Lawson homology for varieties with small Chow groups and the induced filtration on the Griffiths groups

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0. Introduction

Let X be a complex projective variety and let $C_m(X)$ be the disjoint union of the Chow varieties of effective m-cycles of degree $d = 0, 1, 2, \ldots$. The empty cycle $0 \in C_m(X)$ is a natural base point and it also acts as a zero for the addition of cycles, making $C_m(X)$ into a monoid. Let me put

$$\begin{aligned} &\mathcal{Z}_m(X) = C_m(X) \times C_m(X)/\sim, \\ &\text{with } (x, y) \sim (x', y') \Leftrightarrow x + y' = x' + y \quad \text{(the naïve group completion).} \end{aligned}$$

The complex topology induces a natural topology on the monoid $C_m(X)$ and one equips $\mathcal{Z}_m(X)$ with the quotient topology. By [**F1**] this topological space is independent of the chosen embedding of X into a projective space. The induced topology will be called the Chow topology.

The Lawson homology groups are the homotopy groups of this space:

$$L_m H_\ell(X) = \begin{cases} \pi_{\ell-2m} \mathcal{Z}_m(X) & \text{if } \ell \geq 2m \\ 0 & \text{if } \ell < 2m. \end{cases}$$

These groups form an intriguing set of invariants. Despite the vast literature on this subject, the groups have been calculated only in very few cases, such as for the projective spaces. One of the goals of this note is to understand these invariants in case the variety has "few cycles" in the sense that rational and homological equivalence coincide for all m-cycles up to a certain rank $m = s$. Actually, the results only deal with that part of Lawson homology that is not "visible" in ordinary homology in that it is in the kernel of the cycle class maps

$$c_{m,k+2m} : L_m H_{k+2m}(X) \to H_{k+2m}(X),$$

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i.e. the subgroup

$$L_m^{\hom} H_\ell(X) := \ker\{\ c_{m,\ell} : L_m H_\ell(X) \to H_\ell(X)\ \}.$$

These cycle class maps are reviewed in section 3.

The main result can now be formulated as follows (see Theorem 16).

Theorem *Let X be a smooth complex projective variety for which rational and homological equivalence coincide for m-cycles in the range $0 \le m \le s$. Then $L_m^{\hom} H_*(X) \otimes \mathbb{Q} = 0$ in the range $0 \le m \le s+1$.*

While this result holds for $L_m^{\hom} H_\ell(X)$ for m in a restricted range, the second index is arbitrary. If however one considers only even ℓ, there is an interesting translation of the result in terms of a certain filtration on the m-th Griffiths group (i.e. the group of m-cycles homologically equivalent to zero modulo the group of m-cycles algebraically equivalent to zero).

This filtration, the S-filtration from [**F2**] is related to Nori's A-filtration introduced in [**N**]. Very roughly, a Griffiths class is in S_r if it is in the subgroup generated by the images of the action of correspondences from Y to X, equi-dimensional over Y, the action being restricted to r-cycles homologically equivalent to zero. One allows singular projective varieties Y, while Nori only admits smooth projective varieties Y. In the last section I'll prove this translation:

Proposition *Let X be a smooth complex projective variety for which rational and homological equivalence coincide for m-cycles in the range $0 \le m \le s$. Then the last $s+1$ steps induced by the S-filtration on any of the $\mathbb{Q}$-Griffiths groups are constant. In particular, algebraic and homological equivalence coincide for $s+1$-cycles.*

Since the examples constructed by Nori in [**N**] show that in general these steps are non-constant (see also [**F3**]), this is a non-trivial observation. It can be applied to examples of rationally connected varieties and to the examples from [**E-L-V**] as I'll show in the last section.

I'll employ the following notation throughout. Here X is any projective variety.

- $C_m(X)$: the monoid of effective m-cycles, considered as a pointed topological space with the complex topology.
- $\mathcal{Z}_m(X)$: the group of algebraic m cycles on X, considered as a pointed topological space with the Chow topology,
- $L_m H_\ell(X) := \pi_{\ell-2m}\mathcal{Z}_m(X)$, $\ell = m, m+1, \dots$
- $\mathrm{Ch}_m(X)$: the Chow-group of m-cycles modulo rational equivalence,
- $\mathrm{Ch}(X) = \oplus_m \mathrm{Ch}_m(X)$,

- $\mathrm{Griff}_m(X) = \mathrm{Ch}_m^{\mathrm{hom}}(X)/\,\mathrm{Ch}_m^{\mathrm{alg}}(X)$, the m-th Griffiths group. of m-cycles homologically equivalent to zero modulo the group $\mathrm{Ch}_m^{\mathrm{alg}} X$ of m-cycles algebraically equivalent to zero.
- $\mathrm{Griff}_m(X)_{\mathbb{Q}} = \mathrm{Ch}_m^{\mathrm{hom}}(X) \otimes \mathbb{Q}/\,\mathrm{Ch}_m^{\mathrm{alg}}(X) \otimes \mathbb{Q}$, the m-th $\mathbb{Q}$-Griffiths group.

I need some well known facts on correspondences which I collect in the next section. I should warn the reader that in this section only certain correspondences are allowed, the so-called Chow-correspondences which come from equi-dimensional degree 0 correspondences. This action is used to give a precise definition of the S-filtration. The comparison with Nori's filtration necessitates a comparison with the usual action on Chow groups. After a careful review of intersection products on the level of cycle spaces I show that any correspondence induces an action on cycle spaces with good functorial properties (but this action might differ from the action of Chow-correspondences).

Since most algebraic geometers are not familiar with Lawson homology, in §2 I give a synopsis of the main results (in the complex analytic setting). In particular I treat intersection theory and derive some fundamental facts that are probably known among the experts, but for which I did not find a reference.

Using these properties, in §3 the main results are then derived from a certain decomposition of the diagonal (on the level of $\mathrm{Ch}(X) \otimes \mathbb{Q}$) when X has small Chow groups up to a certain rank. This decomposition was originally considered by Bloch and used for instance in [**B-S**].

I am grateful to Robert Laterveer for bringing up this decomposition, and for suggesting that it could be applied to Lawson homology. I am also indebted to E. Friedlander for a critical reading of a first draft and for the suggestions he made to improve the exposition.

1. Correspondences

Let X and Y be complex projective varieties. A correspondence Z from X to Y is a cycle on $X \times Y$. Its class in $\mathrm{Ch}(X \times Y)$ is denoted $[Z]$. The correspondence is called effective, resp. irreducible if the corresponding cycle is effective, resp. irreducible. An effective correspondence of dimension $d + \dim X$ is called a degree d correspondence.

Recall [**Fu, Chapter 16**] that a correspondence Z between smooth projective varieties X and Y acts on Chow groups as follows

$$\begin{aligned}[Z]_* : \mathrm{Ch}(X) &\to \mathrm{Ch}(Y)\\ u &\mapsto (p_2)_* \left(p_1^* u \bullet [Z]\right)\end{aligned}$$

where p_1, resp. p_2 denote the projection onto X, resp. Y, and where $\bullet$ is the intersection product on the Chow group of the smooth variety $X \times Y$. If Z is a degree d correspondence the homomorphism $[Z]$ is homogeneous of degree d:

$$[Z]_* : \mathrm{Ch}_m(X) \to \mathrm{Ch}_{m+d}(Y) \quad \text{if } \deg Z = d.$$

There is a refinement on the level of cycle spaces for a certain class of correspondences, the definition of which necessitates a preliminary

Definition 1. Let Y, Z be projective varieties. A (set-theoretic) map $f : Y \to Z$ is called a continuous algebraic map if its graph is a subvariety of $Y \times Z$ and projection onto Y induces a birational bijective morphism $Y \to Z$. More generally, if Y and Z are (non necessarily finite) disjoint unions of projective algebraic varieties, a map $f : Y \to Z$ is continuous algebraic map if its restriction to each of the corresponding irreducible components Y_α of Y induces a continuous algebraic map $f|Y_\alpha : Y_\alpha \to Z_\beta$, Z_β a component of Z.

Clearly, a continuous algebraic map is the same thing as a rational map that is everywhere defined and continuous (in the Zariski-topology or the complex topology) and for Y normal, it is just a morphism (every bijective birational map from a normal variety Y to Z is a morphism).

Let me next recall the definition of a Chow correspondence.

Definition 2. A Chow correspondence of degree d from X to Y is a continuous algebraic map $f : X \to C_d(Y)$.

Following [**F-M**] I'll outline the relation with correspondences. First note that the incidence correspondence of a Chow correspondence $f : X \to C_d(Y)$ defines a cycle

$$Z_f \subset X \times Y$$

which is effective and equidimensional over X and of degree d. The assignment $f \mapsto Z_f$ is injective, but in general not all correspondences equidimensional over X and of degree d are realized in this way. However, if X is normal the rational map

$$f_Z : X \to C_d(Y)$$

sending a sufficiently general point $x \in X$ to the cycle defined by the scheme-theoretic fibre Z_x over x of the projection $Z \to X$ extends to a morphism realizing the inverse for $f \mapsto Z_f$.

For Chow correspondences there is an induced map on the level of cycle spaces. This concept makes use of the trace of an irreducible subvariety $W \subset C_d(X)$, say of dimension m. Intuitively it is the cycle on X traced out by the d-cycles corresponding to points of W. More formally, recalling that for any morphism $g : Z \to Y$ between projective varieties there are induced maps $g_* : C_m(Z) \to C_m(Y)$ (see [**Fu, p. 11**]), the trace of W, is the cycle $\mathrm{Tr}(W) = (p_2)_* Z_W$ where $Z_W \subset W \times X$ is the incidence correspondence defined by W and $p_2 : Z_W \to X$ is the projection.

Definition 3. Let $f : X \to C_d(Y)$ be a Chow correspondence. The induced action on cycle spaces is defined by

$$\Gamma_f : C_m(X) \xrightarrow{f_*} C_m(C_d(Y)) \xrightarrow{\mathrm{Tr}} C_{m+d}(Y).$$

As to the comparison with the action of the induced incidence correspondence on the level of Chow groups let me first remark that Γ_f sends cycles rationally equivalent to zero to cycles rationally equivalent to zero. Its induced action on Chow groups is denoted $[\Gamma_f]$. One has the following proposition whose proof is contained in [**F2**] (see the proof in loc. cit. of Proposition 2.2 and of Corollary 3.3):

Proposition *Let X and Y be smooth projective varieties.*

1. *Let $f : X \to C_d(Y)$ be a degree d Chow-correspondence from X to Y, Z_f the associated correspondence cycle. Then $[Z_f]_* = [\Gamma_f]$. In other words, the induced action of the Chow correspondence on Chow groups coincides with the induced action of the corresponding incidence cycle.*

2. *If Z is an irreducible degree d correspondence from X to Y such that Z maps onto a codimension c subvariety $V \subset X$ under the projection, there exists a desingularization $\tilde{V}$ of V such that the proper transform Z_V of Z in $\tilde{V} \times Y$ is the incidence correspondence of a morphism $f_V : \tilde{V} \to C_{d+c}(Y)$. Furthermore, with $i : \tilde{V} \to X$ the obvious map, one has*
$$[Z]_* : \mathrm{Ch}_m(X) \xrightarrow{i^*} \mathrm{Ch}_{m-c}(\tilde{V}) \xrightarrow{[Z_V]_* = [\Gamma_{f_V}]} \mathrm{Ch}_{m+d}(Y).$$
Moreover, there are degree d Chow correspondences $f_i : \tilde{V} \times \mathbb{P}^c \to \mathrm{Ch}_d(Y)$, $i = 1, 2$ such that $[Z]_* = [\Gamma_{f_1}] - [\Gamma_{f_2}]$.

This proposition motivates an extension of Nori's A-filtration from [**N**] which we recall at the same time

Definition 4. Let X be a smooth projective variety and m a non-negative integer. Fix an integer r with $0 \le r \le m$.

1. A class $\alpha \in \mathrm{Ch}_m(X)$ belongs to $A_r \mathrm{Ch}_m(X)$ if it is in the subgroup generated by
$$\mathrm{Im}\left([Z]_* : \mathrm{Ch}^{\mathrm{hom}}_{m-r}(Y) \to \mathrm{Ch}^{\mathrm{hom}}_m(X)\right),$$
where Z is an effective degree r correspondence from Y to X.

2. A cycle $W \in \mathcal{Z}_m(X)$ belongs to $A'_r \mathcal{Z}_m(X)$ if it is rationally equivalent to a cycle in the subgroup generated by
$$\mathrm{Im}\left(\Gamma_f : \mathcal{Z}^{\mathrm{hom}}_{m-r}(Y) \to \mathcal{Z}^{\mathrm{hom}}_m(X)\right),$$
where f is a degree r Chow correspondence from Y to X.

Remark That indeed this defines a filtration, i.e. $A_r \subset A_{r+1}$, follows from the fact that the correspondences $Z \subset Y \times X$ and $Z \times \mathbb{P}^1 \subset Y \times X$ have the same image. Similarly $A'_r \subset A'_{r+1}$. Compare [**N, Prop. 5.2**].

The preceding proposition then says that on the level of Chow groups A'_r contains A_r. Indeed, A_r is the subgroup generated by images of Chow correspondences of degree r (acting on cycles homologous to zero) from smooth Y to X.

2. Lawson homology

I'll briefly recall the definition and some main properties of Lawson homology (in the complex-analytic setting). So X denotes a smooth complex projective variety.

A. First Properties

Recall that the Lawson homology groups are the homotopy groups of the cycle space equipped with the Chow topology:

$$L_m H_\ell(X) := \pi_{\ell-2m} \mathcal{Z}_m(X),\ \ell \geq 2m.$$

Examples 5.

1. $L_m H_{2m}(X) = \pi_0 \mathcal{Z}_m(X)$ can be naturally identified with the equivalence classes of algebraic m-cycles under algebraic equivalence, i.e, one has

$$L_m H_{2m}(X) \cong \mathrm{Ch}_m(X)/\,\mathrm{Ch}_m^{\mathrm{alg}}(X).$$

2. The Dold-Thom isomorphism [**D-T**] yields an identification

$$L_0 H_\ell(X) = \pi_\ell \mathcal{Z}_0(X) \cong H_\ell(X)$$

of the zeroth Lawson group with the singular homology group with integral coefficients.

Next, I need the cycle class map for Lawson homology which is induced from the suspension homomorphism

$$s : \pi_k(\mathcal{Z}_m(X)) = L_m H_{k+2m}(X) \to \pi_{k+2}(\mathcal{Z}_{m-1}(X)) = L_{m-1} H_{k+2m}(X)$$

which is defined as follows. Taking $(\infty, 0)$ as a base point on $S^2 \times \mathcal{Z}_m(X)$, the natural continuous map between pointed spaces

$$\begin{aligned} \tau : S^2 \times \mathcal{Z}_m(X) &\to \mathcal{Z}_m(\mathbb{P}^1 \times X) \\ (x, u) &\mapsto (x, u) - (\infty, u) \end{aligned}$$

factors over $S^2 \wedge \mathcal{Z}_m(X)$ and so, identifying $S^2 \wedge S^k$ with S^{k+2}, there is a natural map

$$\begin{aligned} \pi_k(\mathcal{Z}_m(X)) &\to \pi_{k+2}(\mathcal{Z}_m(X \times \mathbb{P}^1) \\ [f] &\mapsto [\tau \circ S^2 f] \end{aligned}$$

The left hand side is isomorphic to $\pi_{k+2}(\mathcal{Z}_{m-1}(X))$ because of Lawson's Algebraic Suspension isomorphism (see [**L**] and [**F-G, Proposition 2.3**]). Combining everything, one obtains the suspension map

$$s : \pi_k(\mathcal{Z}_m(X)) \to \pi_{k+2}(\mathcal{Z}_{m-1}(X)).$$

Iterating this map m times, one arrives at $L_0H_{k+2m}(X) \cong H_{k+2m}(X)$, by [**DT**], thereby defining the class maps

$$c_{m,k+2m} : L_mH_{k+2m}(X) \to H_{k+2m}(X).$$

One deduces from [**F2, Prop. 1.6**] that the class map

$$c_{m,2m} : L_mH_{2m}(X) = \mathrm{Ch}_m(X)/\,\mathrm{Ch}_m^{\mathrm{alg}}(X) \to H_{2m}(X)$$

is the usual class map. One sets

$$L_m^{\mathrm{hom}}H_\ell(X) := \ker\{\ c_{m,\ell} : L_mH_\ell(X) \to H_\ell(X)\ \}.$$

Example 6. Since $c_{0,\ell}$ is the Dold-Thom isomorphism, one has $L_0^{\mathrm{hom}}H_\ell(X) = 0$.

Also, since for an n-dimensional projective variety X (always assumed to be irreducible), one has $\mathcal{Z}_n(X) = \mathbb{Z}$ with the discrete topology, $L_nH_\ell(X) = 0$ unless $\ell = 2n$ and in this case $L_n^{\mathrm{hom}}H_{2n}(X) = 0$.

More generally, there is a filtration on the level of cycle spaces coming from the iterated suspension preceded by the natural quotient morphism

$$\pi : \mathcal{Z}_m(X) \to \mathrm{Ch}_m(X) \to \mathrm{Ch}_m(X)/\,\mathrm{Ch}_m^{\mathrm{alg}}(X) \cong L_mH_{2m}(X).$$

One defines a corresponding S-filtration by

$$S_r\mathcal{Z}_m(X) = \ker\{\ \mathcal{Z}_m(X) \xrightarrow{\pi} L_mH_{2m}(X) \xrightarrow{c_{r,2m}} L_{m-r}H_{2m}X\ \}$$

and so

$$\begin{aligned} S_0\mathcal{Z}_m(X) &\subset S_1\mathcal{Z}_m(X) \cdots \subset S_m\mathcal{Z}_m(X), \\ S_0\mathcal{Z}_m(X) &= m\text{-cycles algebraically equivalent to } 0 \\ S_m\mathcal{Z}_m(X) &= m\text{-cycles homologically equivalent to } 0. \end{aligned}$$

This topologically defined filtration is the same as the previously geometrically defined A'-filtration. See [**F2**]:

Proposition *For any smooth projective variety X one has*

$$A'_r\mathcal{Z}_m(X) = S_r\mathcal{Z}_m(X).$$

In the sequel I need a simple consequence of the definitions

Lemma 7. *One has $L_k^{\mathrm{hom}} H_\ell(X) = 0$ for $k = 1, \ldots, s$ if and only if suspension induces injective homomorphisms*

$$L_s H_\ell(X) \hookrightarrow L_{s-1} H_\ell(X) \cdots \hookrightarrow L_0 H_\ell(X) \cong H_\ell(X).$$

If ℓ is even, say $\ell = 2m$, this is the case if and only if

$$S_{m-s}\mathcal{Z}_m X = S_{m-s+1}\mathcal{Z}_m X = \cdots = S_m \mathcal{Z}_m X.$$

The main functorial properties of Lawson homology derive directly from the corresponding properties on the level of cycles. So the push forward of cycles under proper morphisms $f : X \to Y$ induces $f_* : \mathcal{Z}_m(X) \to \mathcal{Z}_m(Y)$ and the pull back under flat morphism $g : X \to Y$ induces $g^* : \mathcal{Z}_m(Y) \to \mathcal{Z}_{m-c}(X)$ with $c = \dim Y - \dim X$. A good reference for this is [**F1**]. In conclusion, there are (unambiguous) induced homomorphisms

$$\begin{aligned} f_* &: L_m H_\ell(X) \to L_m H_\ell(Y) \\ g^* &: L_m H_\ell(Y) \to L_{m-c} H_{\ell-2c}(X). \end{aligned}$$

B. Intersection products

In this section I recall the approach to intersection theory on cycle spaces from [**F-G**]. It is based on Fulton's approach in [**Fu**]. One of the ingredients in constructing the intersection products is the Gysin morphism associated to any regular embedding $i : X \hookrightarrow Y$ between projective varieties. Let $C = C_X Y$ be the normal cone of X in Y (which is the same as the normal bundle since the embedding is regular) and let $p : C \to X$ be the defining projection. Let $\bar{C} := \mathbb{P}(C \oplus 1)$ be the projective completion. Then C is canonically isomorphic to the complement in $\mathbb{P}(C \oplus 1)$ of $C_\infty := \mathbb{P}(C)$, the "hyperplane at infinity" (see [**Fu, Appendix B.5**]). Fulton shows that the flat pull-back map $p^* : \mathrm{Ch}_{m-c}(X) \to \mathrm{Ch}_m(C)$, $c = \dim Y - \dim X$ is an isomorphism which subsequently must be inverted. To do this on the level of cycle spaces let me first note that for quasi-projective varieties such as C, one does not take for $\mathcal{Z}_m(C)$ the m-cycles on C, but one first embeds C in any projective variety such as $\bar{C}$ and then sets

$$\mathcal{Z}_m(C) := \mathcal{Z}_m(\bar{C})/\mathcal{Z}_m(\bar{C}_\infty), \quad \bar{C}_\infty = \bar{C} \setminus C,$$

and shows that this is independent of the choice of the compactification. For this I refer to [**LF**]. One then proves that the flat pull back

$$p^* : \mathcal{Z}_{m-c}(X) \to \mathcal{Z}_m(C), \quad c = \dim Y - \dim X.$$

is a weak homotopy equivalence. This is a consequence of Lawson's Algebraic Suspension theorem. See [**F-G, Proposition 2.3**]. So one should invert continuous homomorphisms that are weak homotopy equivalences. As pointed out in [**F3**], for X quasi-projective, the cycle spaces $C_m(X)$ and $\mathcal{Z}_m(X)$ admit the structure of CW-complexes so that weak homotopy equivalences are in fact homotopy equivalences ([**Span, 7.6.14**]), so one might as well invert just those. The next construction makes this precise.

Definition 8. Let $\mathfrak{Abtop}$ be the category of abelian topological groups admitting the structure of a CW-complex and let H be the set of those continuous homomorphisms between abelian topological groups that are homotopy equivalences. The category $H^{-1}\mathfrak{Abtop}$ is the category of abelian topological groups in which homotopy equivalences are inverted. This means that the objects are the same as the objects of $\mathfrak{Abtop}$, but the morphisms are "left fractions $g\backslash f$ of maps" $X \xrightarrow{f} \bullet \xleftarrow{g} Y$ with g a homotopy-equivalence and $\bullet$ any object in $\mathfrak{Abtop}$.

Indeed, $\mathfrak{Abtop}$ is an additive category and fiber products can be used to show that H satisfies the conditions in [**Iv, Chapter IX.1**] needed to define (left) fractions. We could equally well work within the homotopy category $\mathfrak{HTop}$ of topological spaces where continuous maps are replaced by homotopy classes of such maps. Indeed, we may replace $g\backslash f$ by $h \circ f$ with h a homotopy-inverse of g. In either category, a morphism $X \to Y$ induces a well-defined homomorphism $\pi_k(X) \to \pi_k(Y)$ between homotopy groups, and hence between the Lawson groups.

Now, just as in Fulton's approach, one can define the Gysin map using the specialization morphism $\tilde{\sigma} : \mathcal{Z}_m(Y) \to \mathcal{Z}_m(C)$ (in [**FG**], this name is reserved for the second map in the decomposition $s_{Y/X} \circ p_1^* : \mathcal{Z}_m(Y) \to \mathcal{Z}_{m+1}(Y \times \mathbb{C}) \to \mathcal{Z}_m(C)$ of σ, where p_1 denotes projection onto the first factor). Next, one defines $i_Y^* = p^* \backslash \sigma : \mathcal{Z}_m(Y) \to \mathcal{Z}_m(C) \leftarrow \mathcal{Z}_{m-c}(X)$.

Finally, one arrives at

Definition 9. The Gysin homomorphism associated to a morphism $f : X \to Y$ between smooth projective varieties is defined as a map in $H^{-1}\mathfrak{Abtop}$ as follows

$$\begin{aligned} f^* : \mathcal{Z}_m(Y) &\to \mathcal{Z}_{m-c}X, \quad c = \dim Y - \dim X \\ v &\mapsto \gamma_f^*(p_2^* v). \end{aligned}$$

where γ_f is the associated graph morphism (which is a regular embedding)

$$\begin{aligned} \gamma_f : X &\to X \times Y \\ x &\mapsto (x, f(x)). \end{aligned}$$

Using the Gysin homomorphism, one introduces then

Definition 10. The intersection product

$$\mathcal{Z}_p(X) \times \mathcal{Z}_q(X) \xrightarrow{\bullet} \mathcal{Z}_{n-p-q}(X)$$

for a smooth projective X of dimension n is defined in the category $H^{-1}\mathfrak{Abtop}$ by taking first the exterior product between cycles and then use the Gysin map for the diagonal embedding $\Delta : X \hookrightarrow X \times X$ (which is a regular embedding since is X smooth).

In particular, these products yield well-defined intersection products on the level of homotopy groups of cycle spaces, i.e. on the level of Lawson homology.

The following three properties of these Gysin maps are used in the sequel.

Lemma 11.

a. *Let X, Y, X', Y' be smooth complex projective varieties and let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a Cartesian square of morphisms. Then with $c = \dim Y - \dim X$, one has $f^* g_* = g'_* f'^* : \mathcal{Z}_m Y' \to \mathcal{Z}_{m-c} X$ in $H^{-1}\mathfrak{Abtop}$.

b. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphism between smooth complex projective varieties, then the* composition formula *holds in $H^{-1}\mathfrak{Abtop}$:*

$$(f \circ g)^* = g^* \circ f^* : \mathcal{Z}_m(Z) \to \mathcal{Z}_{m-d}(X), \quad d = \dim Z - \dim X.$$

c. *Let $f : X \to Y$ be a morphism between smooth complex projective varieties. Then the* projection formula *holds in $H^{-1}\mathfrak{Abtop}$:*

$$f_*(f^* v \bullet u) = v \bullet f_* u, \quad u \in \mathcal{Z}_p(X),\ v \in \mathcal{Z}_q(Y).$$

Proof:

a. This follows directly from the corresponding formulas in case f is a regular embedding (this is Theorem 3.4 d) in [**F-G**]) or a flat morphism (this follows already on the level of cycle spaces by Proposition 1.7 in [**F**]).

b. One uses that a similar composition formula holds for flat morphisms (this follows directly from the definition) and for regular embeddings (this is Theorem 3.4. (c) in [**F-G**]). Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\gamma_f} & X \times Y & \xrightarrow{\gamma'_g = \mathrm{id} \times \gamma_g} & X \times Y \times Z \\ & & \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_{YZ}} \\ & & Y & \xrightarrow{\gamma_g} & Y \times Z \\ & & & & \downarrow{\scriptstyle p_Z} \\ & & & & Z \end{array}$$

One has $p_Y^*(\gamma_g^* p_Z^* u) = (\gamma_g')^* p_{YZ}^* p_Z^* u$ and so, composing with γ_f^*. one gets $f^* \circ g^*(u) = \gamma_f^*\left(p_Y^* \gamma_g^* p_Z^* u\right) = \gamma_f^* (\gamma_g')^*\left(p_{YZ}^* p_Z^* u\right)$, which equals $\gamma_{g\circ f}(p_Z^* u) = (g \circ f)^* u$.

c. Here one considers the diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma_f} & X \times Y \\ \big\downarrow f & & \big\downarrow f \times \mathrm{id} \\ Y & \xrightarrow{\Delta_Y} & Y \times Y \end{array}$$

One gets

$$\begin{aligned} f_* u \bullet v &= \Delta_Y^*(f_* u \times v) \\ &= \Delta_Y^*((f \times \mathrm{id})_*(u \times v)) \\ &= f_*\left((\gamma_f)^*(u \times v)\right) \end{aligned}$$

and it suffices to prove $(\gamma_f)^*(u \times v) = u \bullet f^* v = \Delta_X^*(u \times f^* v)$, but this follows upon inspecting the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times X \\ \big\downarrow \mathrm{id} & & \big\downarrow \mathrm{id} \times f \\ X & \xrightarrow{\gamma_f} & X \times Y. \end{array}$$

Indeed, $\gamma_f^*(u \times v) = \Delta_X^*((\mathrm{id} \times f)^*(u \times v)) = \Delta_X^*(u \times f^* v)$. ■

C. Action of correspondences

Any correspondence between smooth projective varieties induces homomorphisms on the level of cycle spaces by the same formula as the formula on the level of Chow groups, but one has to interpret this in the category $H^{-1}\mathfrak{Abtop}$. So, for a degree d correspondence $\alpha \in \mathcal{Z}_{n+d}(X \times Y)$, n being the dimension of X one puts

$$(*) \qquad \alpha_*(u) = (p_2)_*[p_1^* u \bullet \alpha], \quad u \in \mathcal{Z}_m(X).$$

In this formula $(p_2)_*$ is the proper push-forward, while p_1^* is the flat pull back. In this way, one obtains a correspondence homomorphism in the category $H^{-1}\mathfrak{Abtop}$

$$\alpha_* : \mathcal{Z}_m(X) \to \mathcal{Z}_{m+d}(Y).$$

On the level of Lawson groups this then can be interpreted as follows. The intersection product $\bullet$ is unambiguously defined on the level of homotopy groups. One maps $\alpha \in$

$\mathcal{Z}_{n+d}(X \times Y)$ to the the connected component $[\alpha] \in \pi_0(\mathcal{Z}_m(X \times Y))$ to which it belongs. Then for all $u \in \pi_k(\mathcal{Z}_m(X))$ one has

$$(p_2)_*[p_1^* u \bullet [\alpha]] \in \pi_k(\mathcal{Z}_{m+d}(Y)).$$

Here, by abuse of notation the proper push forward and flat pull backs on the level of cycle spaces are denoted by the same letters as their induced maps on homotopy groups. Summarizing, one has then induced maps

$$\alpha_* : L_m H_\ell(X) \to L_{m+d} H_{\ell+2d}(Y)$$

which, by construction, depend only on the class of α in the Chow group of $X \times Y$ modulo algebraic equivalence.

The following crucial property is needed in the sequel

Lemma 12. *Assume that X and Y are smooth projective varieties and let $\alpha \subset X \times Y$ be an irreducible cycle of dimension* dim $X = n$*, supported on $V \times W$, where $V \subset X$ is a subvariety of dimension v and $W \subset Y$ a subvariety of Y of dimension w. Let $\tilde{V}$, resp. $\tilde{W}$ be a resolution of singularities of V, resp. W and let $i : \tilde{V} \to X$ and $j : \tilde{W} \to Y$ be the corresponding morphisms. With $\tilde{\alpha} \in \tilde{V} \times \tilde{W}$ the proper transform of α, and p_1, resp. p_2 the projections from $X \to Y$ to the first. resp. the second factor, there is a commutative diagram*

$$\begin{array}{ccc}
L_{m-n+v+w} H_{\ell+2(v+w-n)}(\tilde{V} \times \tilde{W}) & \xrightarrow{\tilde{\alpha}_*} & L_m H_\ell(\tilde{V} \times \tilde{W}) \\
\uparrow p_1^* & & \downarrow (p_2)_* \\
L_{m-n+v} H_{\ell-2n+2v}(\tilde{V}) & & L_m H_\ell(\tilde{W}) \\
\uparrow i^* & & \downarrow j_* \\
L_m H_\ell(X) & \xrightarrow{\alpha_*} & L_m H_\ell(Y).
\end{array}$$

Here i^* and p_1^* are induced by the Gysin homomorphisms and $(p_2)_*$ and j_* come from proper push forward.

In particular, $\alpha_* = 0$ if $m < n-v$ or if $m > w$. Moreover, α_{n-v} acts trivially on $L_{n-v}^{\mathrm{hom}} H_* X$, while α_w acts trivially on $L_w^{\mathrm{hom}} H_* X$.

Proof: The proof is based on the projection and composition formulas established above. Let $p_{\tilde{V}}$, resp. $p_{\tilde{W}}$ be the projection from $\tilde{V} \times \tilde{W}$ onto the first, resp. the second factor. As before, $[\alpha] \in \pi_0(\mathcal{Z}_n(X \times Y))$ is the connected component to which α belongs and similarly for $[\tilde{\alpha}] \in \pi_0(\mathcal{Z}_n(\tilde{V} \times \tilde{W}))$. Note that one may assume that the cycle α is not entirely contained in the singular locus of $V \times W$ by moving it in its algebraic equivalence class (the action of $[\alpha]$ depends only on this equivalence class). This ensures that the push-forward of the cycle $\tilde{\alpha}$ under $i \times j$ equals α.

We find for any $u \in \mathcal{Z}_m(X)$:

$$\begin{aligned}\alpha_* u &= (p_2)_*[p_1^* u \bullet [\alpha]] \\ &= (p_2)_* \left(p_1^* u \bullet (i \times j)_*[\tilde{\alpha}]\right) \\ &= (p_2)_*(i \times j)_* \left((i \times j)^* p_1^* u \bullet [\tilde{\alpha}]\right) \\ &= j_*(p_{\tilde{W}})_* \left(p_{\tilde{V}}^* i^* u \bullet [\tilde{\alpha}]\right).\end{aligned}$$

The one but last assertion follows from the fact that $L_{m-n+v} H_* \tilde{V} = 0$ for $m-n+v < 0$ and $L_m H_* \tilde{W} = 0$ if $m > w$. The final assertion follows from the fact that for all varieties Z, one has $L_t^{\text{hom}} H_* Z = 0$ for $t = \dim Z$, while also $L_0^{\text{hom}} H_* Z = 0$. ∎

Corollary 13. *An irreducible cycle $\alpha \subset X \times X$ supported on a product variety $V \times W$ with* $\dim V + \dim W = n = \dim X$ *acts trivially on $L_*^{\text{hom}} H_*(X)$.*

3. The main result

Before stating the main result, I need some terminology.

Definition 14. A variety X has small Chow groups up to rank s, if there exists a closed subvariety $Y \subset X$ of dimension $\leq s$ which supports all k-cycles with $k \leq s$. In other words, with $j : X \setminus Y \to X$ the inclusion, $j^* \operatorname{Ch}_k(X) = 0$ for $k = 0, \ldots, s$.

As shown essentially in [**B-S**], this notion means that rational equivalence and homological equivalence coincide in degrees $\leq s$. See also [**La**].

Characterisation 15. X has small Chow groups up to rank s if and only if rational equivalence and homological equivalence on $\operatorname{Ch}(X) \otimes \mathbb{Q}$ coincide in degrees $\leq s$.

For such varieties X, a result of Paranjape ([**Pa**]) (see also Laterveer ([**La**]) generalizing previous work of Bloch-Srinivas ([**B-S**]), Jansen ([**J**]) and Esnault-Levine-Viehweg ([**E-L-V**]) states that the diagonal $\Delta \subset X \times X$ can be decomposed modulo rational equivalence (allowing also rational coefficients) as follows.

Theorem *Let X have small Chow groups up to rank s. Then there are subvarieties $V_{n-k} \subset X$, $k = 0, \ldots, s+1$ and $W_k \subset X$, $k = 0, \ldots s$ (subscripts indicating the dimension), and a decomposition in* $\operatorname{Ch}(X \times X) \otimes \mathbb{Q}$

$$\Delta = \alpha^{(0)} + \cdots + \alpha^{(s)} + \beta$$

with $\alpha^{(k)}$ having support in $V_{n-k} \times W_k$, $k = 0, \ldots, s$ and β having support in $V_{n-s-1} \times X$.

Using Corollary 13, one deduces that the identity acts as β on Lawson homology $L^{\hom}$. Applying Lemma 12 to β yields

Theorem 16. *Let X be a smooth complex variety having small Chow groups up to rank s. Then $L_m^{\hom}H_*(X) \otimes \mathbb{Q} = 0$ for $m = 0, \ldots, s+1$.*

This has the desired implication on the level of S-filtrations, using Lemma 7.

Corollary 17. *Let X be a smooth complex variety having small Chow groups up to rank s. Then $S_{m-s-1}\mathcal{Z}_m X \otimes \mathbb{Q} = S_{m-s}\mathcal{Z}_m X \otimes \mathbb{Q} = \cdots = S_m \mathcal{Z}_m X \otimes \mathbb{Q}$. In particular the last $s+1$ steps induced by the S-filtration on the any of the $\mathbb{Q}$-Griffiths groups are constant. In particular, algebraic and homological equivalence coincide for $s+1$-cycles.*

Let me end this section by giving some examples of varieties to which these results can be applied.

Examples 18.

1). Any variety with $\mathrm{Ch}_0^{\hom} X = 0$ has $S_{m-1}\mathcal{Z}_m(X) \otimes \mathbb{Q} = S_m \mathcal{Z}_m(X) \otimes \mathbb{Q}$, $m = 0, \ldots$. In particular this holds for rationally connected varieties (in particular Fano varieties up to dimension 5).

2). In [**E-L-V**] are examples of varieties having small Chow groups up to arbitrary rank s (the dimension of these varieties grows astronomically). The number s is bounded by the degrees $d_1 \leq d_2 \cdots \leq d_r$ of the defining equations for X in some projective space. For instance $s \geq 2$ for a degree d hypersurface in $\mathbb{P}^{n+1}$ whenever $d \geq 3$ and $\binom{d+2}{3} \leq n+1$.

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