plane presentation of Enriques surfaces to compute explicitly the automorphism groups for some examples. The observations that $\mathrm{Aut}(Y)$ is finite in case 3 below is due to Dolgachev [6].

4.1. **The Branch Curve.** Denote by Q the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ and let $((u_0 : u_1), (v_0 : v_1))$ be bihomogeneous coordinates on Q. By a line on Q we mean a smooth rational curve belonging to one of the two rulings on Q.

Take constants $a, b, c, d \in \mathbb{C}$ satisfying

$$a \neq 0, \quad c \neq 0 \neq d, \quad c \neq d$$

and consider the curve $B \subset Q$ of bidegree (4, 4) with equation

$$(v_0^2 - v_1^2)\{a(v_0^2 - v_1^2)u_0^4 + 2b(v_0^2 - v_1^2)u_0^2 u_1^2 + (cv_0^2 - dv_1^2)u_1^4\} = 0.$$

Then B splits as $B=N^+ + N^- + C$ with the two lines

$$N^\pm : v_0 \pm v_1 = 0$$

and C a curve of bidegree (4, 2). This curve C meets the line $N^\pm$ at

$$P^\pm = (1:0), (1: \mp 1) \in N^\pm$$

with multiplicity 4. Since $a \neq 0$, C is smooth in these points, so they are A_7-singularities on B.

It turns out that one has to distinguish between the following three cases:

Case 1 (general case): $b \neq 0$, $ac \neq b^2 \neq ad$.
Case 2 (symmetric case): $b=0$.
Case 3 (special case): $ac=b^2$ or $ad=b^2$.

Here Case 3 leads to the surface first considered by Dolgachev [6].

(4.1) **Lemma.** *Each line $(v_0 : v_1) = \text{const} \neq \pm 1$ meets C at four distinct points, unless it is one of the two lines*

$$L^\pm : (v_0 : v_1) = (\sqrt{ad-b^2} : \pm\sqrt{ac-b^2}).$$

Case 1: *The two lines are different and C is smooth at the two distinct points of contact.*
Case 2: *The two lines are different and C meets them with multiplicity four at a smooth point of C.*
Case 3: *The two lines coincide and C has two ordinary nodes on this line.*

(4.2) **Corollary.** *Away from $P^\pm$ the curve B is smooth in case 1 and 2, and has two A_1-singularities on the line $L^+ = L^-$ in case 3.*

Proof of the Lemma. An arbitrary line L with equation $v_0 : v_1 = t_0 : t_1 \neq \pm 1$ intersects C at four distinct points unless $a(ct_0^2 - dt_1^2) = b^2(t_0^2 - t_1^2)$, i.e.

$$(ac-b^2)t_0^2 = (ad-b^2)t_1^2.$$

This condition determines the lines $L^\pm$. The restriction of C of $L^\pm$ has equation $(au_0^2 + bu_1^2)^2 = 0$, so the points of contact are

$$((u_0 : u_1), (v_0 : v_1)) = ((\sqrt{b} : \pm i\sqrt{a}), (\sqrt{ad-b^2} : \pm\sqrt{ac-b^2})).$$

In these points we differentiate the equation for C

$$\begin{aligned} \partial/\partial v_1 &= -2av_1 u_0^4 - 4bv_1 u_0^2 u_1^2 - 2dv_1 u_1^4 \\ &= -2v_1(ab^2 - 2ab^2 + da^2) \\ &= -2av_1(ad-b^2). \end{aligned}$$

So C is smooth here in case 1 and 2, but singular in case 3.

In case 3 let e.g. $ac=b^2$, hence $L^+=L^-$ is the line $v_1=0$. We use inhomogeneous coordinates $u=u_1/u_0$ and $v=v_1/v_0$ to form partial derivatives of the equation for C in the points $\left(\left(1:\pm i\sqrt{\frac{a}{b}}\right),(1:0)\right)$.

$$\begin{aligned}
\partial^2/\partial u^2 &= 4b+12cu^2=4\left(b-3c\frac{a}{b}\right)=-8b\neq 0,\\
\partial^2/\partial u\partial v &= 0,\\
\partial^2/\partial v^2 &= -2a-4bu^2-2du^4\\
&= -2\left(a-2a+d\frac{a^2}{b^2}\right)=-2a(ad-b^2)\neq 0.
\end{aligned}$$

So the singularities are ordinary nodes. □

The equation for B is invariant under the group $\mathbb{Z}_2\times\mathbb{Z}_2$ generated by

$$\begin{aligned}
\tau_1&: ((u_0:u_1),(v_0:v_1))\mapsto((u_0:-u_1),(v_0:v_1)),\\
\tau_2&: ((u_0:u_1),(v_0:v_1))\mapsto((u_0:u_1),(v_0:-v_1)).
\end{aligned}$$

We put

$$\tau=\tau_1\tau_2.$$

If $b=0$ the group $\mathbb{Z}_2\times\mathbb{Z}_2$ can be enlarged to $\mathbb{Z}_4\times\mathbb{Z}_2$ generated by

$$\rho: ((u_0:u_1),(v_0:v_1))\mapsto((u_0:iu_1),(v_0:v_1))$$

and τ_2, i.e., $\rho^2=\tau_1$.

(4.3) **Lemma.** *In case 1 and 3 the group $\mathbb{Z}_2\times\mathbb{Z}_2$ generated by τ_1,τ_2 and in case 2 the group $\mathbb{Z}_4\times\mathbb{Z}_2$ generated by ρ,τ_2 is the full automorphism group of the pair $B\subset Q$.*

Proof. Any automorphism α of (B,Q) respects the pair of lines $N^\pm$, hence does not interchange u and v. Therefore $\alpha=\alpha_1\alpha_2$ with α_1 acting on u and α_2 on v. Additionally α respects the line $u_1=0$, the pair $L^\pm$ and the pair of lines $au_0^2+bu_1^2=0$.

Case 1 and 3. Here the equation $au_0^2+bu_1^2=0$ defines two distinct lines, interchanged by τ_1. This implies $\alpha_1=\mathrm{id}$ or $\alpha_1=\tau_1$. Now either α_2 or $\alpha_2\tau_2$ leaves invariant both the points $(1:\pm 1)$, so it is of the form

$$(v_0:v_1)\mapsto(t_0v_0+t_1v_1:t_1v_0+t_0v_1),\qquad t_0^2\neq t_1^2.$$

This substitution changes

$$\begin{aligned}
v_0^2-v_1^2&\mapsto(t_0^2-t_1^2)(v_0^2-v_1^2),\\
cv_0^2-dv_1^2&\mapsto(ct_0^2-dt_1^2)v_0^2+2(c-d)t_0t_1v_0v_1+(ct_1^2-dt_0^2)v_1^2,
\end{aligned}$$

and the invariance of C under $\alpha_1\alpha_2$ implies first $t_0t_1=0$ and then $t_1=0$. So either $\alpha_2=\mathrm{id}$ or $\alpha_2=\tau_2$.

Case 2. Now the line $u_0=0$ is fixed under α, because on it C touches $L^{\pm}$. This implies

$$\alpha_1:(u_0:u_1)\mapsto(u_0:su_1), \qquad s\neq 0.$$

As in cases 1 and 3 either α_2 or $\alpha_2\tau_2$ is of the form

$$(v_0:v_1)\mapsto(t_0v_0+t_1v_1:t_1v_0+t_0v_1).$$

The invariance of the point pair $cv_0^2=dv_1^2$ implies $t_1=0$, so $\alpha_2=\mathrm{id}$ or $\alpha_2=\tau_2$. Considering the equation for C we find $s^4=1$, hence α_1 is a power of ρ. $\square$

4.2. **The K3-Surface X.** Let $\bar{X}\to Q$ be the double covering branched over B and $q: X\to Q$ its minimal desingularisation. On X we have the following curves:

In all three cases:

$F_1^{\pm},\dots,F_7^{\pm}$	(-2)-curves resolving the A_1-singularities over $P^{\pm}$
$F_8^{\pm}$	two (-2)-curves over the line $u_1=0$
$N^{\pm}$	two (-2)-curves in the branch locus
F	smooth elliptic curve over $u_0=0$

In case 1 and 2 additionally:

$L_1^{\pm}, L_2^{\pm}$	(-2)-curves over $L^{\pm}$ ($L_1^{\pm}$ and $L_2^{\pm}$ touch in case 2)
E, E'	smooth elliptic curves over $v_0=0$, $v_1=0$

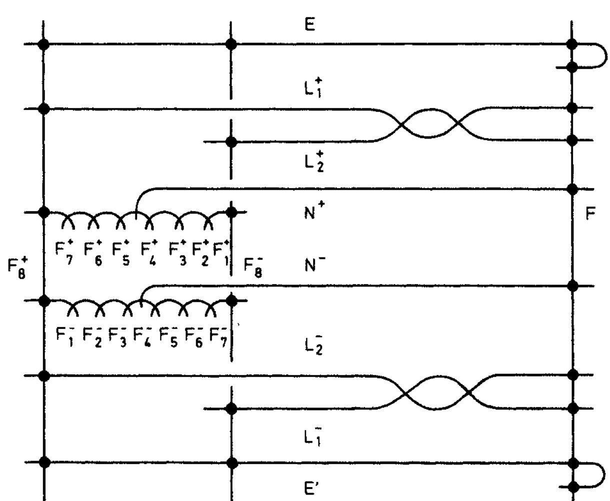
In case 3 additionally:

$E_1^{\pm}$	two (-2)-curves over the line $L^+=L^-$
$E_2^{\pm}$	(-2)-curves resolving the A_1-singularities of $\bar{X}$
E'	smooth elliptic curve over $v_0=0$, resp. $v_1=0$.

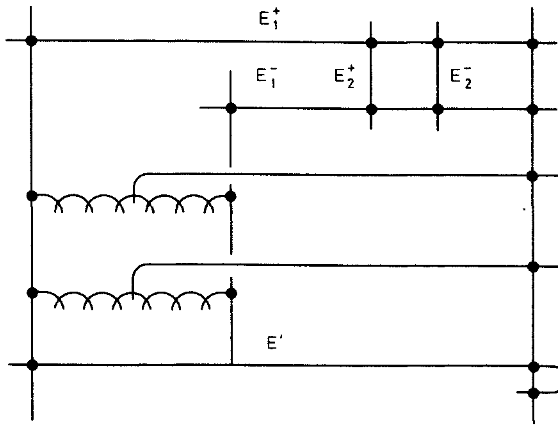
Let $\sigma_3\in\mathrm{Aut}(X)$ be the covering involution interchanging the two sheets of $q: X\to Q$. The automorphisms of Q from Lemma (4.3) lift to X in the following way:

σ_1 is an involution lifting τ_1 and having F, $F_2^{\pm}$, $F_4^{\pm}$, $F_6^{\pm}$, and $F_8^{\pm}$ as curves of fixed points.

σ_2 is an involution lifting τ_2 and having E (resp. $E_1^{\pm}$) and E' as curves of fixed points.



Case 1 (in case 2, $L_1^{\pm}$ and $L_2^{\pm}$ touch on F)



Case 3

σ_1 and σ_2 commute with σ_3, so the involutions $\sigma_1, \sigma_2, \sigma_3$ generate a subgroup $(\mathbb{Z}_2)^3 \subset \mathrm{Aut}(X)$. This group contains in particular $\sigma = \sigma_1 \sigma_2 \sigma_3$, the involution without fixed points.

In case 2 this group is enlarged by lifting ρ to $\tilde{\rho} \in \mathrm{Aut}(X)$ with $F_8^{\pm}$ being curves of fixed points for $\tilde{\rho}$. Then necessarily $F_2^{\pm}$, $F_4^{\pm}$ and $F_6^{\pm}$ are curves of fixed points, and $\tilde{\rho}|F$ is an involution with four isolated fixed points. Further $(\tilde{\rho})^2 = \sigma_1, \tilde{\rho}$ commutes with σ_2 and σ_3, and we have a subgroup $\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$ in $\mathrm{Aut}(X)$.

Notice that the involutions σ_1 and $\sigma_1 \sigma_3$ interchange E_2^+ and E_2^-. The involution $\sigma_2 \sigma_3$ interchanges all other pairs of curves differing by a $\pm$-sign.

4.3. **The Elliptic Pencil $|F|$ on X.** The elliptic curve F in X is linearly equivalent with $\sum_1^8 (F_i^+ + F_i^-)$. We denote by $\phi: X \to \mathbb{P}_1$ the elliptic fibration defined by the pencil $|F|$. We know already the following sections for this pencil:

$$N^+, N^-, L_1^+, L_1^-, L_2^+, L_2^- \qquad \text{(case 1 and 2)}$$
$$N^+, N^-, E_1^+, E_1^- \qquad \text{(case 3).}$$

We denote by $\mathfrak{S}$ the set of all sections and introduce on it the structure of an abelian group by distinguishing N^- as origin. For any of the curves $N^-, N^+, L_i^{\pm}, E_1^{\pm}$ we denote the corresponding group element by $\mathfrak{o}, \mathfrak{n}, \mathfrak{l}_i^{\pm}, \mathfrak{e}^{\pm}$.

With N^- as origin the 2-torsion elements on every elliptic curve in $|F|$ are the intersection points with the ramification divisor of q. Since C does not split, the only non-trivial 2-torsion element in $\mathfrak{S}$ is $\mathfrak{n}$. The involution $\sigma_2 \sigma_3$ acts on $\mathfrak{S}$ as addition by $\mathfrak{n}$.

(4.4) **Proposition.** *The torsion subgroup $\mathfrak{S}_{\mathrm{tors}} \subset \mathfrak{S}$ is*

$$\mathbb{Z}_2 \text{ generated by } \mathfrak{n} \qquad \text{(cases 1 and 2)}$$
$$\mathbb{Z}_4 \text{ generated by } \mathfrak{e}^{\pm} \qquad \text{(case 3).}$$

Proof. If $\mathfrak{s}\in\mathfrak{S}$ is any torsion element represented by the section $S\subset X$, then $qS\subset Q$ is a smooth rational curve of bidegree $(1,n)$. If qS intersects B in a smooth point, then necessarily $2\mathfrak{s}=0$, i.e., $S=N^{\pm}$. The only way to avoid meeting B in a smooth point (on N^+ or N^-) is $n=0$ and $qS=q(E_1^{\pm})$ in the case 3. So $S=E_1^+$ or E_1^-. Since $\mathfrak{S}$ contains only one 2-torsion element $\mathfrak{n}\neq 0$, necessarily $\mathfrak{S}_{\text{tors}}=\mathbb{Z}_4$ generated by $\mathfrak{e}^+$ or $\mathfrak{e}^-$ in this case. □

(4.5) **Corollary.** *In case* 1 *and* 2 *we have* rank $\mathfrak{S}\geqq 1$.

In fact $\mathfrak{S}$ contains the non-torsion elements $\mathfrak{l}_i^{\pm}$. Notice that $\mathfrak{n}+\mathfrak{l}_i^+=\mathfrak{l}_i^-$ and $-\mathfrak{l}_1^{\pm}=\mathfrak{l}_2^{\pm}$.

The automorphisms σ_1 (resp. $\tilde{\rho}$ in case 2) and $\sigma_3\in\mathrm{Aut}(X)$ respect $|F|$ and leave N^- fixed.

(4.6) **Proposition.** *The subgroup of* $\mathrm{Aut}(X)$ *respecting* $|F|$ *and leaving* N^- *fixed is*

$$\mathbb{Z}_2\times\mathbb{Z}_2 \text{ generated by } \sigma_1 \text{ and } \sigma_3 \qquad (\text{case 1 and 3}),$$

$$\mathbb{Z}_4\times\mathbb{Z}_2 \text{ generated by } \tilde{\rho} \text{ and } \sigma_3 \qquad (\text{case 2}).$$

Proof. Any $\alpha\in\mathrm{Aut}(X)$ respecting the pencil $|F|$ fixes the cycle $\sum_1^8(F_i^++F_i^-)$. If $\alpha N^-=N^-$, then (after replacing α by $\alpha\sigma_3$) we may assume $\alpha F_i^+=F_i^+$, $\alpha F_i^-=F_i^-$ for $i=1,\dots,8$. So α leaves invariant the $\tilde{E}_7$-fundamental cycle

$$Z=F_1^-+2F_2^-+3F_3^-+4F_4^-+3F_5^-+2F_6^-+F_7^-+2N^-$$

linearly equivalent with E (resp. $E_1^++E_1^-+E_2^++E_2^-$) and E'. This means that the map $X\to\mathbb{P}_1$ defined by the elliptic pencil $|E'|$ is α-equivariant too, hence α is induced by some symmetry of (B,Q). The assertion follows from Lemma (4.3). □

This group $\mathbb{Z}_2\times\mathbb{Z}_2$ (resp. $\mathbb{Z}_4\times\mathbb{Z}_2$) acts naturally on $\mathfrak{S}$. On the sections given above this action can be traced easily:

σ_3 being the covering involution induces $-\mathrm{id}$ on all elliptic curves in the pencil $|F|$, hence acts on $\mathfrak{S}$ as $-\mathrm{id}$.

σ_1 (resp. $\tilde{\rho}$) leaves invariant each of the sections $\mathfrak{n}$, $\mathfrak{l}_i^{\pm}$, $\mathfrak{e}^{\pm}$ so it acts trivially on $\mathfrak{S}_{\text{tors}}$ and on the rank-1 subgroup generated by the $\mathfrak{l}_i^{\pm}$.

Now let

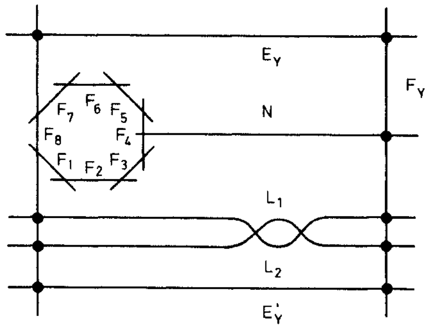
$$\mathfrak{R}=(\mathbb{Z}_2\times\mathbb{Z}_2)\ltimes\mathfrak{S}, \qquad \text{resp. } (\mathbb{Z}_4\times\mathbb{Z}_2)\ltimes\mathfrak{S}$$

be the semidirect product w.r. to this action.

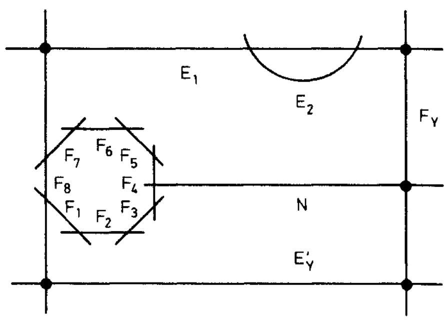
(4.7) **Proposition.** $\mathfrak{R}\subset\mathrm{Aut}(X)\subset O(L)$ *is just the stabilizer subgroup of the class* $f\in L$ *of* F.

Proof. Assume $\alpha f=f$ for some $\alpha\in\mathrm{Aut}(X)$. Then $\phi: X\to\mathbb{P}_1$ is α-equivariant. After replacing α by $\alpha\circ\{\text{translation by } -\alpha(\mathfrak{o})\in\mathfrak{S}\}$ we may assume $\alpha N^-=N^-$. The assertion follows from Proposition (4.6) above. □

4.4. **The Enriques Surface** $Y=X/\sigma$. Since $\sigma=\sigma_1\sigma_2\sigma_3\in\mathrm{Aut}(X)$ has no fixed points on X, the surface $Y=X/\sigma$ is an Enriques surface. As usual denote the projection by $\pi: X\to Y$. Under σ all the curves $E_i^{\pm}$, $F_i^{\pm}$, $L_i^{\pm}$, $N_i^{\pm}$ differing by a $\pm$sign are identified. So on Y we have the following curves:



Case 1 (in case 2, L_1 and L_2 touch on F_Y)



Case 3

First we determine divisors representing a $\mathbb{Z}$-basis of $H^2(Y, \mathbb{Z})_f = M$. Consider the cycle

$$Z = 2F_2 + 4F_3 + 6F_4 + 5F_5 + 4F_6 + 3F_7 + 2F_8 + 3N.$$

It is the fundamental cycle of an E_8-configuration, hence $Z^2 = -2$. We complete Z to $\tilde{E}_8$-configurations

$$Z_1 = L_1 + Z, \quad Z_2 = L_2 + Z \qquad \text{(case 1 and 2)},$$
$$Z_1 = E_1 + Z \qquad \text{(case 3)}.$$

Then there are classes $h_1, h_2 \in M$ with $h_1^2 = h_2^2 = 0$ and

$$z_1 = 2h_1, \quad z_2 = 2h_2 \qquad \text{(case 1 and 2)},$$
$$z_1 = 2h_1, \quad h_1 \cdot e_2 = 1 \qquad \text{(case 3)}.$$

(Here as usual we denote the class in M represented by a cycle with the corresponding small letter.) Putting $h_2 = h_1 + e_2$ in case 3, we have in all cases

$$h_1^2 = h_2^2 = 0, \quad h_1 \cdot h_2 = 1,$$
$$h_i \cdot f_j = h_i \cdot n = 0, \qquad i = 1, 2, \ j = 2, \dots, 8.$$

This proves: $h_1, h_2, f_2, \ldots, f_8, n$ form a $\mathbb{Z}$-basis of M.

Recall that $\mathrm{Aut}(Y) = \mathrm{Aut}(X, \sigma)/\sigma$. Putting

$$\mathfrak{R}(\sigma) = \mathfrak{R} \cap \mathrm{Aut}(X, \sigma) = \{\alpha \in \mathfrak{R}: \alpha\sigma = \sigma\alpha\}$$

we have $\mathfrak{R}(\sigma)/\sigma$ as subgroup of $\mathrm{Aut}(Y)$. To describe this group more explicitely recall that $\sigma = \sigma_1 \cdot (\sigma_2 \sigma_3)$ and that $\sigma_2 \sigma_3$ acts on $\mathfrak{S}$ as translation by the unique 2-torsion element $\mathfrak{n}$. So $\sigma_2 \sigma_3$ commutes with all elements of $\mathfrak{R}$ and $\mathfrak{R}(\sigma)$ is the centralizer of σ_1. We observed that σ_1 centralizes $\mathfrak{S}_{\mathrm{tors}}$. If σ_1 commutes with translation by a non-torsion element $\mathfrak{s} \in \mathfrak{S}$, then necessarily $\sigma_1(\mathfrak{s}) = \mathfrak{s}$. So putting

$$\mathfrak{S}(\sigma) = \{\mathfrak{s} \in \mathfrak{S}: \sigma_1(\mathfrak{s}) = \mathfrak{s}\}$$

we have $\mathfrak{R}(\sigma) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathfrak{S}(\sigma)$, resp. in case 2 $\mathfrak{R}(\sigma) = (\mathbb{Z}_4 \times \mathbb{Z}_2) \ltimes \mathfrak{S}(\sigma)$. We put

$$s = \mathrm{rank}\, \mathfrak{S}(\sigma),$$

then $\mathfrak{S}(\sigma) = \mathfrak{S}_{\mathrm{tors}} \times \mathbb{Z}^s$. Notice that we do not yet know s, but in the cases 1 and 2 we have $\mathfrak{l}_i^{\pm} \in \mathfrak{S}(\sigma)$, hence $s \geqq 1$. So we obtain the following description of $\mathfrak{R}(\sigma)/\sigma \subset \mathrm{Aut}(Y)$.

Case 1: $\mathbb{Z}_2(\sigma_1) \times \mathbb{Z}_2(\sigma_2 \sigma_3) \times (\mathbb{Z}_2(\sigma_3) \ltimes \mathbb{Z}^s)/\sigma_1 \sigma_2 \sigma_3 = \mathbb{Z}_2 \times (\mathbb{Z}_2 \ltimes \mathbb{Z}^s)$.

Case 2: $\mathbb{Z}_2(\sigma_2 \sigma_3) \times ((\mathbb{Z}_4(\tilde{\rho}) \times \mathbb{Z}_2(\sigma_3)) \ltimes \mathbb{Z}^s)/\sigma_1 \sigma_2 \sigma_3 = (\mathbb{Z}_4 \times \mathbb{Z}_2) \ltimes \mathbb{Z}^s$.

Case 3: $\mathbb{Z}_2(\sigma_1) \times (\mathbb{Z}_2(\sigma_3) \ltimes (\mathbb{Z}_4(\mathfrak{e}^{\pm}) \times \mathbb{Z}^s))/\sigma_1 \sigma_2 \sigma_3 = \mathbb{Z}_2 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}^s)$.

The aim of this section is to prove that there are no other automorphisms of Y. For $\alpha \in \mathrm{Aut}(X, \sigma)$ let us denote by α mod σ the induced automorphism of Y. The key observation is the following one.

(4.8) **Proposition.** a) *The involution* σ_1 mod σ *acts trivially on* $H^2(Y, \mathbb{Z})$ *and generates the kernel of the representation of* $\mathrm{Aut}(Y)$ *on* $H^2(Y, \mathbb{Z})$.

b) *In case* 2 *the automorphism* $\tilde{\rho}$ *mod* σ *acts trivially on* $H^2(Y, \mathbb{Z})_f$ *and generates the kernel of the representation of* $\mathrm{Aut}(Y)$ *on* $H^2(Y, \mathbb{Z})_f$.

Proof. σ_1 (as well as $\tilde{\rho}$ in case 2) leaves invariant all the curves on X specified above, except for interchanging E_2^+ and E_2^-. This proves that σ_1 mod σ (as well as $\tilde{\rho}$ mod σ) acts trivially on the basis of $H^2(Y, \mathbb{Z})_f$ considered above.

To prove that σ_1 mod σ acts trivially on $H^2(Y, \mathbb{Z})$ already (and $\tilde{\rho}$ mod σ does not do it) we observe that not only the classes $h_1, h_2, f_2, \ldots, f_8, n$, but also the curves $Z_1, Z_2, F_2, \ldots, F_8, N$ are left invariant under σ_1 mod σ, resp. $\tilde{\rho}$ mod σ. So it suffices to consider the action on the two half-pencils in the linear systems $|Z_1|$ and $|Z_2|$.

Let us denote by p'_1, $p'_2 \in E'_Y$ the points where F_8, F_Y meet the smooth elliptic curve E'_Y. Then $|2p'_1| = |2p'_2|$ is the linear system cut out on E'_Y by both $|Z_1|$ and $|Z_2|$. The two half-pencils in $|Z_1|$ and $|Z_2|$ intersect E'_Y in the two other points p'_3, p'_4 with $\mathcal{O}_{E'_Y}(p'_i) = \mathcal{O}_{E'_Y}(p'_j)$, $i = 1, 2$, $j = 3, 4$. Now σ_1 mod σ fixes the points p'_3, p'_4 and the corresponding half-pencils, whereas $\tilde{\rho}$ interchanges p'_3 and p'_4.

Conversely, consider an arbitrary $\alpha \in \mathrm{Aut}(X, \sigma)$ with α mod σ acting trivially on $H^2(Y, \mathbb{Z})_f$. After replacing α by $\alpha\sigma$ we may assume $\alpha N^- = N^-$. By Proposition (4.6) this α is one of the following

$$\mathrm{id}_X, \tilde{\rho}, \sigma_1, \tilde{\rho}^3, \sigma_3, \tilde{\rho}\sigma_3, \sigma_1\sigma_3, \tilde{\rho}^3\sigma_3.$$

But the last four automorphisms in this list reverse the orientation in the cycle $\sum_1^8 (F_i^+ + F_i^-)$. This proves the assertion. □

(4.9 a) **Corollary.** σ_1 mod σ *belongs to the center of* $\mathrm{Aut}(Y)$ *and its fixed point set is stable under each automorphism of* Y.

Now σ_1 mod σ has the following set of fixed points

Case 1 and 2: $F_Y, F_2, F_4, F_6, F_8, \{p_3, p_4, p_3', p_4'\}$ $p_i \in E_Y$, $p_i' \in E_Y'$,

Case 3: $F_Y, F_2, F_4, F_6, F_8, E_2, \{p_3', p_4'\}$.

It follows that any automorphism of Y is of the form α mod σ with $\alpha \in \mathfrak{N}(\sigma)$.

(4.9 b) **Corollary.** $\mathrm{Aut}(Y) = \mathfrak{N}(\sigma)/\sigma$.

It remains to determine the rank s of the abelian subgroup $\mathbb{Z}^s \subset \mathfrak{S}(\sigma)$.

(4.10) **Lemma.** a) *The subgroup of* $O(M)$ *leaving invariant* $f_1, \ldots, f_8$ is $D_\infty = \mathbb{Z}_2 \ltimes \mathbb{Z}$, *the infinite dihedral group.*

b) *The subgroup leaving in addition* e_2 *invariant is trivial.*

Proof. Any $\alpha \in O(M)$ leaving invariant $f_2, \ldots, f_8$ is determined by its action on $\{f_2, \ldots, f_8\}^\perp$. Now h_1, h_2 and $f = \sum_1^8 f_i$ belong to this orthogonal complement. Since

$$h_1^2 = h_2^2 = f^2 = 0, \quad h_1 \cdot h_2 = 1, \quad h_1 \cdot f = h_2 \cdot f = 2,$$

their 3×3 intersection matrix has determinant -8, hence equals $\det(f_i \cdot f_j)_{2 \leq i, j \leq 8}$. This shows that h_1, h_2, f are a $\mathbb{Z}$-basis of $\{f_2, \ldots, f_8\}^\perp$.

Now α acts as

$$\begin{aligned} h_1 &\mapsto r_1 h_1 + r_2 h_2 + r f, \\ h_2 &\mapsto s_1 h_1 + s_2 h_2 + s f, \\ f &\mapsto f, \end{aligned}$$

with $r, s, r_i, s_i \in \mathbb{Z}$ and determinant

$$r_1 s_2 - r_2 s_1 = \pm 1.$$

Denote by $\alpha_0 \in O(M)$ the permutation $h_1 \leftrightarrow h_2$. It generates a subgroup $\mathbb{Z}_2 \subset O(M)$ and modulo this subgroup we may assume the determinant above to be $+1$. Now orthogonality of α implies

$$\begin{aligned} 2 = h_1 \cdot f = 2(r_1 + r_2) &\Rightarrow r_2 = 1 - r_1, \\ 2 = h_2 \cdot f = 2(s_1 + s_2) &\Rightarrow s_2 = 1 - s_1 \end{aligned}$$

and from the determinant condition we conclude $r_1 = s_1 + 1$. Putting $s_1 = t \in \mathbb{Z}$ we find

$$h_1 \mapsto (t+1)h_1 - th_2 + rf,$$
$$h_2 \mapsto th_1 + (1-t)h_2 + sf$$

and orthogonality of α is equivalent with

$$0 = h_1^2 = -2t(t+1) + 4r \Rightarrow r = t(t+1)/2,$$
$$0 = h_2^2 = 2t(1-t) + 4s \Rightarrow s = t(t-1)/2.$$

So $\alpha = \alpha_t$ is an element of the group

$$\mathbb{Z} \ni t \mapsto \alpha_t = \begin{pmatrix} 1+t & -t & t(t+1)/2 \\ t & 1-t & t(t-1)/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

On this group α_0 acts by $\alpha_t \to \alpha_{-t}$. So α_0 and $\mathbb{Z}$ generate D_∞.

b) We have $2E_2 = Z_2 - Z_1$ and hence $e_2 = h_2 - h_1$. So $\alpha_t(e_2) = e_2 - 2t(e_1 + e_2)$. So $\alpha_t(e_2) = e_2$ implies $t = 0$. Since $\alpha_0(e_2) = -e_2$, the assertion follows. □

(4.11) **Corollary.** *In case* 1 *or* 2 *we have* $s = 1$, *and* $s = 0$ *in case* 3.

Proof. If $\mathfrak{s} \in \mathbb{Z}^s \subset \mathfrak{S}(\sigma)$ then addition by $8\mathfrak{s}$ leaves invariant $f_1, \ldots, f_8$ and in case 3 also e_2. So $s \leqq 1$, resp. $s = 0$ in case 3, by the lemma above. But $s \geqq 1$ in case 1 or 2 was observed already. □

The final result is the following.

(4.12) **Theorem.** *Let* Y *be an Enriques surface as considered above. Then* $\operatorname{Aut}(Y)$ *is isomorphic with*

$$\mathbb{Z}_2 \times D_\infty \quad \text{(case 1)},$$
$$\mathbb{Z}_4 \times D_\infty \quad \text{(case 2)},$$
$$D_4 \quad \text{(case 3)}.$$

4.5. **Invariants.** In this section we compute the Picard number $\rho(X)$ and the nodal type N for the general surfaces in our family. As above we use the decomposition $L = \mathbb{H} \perp \mathbb{H} \perp \mathbb{H} \perp \mathbb{E} \perp \mathbb{E}$. In 4.4 we observed that the nodal classes $f_2, \ldots, f_8, n \in M = \mathbb{H} \perp \mathbb{E}$ form a $\mathbb{Z}$-basis for $\mathbb{E}$. Their inverse images on X decompose into 16 nodal classes forming two E_8-configurations. In fact these two configurations are bases for the two factors $\mathbb{E} \subset L$, see [7, Sect. 5]. Since the algebraic lattice S_X contains $L^+ \subset L$, this proves the following.

(4.13) **Lemma.** *For all surfaces* X *considered here the algebraic lattice* $S_X \subset L$ *contains the* rank-18 *sublattice*

$$\mathbb{H}(2) \perp \mathbb{E} \perp \mathbb{E} = \{(0, h, h, x, y) : h \in \mathbb{H},\ x, y \in \mathbb{E}\}.$$

The orthogonal complement in L of this rank-18 lattice is

$$N_1 = \mathbb{H} \perp \mathbb{H}(2) = \{(h, h', -h', 0, 0) : h, h' \in \mathbb{H}\}.$$

In case 3 also the class $e_2 = h_2 - h_1$ is nodal. Then $\pi^* e_2 = d + s(d)$ with $d \in \mathbb{H} \perp \mathbb{H} \perp \mathbb{H} \subset L$ a nodal class such that $\langle d, s(d) \rangle = 0$ and

$$\begin{aligned} d + s(d) &= (0, h_2 - h_1, h_2 - h_1), \\ d - s(d) &= (h, h', -h') \in N_1, \\ 2d &= (h, h' + h_2 - h_1, -h' + h_2 - h_1). \end{aligned}$$

This shows $h \in 2\mathbb{H}$. By Corollary (1.2) and Lemma (1.8) it follows that there is an automorphism of N_1 extending to $\mathbb{H} \perp \mathbb{H} \perp \mathbb{H}$ as identity on $N_1^{\perp}$ mapping $d - s(d)$ to $(0, h_2 - h_1, h_1 - h_2)$. If we change the fixed marking of Y by this automorphism we have

$$d + s(d) = (0, h_2 - h_1, h_2 - h_1) \quad d - s(d) = (0, h_2 - h_1, h_1 - h_2)$$

and therefore

$$d = (0, h_2 - h_1, 0), \qquad s(d) = (0, 0, h_2 - h_1).$$

Let $N_3 \subset N_1$ be the rank-3 sublattice orthogonal to d. Then we have shown the following.

(4.14) **Lemma.** *In case* 3 *for all surfaces X the algebraic lattice S_X contains the rank*-19 *lattice* $N_3^{\perp}$.

In other words: If $N \subset L$ is the nodal type of X, then always $N \subset N_1$ and in case 3 even $N \subset N_3$.

(4.5) **Proposition.** *For general X in our family we have $T_X = N = N_1$. For general X under case* 3 *we have $T_X = N = N_3$. In particular $\rho(X) = 18$ for general X, and $\rho(X) = 19$ for general X in case* 3.

Proof. The cup product on N_1 is non-degenerate. So there is no 3-dimensional isotropic linear subspace of $N_1 \otimes \mathbb{C}$. If $T_X \neq N_1$ for all X, then the period point $\tau(Y, \varphi)$ would vary in a countable union of 1-dimensional quadrics. This is impossible, because we show that in D^0/Γ our family has an image containing a (local analytic) variety of dimension two. Similarly, in case 3, we show that the image contains a curve.

Case 1. Consider the map $\Phi_Y \colon Y \to \mathbb{P}_1$ given by the pencil $|2F_Y|$. We have $\Phi_Y \circ \pi = \Phi$, and $\Phi | E$ is the quotient map w.r.t. $\mathbb{Z}_2(\sigma_1 | E) \times \mathbb{Z}_2(\sigma_3 | E)$. So $\Phi | E$ factors as $E \xrightarrow{q} \mathbb{P}_1(u_0 : u_1) \to \mathbb{P}_1(w_0 : w_1)$, $w_i = u_i^2$, and similarly for $\Phi | E'$. The four isolated fixed points $p_3, p_4 \in E_Y$, $p_3', p_4' \in E_Y'$ and their images under Φ_Y (modulo the natural $\mathbb{C}^*$-action on $\mathbb{P}_1$) depend only on the isomorphism class of Y. These four image points are the four roots of the polynomial

$$\begin{aligned} &(a w_0^2 + 2b w_0 w_1 + c w_1^2)(a w_0^2 + 2b w_0 w_1 + d w_1^2) \\ &\quad = a^2 \left(w_0^4 + 4 \frac{b}{a} w_0^3 w_1 + \left(4 \frac{b^2}{a^2} + \frac{c+d}{a} \right) w_0^2 w_1^2 + 2 \frac{b}{a} \left(\frac{c+d}{a} \right) s_0 w_1^3 + \frac{cd}{a^2} w_1^4 \right). \end{aligned}$$

Since $\frac{b}{a}$, $\frac{c}{a}$, and $\frac{d}{a}$ vary independently, we see that these 4-tuples in $\mathbb{P}_1^4$ form a 3-dimensional set and dividing by the $\mathbb{C}^*$-action the dimension is two.

Case 3. It suffices to show that the elliptic curve F varies. It is ramified over the four points ± 1, $\pm\sqrt{\frac{c}{d}}$, or after multiplying with $\sqrt{d}$ over $\pm\sqrt{c}$, $\pm\sqrt{d}$. The cross-ratio

$$\frac{\sqrt{d}-\sqrt{c}}{\sqrt{d}+\sqrt{c}} : \frac{-\sqrt{d}-\sqrt{c}}{-\sqrt{d}+\sqrt{c}} = \frac{(\sqrt{d}-\sqrt{c})^2}{(\sqrt{d}+\sqrt{c})^2}$$

varies with c/d. □

The lattices $L_1, \ldots, L_4$ for general X then are as follows

$$\begin{aligned} &L_1 = L^+, \\ &L_2 = N_1^\perp \cap LL^-, \quad L_3 = 0, \quad L_4 = N_1 \qquad \text{in case 1}, \\ &L_2 = N_3^\perp \cap L^-, \quad L_3 = 0, \quad L_4 = N_3 \qquad \text{in case 3}. \end{aligned}$$

4.6 **The action of $\mathrm{Aut}(X,\sigma)$ on N.** Let us finish by giving a few properties of the representation of $\mathrm{Aut}(X,\sigma)$ on N_1 (cases 1 and 2) and N_3 (case 3). In the proof of (4.15) we showed that the period point $\tau(Y,\varphi)$ moves in an open set of $D^0 \cap \mathbb{P}(N_1)$ (resp. $D^0 \cap \mathbb{P}(N_3)$) if Y moves in our family (resp. in the part of the family under case 3). So the argument of (2.9) applies and shows that the image of

$$\mathbb{Z}_2(\sigma_1) \times \mathbb{Z}_2(\sigma_2\sigma_3) \times (\mathbb{Z}_2(\sigma_3) \ltimes \mathbb{Z}) \to \mathrm{Aut}(N_1)$$
$$\mathbb{Z}_2(\sigma_1) \times (\mathbb{Z}_2(\sigma_3) \ltimes \mathbb{Z}_4(e^\pm)) \to \mathrm{Aut}(N_3)$$

in general is the group $\{\pm \mathrm{id}\}$. But this assertion is invariant under deformations, so it holds for *all our surfaces Y.*

Now for $\tilde{\rho}$ the situation is quite different.

(4.16) **Proposition.** *We have $(\tilde{\rho}|N_1)^2 = -\mathrm{id}$, in particular the order of $\tilde{\rho}$ on N is four.*

Proof. It suffices to show that $\sigma_1 = \tilde{\rho}^2$ acts on N_1 as $-\mathrm{id}$. To do this, we use the Lefschetz fixed point formula. It reads

$$\mathrm{Trace}(\sigma_1 \text{ on} \bigoplus H^{2q}(X, \mathbb{R})) = e(\mathrm{Fix}(\sigma_1)),$$

where on the right-hand side we add over the Euler numbers of all components of the fixed point set of σ_1. Now σ_1 fixes point-wise the eight rational curves $F_2^\pm$, $F_4^\pm$, $F_6^\pm$, and $F_8^\pm$ as well as the smooth elliptic curve F. So $e(\mathrm{Fix}(\sigma_1)) = 16$ and $\mathrm{Trace}\,(\sigma_1|L_{\mathbb{R}}) = 14$. Since σ_1 leaves invariant all the curves $F_i^\pm$, $L_i^\pm$, $N^\pm$, from which a basis for the lattice $N_1^\perp$ can be chosen (cf. (4.13)), we have $(\sigma_1|N_1^\perp) = \mathrm{id}$ and $\mathrm{Trace}\,(\sigma_1|(N_1^\perp) \otimes \mathbb{R}) = 18$.

This implies

$$\mathrm{Trace}(\sigma_1|N_1 \otimes \mathbb{R}) = 14 - 18 = -4,$$

hence σ_1 acts on N_1 as $-\mathrm{id}$. □

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