Introduction to Lawson homology

Chris PETERS, email: chris.peters@ujf-grenoble.fr
Siegmund KOSAREW email: siegmund.kosarew@ujf-grenoble.fr
Department of Mathematics, University of Grenoble I
UMR 5582 CNRS-UJF, 38402-Saint-Martin d'Hères
France

Abstract

Lawson homology has quite recently been proposed as an invariant for algebraic varieties. Various equivalent definitions have been suggested, each with its own merit. Here we discuss these for projective varieties and we also derive some basic properties for Lawson homology. For the general case we refer to Paulo Lima-Filho's lectures in this volume.

Keywords: Lawson homology, cycle spaces
MSC2000 classification: 14C25, 19E15, 55Qxx

Introduction

This paper is meant to serve as a concise introduction to Lawson homology of projective varieties. For another introduction the reader should consult [14].

It is organized as follows. In the first section we recall some basic topological tools needed for a first definition of Lawson homology. Then some basic examples are discussed. In the second section we discuss the topology of the so-called "cycle spaces" in more detail in order to understand functoriality of Lawson homology. In the third and final section we relate various equivalent definitions. Here the language of simplicial spaces is needed and we only summarize some crucial results from the vast literature on this highly technical subject.

Finally we want to thank the referee for his suggestions to improve the exposition.

1 Basic Notions

1.1 Homotopy groups

We start by recalling the definition and the basic properties of the homotopy groups. For any two pairs of topological spaces (X, A) and (Y, B) we use the notation $[(X, A), (Y, B)]$ for the set of homotopy classes of maps $X \to Y$ sending A to B (any homotopy is supposed to send A to B as well). Then, fixing a point s on the k-sphere S^k, we have

$$\pi_k(X, x) = [(I^k, \partial I^k), (X, x)] = [(S^k, s), (X, x)].$$

There is a natural product structure on these sets (divide I^k in two and use the first map on one half and the second map on the other half). This makes $\pi_k(X, x)$ into a group, which turns out to be abelian for $k \geq 2$.

Homotopy and homology are related through the *Hurewicz homomorphism*

$$h_k : \pi_k(Y, y) \to H_k(Y),$$

defined by associating to the class of a map $f : S^k \to Y$ the image under f_* of a generator of $H_k(S^k)$. The following important result tells us when the Hurewicz homomorphism actually is an isomorphism:

Theorem 1 (Hurewicz' Theorem) Suppose that (X, x) is $(k-1)$-connected, i.e. $\pi_s(X, x) = 1$, $s = 0, \ldots, k-1$. Then h_k is an isomorphism.

One can show that homotopic maps induce the same map in homology. Hurewicz' theorem tells us that any map inducing isomorphisms on the homotopy groups will also induce isomorphisms on the homology groups. This motivates the following definitions.

Definition 2

(1) A continuous map $f : X \to Y$ is a *homotopy equivalence* if there is a continuous map $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity.

(2) A continuous map $f : X \to Y$ is a *weak homotopy equivalence* if the induced maps on the homotopy groups are all isomorphisms.

(3) Two topological spaces are *(weakly) homotopically equivalent* if there exist a (weak) homotopy equivalence between them.

Example 3 A space X is said to be an *Eilenberg-Maclane* $K(\pi, k)$*-space* if its only non-trivial homotopy group is $\pi_k(X) = \pi$. Hence any space homotopy equivalent to a $K(\pi, k)$-space is again a $K(\pi, k)$-space. For instance S^1 is a $K(\mathbb{Z}, 1)$, and the inductive union of projective spaces, $\mathbb{P}^\infty$ is a $K(\mathbb{Z}, 2)$.

An important class of topological spaces is the class of *CW-complexes*. Here we don't give the precise definition, but refer to [20]. Roughly speaking, a CW-complex is inductively defined by specifying its cells in a given dimension $(k+1)$ together with the attaching maps of the cells to the union of the cells of lower dimension, the k-skeleton. In general one needs infinitely many cells, but a CW-complex has to satisfy a certain local finiteness condition. Any topological space admitting a triangulation gives an example. For instance (see [17]) any differentiable manifold has the structure of a CW-complex. An important remark is that although an Eilenberg-Maclane space need not be a CW-complex, it has the homotopy type of a CW-complex.

For CW-complexes any weak homotopy equivalence is a homotopy equivalence. This result is due to Whitehead [20, p. 405]). Another important result (loc. cit) for CW-complexes is the fact that $K(\mathbb{Z},m)$ classifies cohomology:

$$[X, K(\mathbb{Z},m)] \cong H^m(X;\mathbb{Z}).$$

For CW-complexes with finitely many cells, there is a homomorphism refining the Hurewicz map, due to Almgren ([1]). Recall that singular homology is computed as the cohomology of the singular complex $S_\bullet(X)$. Instead, one can also use the complex of $I_\bullet(X)$ "integral currents", a refinement due to Federer and Fleming [4] of the well known result that one can use the complex of currents to compute real (co)homology. The so called "flat-norm topology" makes the spaces of integral currents into topological spaces so that we can speak of the homotopy group of the cycle spaces $Z'_k = \ker(\partial : I_k(X) \to I_{k-1}(X))$. We shall summarize this by saying that we give $Z'_k(X)$ the *Federer topology.* Almgren's thesis tells us that any continuous map $f : S^r \to Z'_k(X)$ of the r-sphere into this space can be seen as a $(k+r)$-cycle $f_k \in Z'_{k+r}(X)$ and so by the afore mentioned result by Federer and Fleming, yields a class in the singular homology group $H_{k+r}(X)$. This map is an isomorphism:

Theorem 4 *For a CW-complex X with finitely many cells (such as a projective manifold) equip the set of k-cycles $Z_k(X)$ with the Federer topology. The map (as defined above)*

$$\begin{array}{ccc} \pi_r Z_k(X) & \to & H_{r+k}(X) \\ [f] & \mapsto & [f_k] \end{array}$$

is an isomorphism.

Intuitively, f is a continuous family of k-cycles on X and sweeps out a $(k+r)$-cycle. Homotopic maps give rise to homologous cycles. For $k=0$

we can work with the usual notion of 0-cycles, obtaining a refinement of the Hurewicz map. This gives the Dold-Thom theorem [3]

We next discuss the concept of a *(Hurewicz) fibration*. This is a continuous surjective map between topological spaces $p : E \to B$ which has the homotopy lifting property: given a map $g : X \to E$, every homotopy of $p \circ g$ can be lifted to a homotopy of g. For such a fibration any two fibers are homotopy equivalent [20, p. 101] and with $e \in E$ the base point and F the typical fiber, one has the homotopy exact sequence ([20, p. 377])

$$\cdots \pi_n(F, e) \to \pi_n(E, e) \to \pi_n(B, p(e)) \to \pi_{n-1}(F, e) \cdots$$

Examples include locally trivial fiber bundles such as smooth Kähler families or smooth projective families.

Although fibrations look rather special, in homotopy theory all maps are fibrations. Indeed, one can functorially replace any continuous map $f : X \to Y$ by a Hurewicz fibration. Using the path space PY of continuous paths in Y, the total space of the fibration is

$$E_f = \{(x, \gamma) \in X \times PY \mid \gamma(0) = f(x)\}$$

and the map $\pi_f : E_f \to Y$ given by sending a pair (x, γ) to the endpoint $\gamma(1)$ of γ gives it the structure of a fibration. The map $s : X \to E_f$ which sends x to the pair (x, constant path at $f(x)$) is a homotopy equivalence so that indeed $f : X \to Y$ may be replaced by the fibration $\pi_f : E_f \to Y$. The *homotopy fiber* $E_f(y)$ of f above y by definition is the fiber of π_f above y. Its homotopy type depends only on the path component to which y belongs. So, if we start with a Hurewicz fibration over a path connected space, any fiber is homotopy equivalent to the homotopy fiber.

1.2 Lawson homology

Let X be a complex projective variety. An *effective (algebraic) m-cycle* on X is a finite formal linear combination $Z = \sum n_V[V]$, $n_V \in \mathbb{N}$ of (irreducible) subvarieties $V \subset X$ of dimension m. The union $\cup V$ is called the *support of* Z and will be denoted $\mathrm{supp}(Z)$. If a projective embedding is fixed, one can speak of the *degree* of the cycle $\deg Z = \sum n_V \deg V$. The set $C_{m,d}(X)$ which parametrizes the effective m-cycles of degree d is known to be a projective variety (see Elizondo's lectures in this volume). It comes with the complex topology. Now we take the the disjoint union $\mathcal{C}_m(X)$ of the Chow varieties of effective m-cycles of degree $d = 0, 1, 2, \ldots$. The cycle $0 \in \mathcal{C}_m(X)$ by definition is the cycle with empty support and serves as a natural base point. It also acts as a zero for the addition of cycles, making $\mathcal{C}_m(X)$ into a monoid. Let us put

$$\begin{array}{ll} \mathcal{Z}_m(X) & = C_m(X) \times C_m(X)/\sim \\ (x,y) & \sim (x',y') \Leftrightarrow\ x+y' = x'+y \quad (\textit{the naïve group completion}). \end{array}$$

The complex topology induces a natural topology on the monoid $C_m(X)$ and one equips $\mathcal{Z}_m(X)$ with the quotient topology. We can be more specific. Introduce the compact sets

$$K_d = \bigcup_{d_1+d_2\leq d} C_{m,d_1}(X) \times C_{m,d_2}(X)/\sim$$

which filter $\mathcal{Z}_m(X)$. Then $B \subset \mathcal{Z}_m(X)$ is closed if and only if its intersection which each of the K_d is closed.

In the next chapter we'll review the proof that this topological space is independent of the chosen embedding of X into a projective space. The induced topology will be called the *Chow topology.*

An algebraic cycle on a projective variety defines an integral current (via integration over its smooth locus), and the inclusion $\mathcal{Z}_m(X) \subset Z'_{2m}(X)$ is continuous. Using Federer's theorem (Theorem 4) this yields maps

$$\pi_{\ell-2m}\mathcal{Z}_m(X) \xrightarrow{c_{m,\ell}} H_\ell(X)$$

motivating the following definition.

Definition 5 *The Lawson homology groups of a complex projective algebraic variety X are the homotopy groups of the cycle space:*

$$L_mH_\ell(X) = \begin{cases} \pi_{\ell-2m}\mathcal{Z}_m(X) & \textit{if } \ell \geq 2m \\ 0 & \textit{if } \ell < 2m. \end{cases}$$

Lawson homology incorporates usual homology in view of an old result of Dold and Thom (see [3]) which can be reformulated as a natural isomorphism

$$c_{o,\ell} : L_0H_\ell(X) \xrightarrow{\sim} H_\ell(X).$$

This reformulation will be explained in Chapter 2 (see Example 26 and 33).

As with the Chow groups, there are relatively few classes of varieties for which one can compute Lawson homology.

One of the breakthroughs was Lawson's computation [13] of these groups for projective spaces. He used a powerful tool, the Suspension Theorem. To formulate it, we introduce some notation. Let $X \subset \mathbb{P}^N$ be a projective variety and a point $v \in \mathbb{P}^{N+1}$ not in the subspace $\mathbb{P}^N$ in which X lies. Let $\Sigma X \subset \mathbb{P}^{N+1}$ be the projective cone over X with v as its vertex.

Theorem 6 (Lawson's Suspension Theorem) *The map which associates to an m-cycle Z of degree d on X its projective cone ΣZ, an $(m+1)$-cycle of degree d on ΣX, induces a weak homotopy equivalence*

$$\Sigma : \mathcal{Z}_m(X) \to \mathcal{Z}_{m+1}(\Sigma X).$$

In particular we have

$$L_m H_\ell(X) \xrightarrow{\sim} L_{m+1} H_{\ell+2}(X).$$

1.3 Examples

1. Two cycles that are algebraically equivalent belong to the same connected component of the cycle space. One can show ([5]) that the converse also holds. So $\pi_0(\mathcal{Z}_m(X))$ can be identified with the group of equivalence classes of cycles modulo algebraic equivalence:

   $$L_{2m}H_m(X) = \pi_0(\mathcal{Z}_m(X)) = \mathrm{Ch}_m^{\mathrm{alg}}(X) = \mathrm{Ch}_m(X)/\{\text{algebraic equivalence}\}.$$

   This is the first indication that Lawson homology reflects also the algebraic nature of the variety, for a priori it could only be a topological invariant of its analytic topology.

2. Look at zero cycles on curves X. The Abel-Jacobi map

   $$C_{0,d}(X) \to \mathrm{Jac}(X)$$

   is surjective for d large enough with fiber a projective space $\mathbb{P}^{d-g}$. This shows that

   $$\lim_{d\to\infty} C_{0,d}(X) \sim \mathrm{Jac}(X) \times \mathbb{P}^\infty$$

   and so we know its homotopy type: its is a $K(\mathbb{Z}^{2g}, 1) \times K(\mathbb{Z}, 2)$. Later we shall show that the limit computes the homotopy type of the cycle space $\mathcal{Z}_0(X)$ and so we have shown:

   $$\pi_k(\mathcal{Z}_0(X)) \cong H_k(X)$$

   and this is a special case of the Dold-Thom theorem.

3. Look at irreducible curves of degree d in the projective plane $\mathbb{P}^2$. Choose a point p not on a given curve C and choose coordinates such that $p = (0 : 0 : 1)$. Projection from this point is then given by $(x : y : z) \mapsto (x : y : 0)$ and it fits in a one-parameter family of maps $\pi_t : \mathbb{P}^2 \to \mathbb{P}^2$, $t \in \mathbb{C}$ given by

   $$\pi_t(x : y : z) = (x : y : tz).$$

For $t \neq 0$ this is an automorphism and for $t = 0$ we get our projection. The curve $C_t = \pi_t C$ degenerates to dL where L is the line $z = 0$. If $p \in C$, one can see that the curve degenerates into $aL + \sum_j b_j M_j$, where $a + \sum_j b = d$ and M_j are the tangents of C in p. So the space $C_{1,d}(\mathbb{P}^2)$ is connected and simply connected. It follows that $\pi_0(\mathcal{Z}_1(\mathbb{P}^2)) \cong \mathbb{Z}$ and that also $\mathcal{Z}_1(\mathbb{P}^2)$ is simply connected.

4. The Suspension theorem generalizes this to arbitrary varieties. The role of L being played by an arbitrary variety X and $\mathbb{P}^2$ is seen as the projective cone ΣL over the line L. As an application of this, recall that the Dold-Thom theorem states that the m-th homotopy group of cycle space $\mathcal{Z}_0(X))$ is equal to $H_m(X)$ and so $\mathcal{Z}_0(X)$ is homotopy equivalent to the product of the Eilenberg-Maclane spaces $K(H_m(X), m)$. If $X = \mathbb{P}^N$ the homology groups being $\mathbb{Z}$ (for $m \leq 2N$ even) or 0 (otherwise) we get:

$$\mathcal{Z}_0(\mathbb{P}^{n-m}) = K(\mathbb{Z}, 0) \times K(\mathbb{Z}, 2) \times \cdots \times K(\mathbb{Z}, 2(n-m)).$$

Using the Suspension theorem we thus find

$$\mathcal{Z}_m(\mathbb{P}^n) = K(\mathbb{Z}, 0) \times K(\mathbb{Z}, 2) \times \cdots \times K(\mathbb{Z}, 2(n-m))$$

and thus

$$L_m H_\ell(\mathbb{P}^n) = \pi_{\ell-2m}(\mathbb{P}^n) = \begin{cases} 0 & \text{if } \ell \text{ odd} \\ \mathbb{Z} & \text{if } \ell = 2k,\ k = m, \ldots, n. \end{cases}$$

2 Chow-varieties and cycle spaces

2.1 Chow-varieties

In this section we recall the definition of the Chow-variety of a projective variety X in a fixed embedding

$$i : X \subset \mathbb{P}^N = \mathbb{P}.$$

This variety $C_{m,d}(X)$ parametrizes the effective degree d cycles on X. For more details see Elizondo's lectures in the present volume.

First of all, one constructs the Chow variety for $\mathbb{P}$ itself as follows. Let Z be an m-dimensional subvariety of $\mathbb{P}$. The $(m+1)$-tuples of hyperplanes whose intersection meets Z form a hypersurface in $(\mathbb{P}^\vee)^{(m+1)}$ of multidegree $(d, \ldots, d)$, where d is the degree of $Z \subset \mathbb{P}$. This hypersurface F_Z defines the *Chow point* in $\mathbb{P}(m, d)$, the projective space of forms of multidegree $(d, \ldots, d)$.

The *Chow-coordinates* of this point are the corresponding $(m+1)d$ homogeneous coordinates. The variety Z can be reconstructed from F_Z, or its Chow-point since one can show that $z \in Z$ if and only if for any $m+1$-tuple $(l_0, \ldots, l_m)$ of hyperplanes passing through z, one has $F_Z(l_0, \ldots, l_m) = 0$. If now Z is a cycle $\sum n_V[V]$, its Chow-point is the point corresponding to the hypersurface $\prod F_V^{n_V}$. All cycles of $\mathbb{P}$ of fixed degree d and dimension m fill up the Chow variety of $\mathbb{P}$, a projective subvariety $C_{m,d}(\mathbb{P}) \subset \mathbb{P}(m,d)$. Cycles belonging to a fixed subvariety $X \subset \mathbb{P}$ belong to the Chow-variety of X, a subvariety $C_{m,d}(X) \subset C_{m,d}(\mathbb{P})$.

Let us next discuss the functoriality of this construction. First of all, there is no universal family over the Chow variety. A weak approximation of this is the *incidence correspondence* $\Gamma_{n,d}(X) \subset C_{n,d} \times X$, which is the closure inside $C_{m,d}(X)$ of the variety of pairs (Chow point of Z, x) of irreducible m-dimensional subvarieties $Z \subset X$ of degree d such that x belongs to Z. The scheme-theoretic fiber over the Chow point of a possibly reducible $Z = \sum n_V[V]$ does have support in Z but the multiplicities are not necessarily the same. This causes failure of universality and is the source of a lot of technical problems when dealing with Chow-varieties.

Let us briefly describe what sort of families of m-cycles over a base T do give rise to a morphism $T \to C_{m,d}(X)$. The following concept is crucial.

Definition-Lemma 7 *Let T be a projective variety. A subscheme $Z \subset T \times X$ is an* equi-dimensional effective relative m-cycle of X over T *if the projection $p_1 : Z \to T$ is surjective and for all $t \in T$ the scheme-theoretic fiber Z_t over t has dimension m. The set of these cycles forms a Zariski-open subset*

$$C_{m,d}(X;T) \subset C_{m+t,d}(T \times X), \quad t = \dim T.$$

The fact that the complement is a Zariski-closed set follows from upper-semi-continuity of the dimensions of the fibers of the incidence correspondence $\{(Z,x,y) \in C_{m+t,d}(T \times X) \times (X \times Y) \mid (x,y) \in \text{support of } Z\}$ over the first factor.

A standard example of such a situation arises when Z dominates T and $p_1 : Z \to T$ is *flat*. To any scheme like Z_t one can associate a cycle $[Z_t] = \sum n_V[V]$, where V runs over the irreducible components of Z_t and n_V is the length of the local artinian ring $\mathcal{O}_{V,Z_t}$ (see [10], 1.5]). In our situation, if Z is flat over T the degree of the cycle $[Z_t]$ is constant and one indeed does have an associated map $\varphi_Z : T \to C_{m,d}(X)$ which is a morphism as we'll indicate below. In fact, one has

Lemma 8 *Let Z be a flat relative m-cycle of degree d dominating a smooth projective variety T. The map*

$$\varphi_Z : T \to C_{m,d}$$

sending t to the cycle associated to the scheme-theoretic fiber of $Z \to T$ over t is a morphism. Conversely, a morphism $\varphi : T \to C_{m,d}$ determines (by taking the fiber product over $C_{m,d}$ of T and the incidence correspondence $\Gamma_{m,d}(X)$) a flat relative m-cycle of degree d whose associated morphism is φ.

The following example, taken from [9, p.29] illustrates what happens in the non-flat case.

Example 9 Take $T = \mathbb{C}^2$ with coordinates (x_1, x_2), $X = \mathbb{P}^2$ with homogeneous coordinates (X_0, X_1, X_2) and Z given by the union of the two varieties $Z_1 = \{X_1 = x_1 X_0, X_2 = x_2 X_0\}$ and $Z_2 = \{X_1 = -x_1 X_0, X_2 = -x_2 X_0\}$. The scheme-theoretic fiber $Z_{a,b}$ consists of two points $((1 : \pm a : \pm b) \in \mathbb{P}^2$ for $(a, b) \neq (0, 0)$, while the scheme-theoretic fiber over the origin is $(1 : 0 : 0) \in \mathbb{P}^2$ with multiplicity 3.

To obtain a constant degree cycle over the smooth locus, one has to replace $[Z_t]$ by the *intersection-theoretic fiber*

$$Z \cdot [t] = \sum e_V [V], \quad e_V = (e_{Z_t} Z)_V = i(V, \{t\} \cdot Z; T)$$

as in [10, Ch. 7]. Here e_V is Samuel's multiplicity of the primary ideal determined by Z_t in the local ring $\mathcal{O}_{V,Z}$. This multiplicity is bounded above by $\text{length}(\mathcal{O}_{V,Z_t})$ and is equal to it if for instance Z_t is a regular subscheme of Z, which happens for points $t \in U$, where U is the set of smooth points of T over which f is flat (use [10, Prop. 7.1] and [loc. cit, Example A.5.5]). In the above example, $Z \cdot \{0\} = 2(1 : 0 : 0)$ which has the correct multiplicity $2 < 3$.

To see that this degree is constant we recall that the notion of *degree* of an n-dimensional subvariety $X \subset \mathbb{P}$ defined as the cardinality of $X \cap L$, where L is a general linear space of codimension n in $\mathbb{P}$ can also be defined using the intersection product on Chow-groups (see e.g. [10])

$$\text{Ch}_p(X) \times \text{Ch}_q(X) \xrightarrow{\bullet} \text{Ch}_{p+q-n}$$

as $\deg(X) =$ degree of the cycle $[X] \bullet [L]$. Now one applies the "principle of conservation of number" (see [10, §10.2] which says that the degree of the class of the zero cycle $(Z \cdot [t]) \cdot [L]$, $L \subset \mathbb{P}$ a subvariety of complementary dimension, is constant, say d and is called the *degree of Z over T*. Over the Zariski open subset $U \subset T$ consisting of smooth points over which Z is flat, this is the degree of the scheme-theoretic fiber as defined above.

The rational map

$$\begin{array}{rcl} \varphi_Z : T & \to & C_{m,d} \\ t & \mapsto & \{\text{Chow point of } Z \cdot [t]\} \end{array}$$

is defined over the smooth locus of T. It is a morphism over the locus where Z is flat. To see that it extends as a morphism over the smooth locus, one needs the following continuity property of Chow-varieties.

Lemma 10 *Suppose that S a smooth variety and let Z be a relative effective m-cycle of X over S. Let $U \subset S$ be the Zariski-open over which Z is flat. Fix $s_0 \in S$. For any sequence $\{s_n \mid n = 1, 2, \ldots\}$ of points $s_n \in U$ converging to s_0, the limit of the Chow-points of the cycles Z_{s_n} exists and is equal to the Chow point of the intersection theoretic fiber $Z \cdot [s_0]$. In particular, this limit is independent of the chosen sequence in U.*

The relevance of this Lemma in this context shows itself when one wants to prove that a rational map $f : T \dashrightarrow V$ defined over U is everywhere defined (but only as a continuous map in the complex topology). One considers its graph $\Gamma_f \subset T \times V$ whose points above t are in fact the limit points $(t_n, f(t_n))$ where $t_n \in U$ converges to t. So, if the limit $\lim_{n\to\infty} f(t_n)$ is independent of the chosen sequence, the rational map extends to an everywhere defined continuous map whose graph maps bijectively to V under the projection. This is the meaning of the following

Definition 11 Let T, V be projective varieties. A (set-theoretic) map $f : T \to V$ is called a *continuous algebraic map* if its graph Γ_f is a subvariety of $T \times V$ and projection onto T induces a birational bijective morphism $\Gamma_f \to T$. More generally, if T and V are (not necessarily finite) disjoint unions of projective algebraic varieties, a map $f : T \to V$ is continuous algebraic map if its restriction to each of the corresponding irreducible components T_α of T induces a continuous algebraic map $f|T_\alpha : Y_\alpha \to V_\beta$, V_β a component of V. A *bi-continuous algebraic map* is a bijective continuous algebraic map whose inverse is a continuous algebraic map.

Clearly, a continuous algebraic map is the same thing as a rational map which is everywhere defined and continuous (in the complex topology) and for Y normal, it is just a morphism (every bijective birational map from a normal variety Y to Z is a morphism). Bi-continuous algebraic maps are always homeomorphisms in the complex topology.

To understand continuous algebraic maps, one introduces

Definition-Lemma 12 The *weak normalization* $w : X^{\mathrm{wn}} \to X$ of X is the unique morphism of varieties over which the normalization $n : X^{\mathrm{n}} \to X$ factors and such that w is a homeomorphism. It is characterized by the property that $w_* \mathcal{O}_{X^{\mathrm{wn}}}$ is the sheaf of continuous meromorphic functions on X.

If $X = X^{\mathrm{wn}}$ we say that X is *weakly normal*. Weakly normal curves are precisely the unibranch curves, *i.e.*, those that are locally irreducible. Any continuous algebraic map $f : X \to Y$ gives rise to a morphism $f \circ w : X^{\mathrm{wn}} \to Y$ and conversely. For weakly normal varieties the continuous algebraic maps are precisely the morphisms.

Example 13

1. Let Y be a cuspidal cubic and $X \to Y$ its normalization. This is a bi-continuous algebraic map but not an isomorphism. The curve Y is unibranch (and hence weakly normal).
2. If T is smooth, the map $\varphi_Z : T \to C_{m,d}(X)$ defined in the preceding Lemma is a continuous algebraic map and hence a morphism. In case $t_0 \in T$ is a singular point at which T is locally irreducible, one can still define a cycle Z_{t_0} of degree d, which differs from the cycle associated to the scheme theoretic fiber in such a way that for **weakly normal** T the resulting map
$$\varphi_Z : T \to C_{m,d}(X)$$
is a continuous algebraic map and hence a morphism. See [2] and [8].
3. The addition $\mathcal{C}_m(X) \times \mathcal{C}_m(X) \to \mathcal{C}_{n+m}(X)$ is is clearly an algebraic morphism. In terms of Chow points, this amounts to multiplication of Chow forms (by definition), which is an algebraic map.

2.2 Functoriality

The aim here is to discuss the nature of the map on Chow varieties induced by morphisms $f : X \to Y$ between projective varieties. Let us start with cycles themselves. Recall [10, p. 11] that to any variety $V \subset X$ there is associated a *push-forward cycle* $f_* V$ which by definition is 0 if $\dim f(V) < \dim V$ and otherwise equals $f(V)$ with multiplicity $(\mathbb{C}(V) : \mathbb{C}(W))$. This map then is extended by linearity to all effective cycles on X, yielding a morphism of monoids of effective cycles
$$f_* : C_m(X) \to C_m(Y).$$

Also [10, p. 18]), if $f : X \to Y$ is flat of relative dimension $c = \dim X - \dim Y$, for each variety W the scheme-theoretic inverse image $f^{-1} W$ is a pure $(c+\dim W)$-dimensional scheme whose associated cycle defines the *flat pull back* $f^*[W]$ of the cycle $[W]$. Again, this is extended by linearity to all effective cycles of Y, yielding a morphism of monoids

$$f^* : C_m(Y) \to C_{m+c}(X).$$

We want to discuss now the maps induced on Chow-varieties by a given morphism $f : X \to Y$. In particular, we want that the above maps induce **continuous algebraic** maps.

Proposition 14 *Fix some component $U \subset C_{m,d}(X)$ and consider the proper push-forward cycles $f_*(Z_u)$ where Z_u is the cycle with Chow-point u. Assume that this degree is generically e. Then the degree is e for all $u \in U$ and the map*

$$\begin{array}{rcl} \mathfrak{f} : U & \to & C_{e,m}(Y) \\ u & \mapsto & \text{Chow point of } f_*(u) \end{array}$$

is a continuous algebraic map.

Proof: Note first that $\mathfrak{f}$ is a rational map. To see this, assume for simplicity that $f : X \to Y$ is induced by a morphism $F : P \to P'$ of projective spaces. Then for an irreducible subvariety $V \subset P$ the Chow form F_W of the proper push-forward $f(V) = W$ is characterized by $F_W((l'_0, \dots, l'_m) = 0$ if and only if $l'_0 \cap \cdots \cap l'_m \cap W \neq \emptyset$. This is the case if and only if $F_V(F^*l'_0, \dots, F^*l'_m) = 0$. So the rational map $P(m,d) \to P'(m,d)$ defined by $F^*l'_k = l_k$ sends the component U to $C_{m,d}(Y)$. Next, let us see that this map is defined at any point $u_0 \in U$ and is continuous there. One chooses a pointed smooth curve (C, c_0) and a morphism $(C, c_0) \xrightarrow{\varphi} (U, u_0)$ such that a pointed disk around c_0 maps entirely in the locus where $\mathfrak{f}$ is defined. Let $\{c_n\}_{n=1,\dots}$ be a sequence of points in this neighborhood converging to c_0 and put $x_n = \mathfrak{f}c_n$.

The equidimensional relative m-cycle Z_φ over C yields the equidimensional relative m-cycle $(1 \times f)_* Z_\varphi$ with fiber $f_* Z_{u_0}$ over u_0. This cycle defines in its turn the morphism $\psi = \mathfrak{f} \circ \varphi : C \to C_{m,e}(Y)$. By Proposition 10 the limit of the Chow-points $\mathfrak{f}(u_n)$ is equal to $\lim \psi(c_n)$, the Chow point of $f_* Z_{u_0}$. But this limit is independent of the chosen curve and hence the map $\mathfrak{f}$ is a continuous algebraic map. □

In a similar fashion one can prove (see [5, Prop. 2.9])

Proposition 15 *Let $f : X \to Y$ be a flat morphism of relative dimension $c = \dim X - \dim Y$. Let $V \subset C_{m,d}(Y)$ be a component. If the degree of the flat pull back $f^*[Z_v]$ is e, where $[Z_v]$ is a generic cycle with Chow point*

$v \in V$, the degree of all flat pull backs of cycles with Chow point in V are e and there results a continuous algebraic map

$$V \to C_{m+c,e}(X).$$

2.3 Cycle spaces and their group completions

The next step is to define

$$\mathcal{C}_m(X) = \mathcal{C}_m(X, i) = \coprod_{d \geq 0} C_{m,d}(X)$$

with the topology on each component given by a fixed projective embedding $i : X \hookrightarrow \mathbb{P}$. To show that this topology in fact does not depend on the embedding, one considers another embedding $i' : X \hookrightarrow \mathbb{P}'$ and the Segre embedding $\mathbb{P} \times \mathbb{P}' \to \mathbb{P}''$. This defines an embedding $i'' : X \to \mathbb{P}''$ dominating both embeddings. There are evident bijections $\mathcal{C}_m(X, i'') \to \mathcal{C}_m(X, i)$ and $\mathcal{C}_m(X, i'') \to \mathcal{C}_m(X, i')$ induced by the projections from $P \times P'$ onto its factors. These are continuous algebraic maps. This can be seen as follows. Let X_i, $i = 1, \ldots, N$ resp. Y_j, $j = 1, \ldots, M$ be homogeneous coordinates on $\mathbb{P}$ resp. $\mathbb{P}'$. Then T_{ij}, $i = 1, \ldots, N, j = 1, \ldots, M$ can be taken as homogeneous coordinates on $\mathbb{P}''$ with Segre embedding given by $T_{ij} = X_i Y_j$. If the Chow point of Z in the embedding i'' is given by the form $F(T_{ij}^{(\alpha)})$ separately homogeneous of the same degree d in each of the $m+1$ sets of variables $T_{ij}^{(\alpha)}$, $\alpha = 0, \ldots, m$, we may write $F(X_i^{(\alpha)} Y_j^{(\alpha)})$ as a sum of products $G_s(X_i^{(\alpha)}) H_s(Y_j^{(\alpha)})$, $s = 1, \ldots, S$ such that G_s and H_s are bihomogeneous in each set of variables. The greatest common divisor $G(X_i^{(\alpha)})$ of $G_1, \ldots, G_s$ is a form and this form is the Chow form of Z in the embedding i. The assignment $F \mapsto G$ is algebraic in the sense that the coefficients of G depend rationally on the coefficients of F and hence the bijection $\mathcal{C}_m(X, i'') \to \mathcal{C}_m(X, i)$ is a (birational) morphism and hence a bi-continuous algebraic map. It follows that the induced bijection $\mathcal{C}_m(X, i) \to \mathcal{C}_m(X, i')$ is a bi-continuous algebraic map. It is in particular a homeomorphism and the topology on $\mathcal{C}_m(X, i)$ is therefore independent of the embedding i. In the sequel we therefore omit the reference to i.

From propositions 14 and 15 one immediately gets:

Proposition 16 *A morphism $f : X \to Y$ between projective varieties induces a continuous algebraic map*

$$f_* : \mathcal{C}_m(X) \to \mathcal{C}_m(Y)$$

and if f is flat, there is an induced continuous algebraic map

$$f_* : \mathcal{C}_m(Y) \to \mathcal{C}_{m+c}(Y), \quad c = \dim X - \dim Y.$$

The last step consists in considering the group of possibly non-effective m-cycles on projective variety X.

Definition 17 The *naïve group completion* $\mathcal{Z}_m(X)$ of $\mathcal{C}_m(X)$ consists of considering the set of m-cycles as the topological quotient

$$\mu_m : \mathcal{C}_m(X) \times \mathcal{C}_m(X) \to \mathcal{Z}_m(X)$$

under the equivalence relation $(Z_1, Z_2) \sim (Z_1', Z_2')$ if $Z_1 + Z_2' = Z_1' + Z_2$. In other words $\mu_m(Z_1, Z_2) = Z_1 - Z_2$ a is well-defined m-cycle and any m-cycle can be written as a unique equivalence class.

We already explained how to put a topology on the naïve group-completion. As to functoriality, the preceding Proposition implies:

Corollary 18 *A morphism $f : X \to Y$ between projective varieties induces a group homomorphism which is a continuous map*

$$f_* : \mathcal{Z}_m(X) \to \mathcal{Z}_m(Y)$$

and if f is flat, there is an induced group homomorphism which is a continuous map

$$f_* : \mathcal{Z}_m(Y) \to \mathcal{Z}_{m+c}(Y), \quad c = \dim X - \dim Y.$$

3 Defining Lawson homology

3.1 Simplicial Stuff

Apart from the definition given in the introduction, there are various other equivalent definitions, each having its advantage. These definitions all use the language of *simplicial spaces*, and so we'll first briefly review this.

Recall that the standard p-simplex Δ_p is the convex hull in $\mathbb{R}^{p+1}$ of the $p + 1$ standard unit-vectors

$$\Delta_p = \{(x_0, \ldots, x_p) \mid x_i \geq 0, \sum_i x_i = 1\}.$$

Its boundary consists of the $(p - 1)$-simplices $\Delta_p^q = \Delta_p \cap \{x_q = 0\}$, $q = 1, \ldots, p+1$ inducing the embeddings $\Delta^q : \Delta_{p-1} \to \Delta_p$. Its vertices, the $p+1$

standard unit-vectors, are often identified with elements from the ordered set $\{0, \ldots, p\}$ by the correspondence $i \iff e_i$. The standard p-simplex will also be denoted by $[p]$, which means the *ordered* set $\{0, \ldots, p\}$. Thus, the maps Δ^q are examples of non-decreasing maps $[p-1] \to [p]$. Other examples of non-decreasing maps are the degeneration maps $\sigma^q : \Delta_p \to \Delta_{p-1}$, $q = 0, \ldots, p$ defined by $\sigma^q e_0 = e_0, \ldots, \sigma^q e_q = \sigma^q e_{q+1} = e_q, \sigma^q e_{q+2} = e_{q+1}, \ldots \sigma^q e_p = e_{p-1}$. All non-decreasing maps are obtained upon composing face and degeneracy maps.

The standard simplices and all its face- and degeneracy-maps form a *co-simplicial set*

$$\Delta_0 \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \Delta_1 \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \Delta_2 \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \Delta_3 \cdots$$

Non-decreasing maps form the morphisms of a category Δ whose objects are the standard simplices Δ_n. All information of this category is given by the corresponding co-simplicial set. Dually, a *simplicial set* $K_\bullet$ is a collection of sets $K_0, K_1, \ldots$ together with mappings $K(f) : K_p \to K_q$, one for each nondecreasing map $f : [q] \to [p]$ such that

$$K(\mathrm{id}) = \mathrm{id}, \quad K(g \circ f) = K(f) \circ K(g).$$

In other words, if considering the collection of standard simplices with non-decreasing maps as a category, a simplicial set is just a contravariant functor of this category to the category of sets. So, a simplicial set can be given as a diagram as before, but by reversing the arrows.

$$\cdots K_3 \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} K_2 \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} K_1 \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} K_0$$

The arrows correspond to the face and degeneracy maps, so we have face maps $d_j : K_n \to K_{n-1}$ for $j = 0, \ldots, n$ and degeneracy maps $s_j : K_n \to K_{n+1}$, $j = 0, \ldots, n$. These satisfy certain compatibility relations (the *simplicial identities* to be found in [16], §1, they do not play a role here) which can be used to define a simplicial space directly. Using these, one may write any non-decreasing map in a unique way in the form $s_{j_t} \circ \cdots s_{j_1} \circ d_{i_s} \circ \cdots \circ d_{i_1}$, which makes precise how the simplicial set is completely determined by the data of face and degeneracy maps.

Example 19

1. The complex of singular simplices $S_\bullet(X)$ (with the usual face and degeneracy maps).
2. Any simplicial complex can be viewed as a simplicial set by considering its simplices as non-degenerate simplices and by adding to these the

degenerate simplices obtained by letting the face and degeneracy maps act on these. The simplicial unit interval I, or more generally, the ordinary n-simplices Δ^n thus define simplicial sets denoted $\Delta[n]$. The standard boundary maps $\Delta_j : \Delta^{n-1} \to \Delta^n$ (inclusion of the j-th face) and degeneracy maps $\sigma_j : \Delta^{n+1} \to \Delta^n$ (collapsing by leaving out the j-th vertex) induce simplicial maps $(\Delta_j)_* : \Delta[n-1] \to \Delta[n]$ and $(s_j)_* : \Delta[n+1] \to \Delta[n]$.

Using this language two degree-preserving simplicial maps $f, g : K_\bullet \to L_\bullet$ are *homotopic* if there exists a simplicial map $h : K_\bullet \times \Delta[1] \to L_\bullet$ such that $d_0 h = f$ and $d_1 h = g$.

3. Let SS be the category of simplicial sets. Then for any two simplicial sets $K_\bullet$ and $L_\bullet$ the set
$$\mathrm{Hom}_{\mathrm{SS}}(K_\bullet, L_\bullet)$$
is also a simplicial set. Here we define an n-simplex as a simplicial map $f : K_\bullet \times \Delta[n] \to L_\bullet$, the boundary maps are defined by $d_i f(k,t) = f(k, (\sigma_i)_* t)$ and the degeneracy maps by $s_i f(k,t) = f(k, (\Delta_i)_* t)$, $k \in K_q$, $t \in (\Delta[n])_q$.

4. A *Kan complex* is a simplicial set satisfying the extension property: given exactly $n+1$ simplices $\sigma_0, \sigma_1, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_{n+1}$ whose boundaries match ($d_i \sigma_j = d_{j-1} \sigma_i$), $i < j$, $i \neq k$, $j \neq k$), there exists an $(n+1)$-simplex σ, whose i-th boundary is σ_i. In other words, the simplicial set contains a simplex, if all but one of its faces are already in it. A standard example of a Kan complex is the singular complex (any continuous map defined on all but one of the n-dimensional faces of Δ^{n+1} extends to Δ^{n+1}).

A *simplicial map* from $K_\bullet$ to $L_\bullet$ is a degree preserving map which commutes with face and degeneracy operators. Equivalently, it is a natural transformation of functors.

One can define a *pointed simplicial set* as a simplicial set $K_\bullet$ together with a map $[0] \to K_\bullet$. Standard examples are obtained from subcomplexes $K_\bullet \subset L_\bullet$ by considering for each n the equivalence classes L_n/K_n, where all elements of K_n are made equivalent. This yields the *quotient simplicial complex* $(L/K)_\bullet$ with natural base point the equivalence class of points in $K_\bullet$. The standard example is the *simplicial sphere* $S[n]$ obtained by identifying all $(n-1)$-faces except the last in $\Delta[n]$.

It should be clear what is meant by a *homotopy* between two simplicial maps between (pointed) simplicial sets. This however is *not* an equivalence relation in general. For this reason Kan complexes have been introduced, since homotopy is an equivalence on $\mathrm{Hom}_{SS}(K_\bullet, L_\bullet)$ whenever $L_\bullet$ is a Kan complex (see [16, §6]) and so for these we can introduce the set

$$[K_\bullet, L_\bullet]$$

of equivalence classes of pointed simplicial maps from $K_\bullet$ to $L_\bullet$ under the homotopy relation. So for a Kan complex $L_\bullet$, there are *homotopy groups*

$$\begin{aligned}\pi_n(L_\bullet) &= [S[n], L_\bullet] = \\ &= \{\ell \in L_n \mid d_i\ell = *, \forall i\}/\sim,\end{aligned}$$

where $\ell \sim \ell'$ if there exists some $z \in L_{n+1}$ whose boundary components $dz = (d_0z, \ldots, d_{n+1}z)$ are given by $dz = (*, \ldots, *, \ell, \ell', *, \ldots, *))$. In fact, one can introduce a product in these sets as follows. If $[x], [y] \in \pi_n(L)$, by the Kan property, there exists $v \in L_{n+1}$ such that $dv = (x, ?, y, *, \ldots, *)$ and one sets $[xy] = [?]$. This does yield a group structure for $n \geq 1$, which is abelian if $n > 1$ (see [16, §4]).

The functor which to any topological space X associates the simplicial set $S_\bullet(X)$ of singular simplices has a natural adjoint functor, the geometric realization functor. This functor associates to any simplicial set $K_\bullet$ the topological space

$$|K| = \left(\coprod_{p=0}^{\infty} \Delta_p \times K_p\right) \Big/ R,$$

where the equivalence relation R is generated by identifying $(s, x) \in \Delta_q \times K_q$ and $(f(s), y) \in \Delta_p \times K_p$ if $x = K(f)y$ and $f : \Delta_q \to \Delta_p$ is any non-decreasing map. The topology on $|K|$ is the quotient topology under R obtained from the direct product topology, where the sets K_p are given the discrete topology. Observe that $|K|$ has a natural structure as a CW-complex.

Let us make explicit that the geometric realization functor is adjoint to the functor of singular simplices. Given a simplicial complex $K_\bullet$ and a topological space X, there are natural bijections

$$(*)\begin{cases} \mathrm{Hom}_{\mathrm{SS}}(K, S_\bullet X) & \xrightarrow{\phi} \quad \{\text{ Continuous maps } |K| \to X\} \\ \{\text{ Continuous maps } |K| \to X\} & \xrightarrow{\psi} \quad \mathrm{Hom}_{\mathrm{SS}}(K, S_\bullet X) \end{cases}$$

given by $\phi(f)(t, k) = f(k)t$ and $\psi(g)(k)t = g(t, k)$. These preserve homotopies and so in particular, taking for K the simplicial n-sphere $S[n]$, one has

Lemma 20 *For any topological space X, the bijection* (*) *induces an isomorphism*

$$\pi_n(X) \cong \pi_n(S_\bullet X).$$

While this assertion is quite straightforward (see [16, §16]), the fact that for a Kan complex $K_\bullet$ the analogous isomorphism

$$\pi_n(K_\bullet) \cong \pi_n(|K|)$$

holds, is more difficult. See loc. cit.

Even more is true. The above bijections induce adjunction morphisms

$$\begin{array}{ll} K_\bullet & \to S_\bullet|K| \\ X & \to |S_\bullet X| \end{array}$$

(take $K = S_\bullet X$ and the identity, resp. $X = |K|$ and the identity in (*) above). These induce isomorphisms on the level of homotopy (see [19, §8.6]. We have seen that a weak homotopy equivalence between CW-complexes is a homotopy equivalence. In particular, any Kan complex K has the same homotopy type as the singular complex of its geometric realization (i.e. $K_\bullet$ is homotopy equivalent to $S_\bullet(|K|)$), and any CW-complex X has the same homotopy type as the geometric realization of its associated singular complex (i.e X is homotopy equivalent to $|S_\bullet X|$). So, in homotopy theory CW-complexes can be replaced by Kan complexes.

Simplicial sets are very flexible. For instance the K_p could have extra structure, i.e. they could be topological spaces, complex varieties, groups, monoids etc. In fact, a *simplicial object in a category* $\mathfrak{C}$ is a contravariant functor from the category Δ of standard simplices to the category $\mathfrak{C}$. We then speak of a *simplicial topological space, simplicial complex variety, simplicial group, simplicial monoid* etc.

A simplicial group is a Kan complex [19, Theorem 8.3.1.] Its homotopy groups π_i are all abelian for $i \geq 1$; in fact this is true for a Kan monoid complex [16, Prop. 17.3.]. The homotopy groups can be calculated as the homology groups of a (not necessary Abelian) chain complex the *normalized chain complex*

$$NG_\bullet = \{\cdots \to NG_n \xrightarrow{d_0} NG_{n-1} \xrightarrow{d_0} \cdots NG_2 \xrightarrow{d_0} NG_1\},$$

where $NG_n \triangleleft G_n$ is the intersection of the kernel of all face maps except d_0. Since $\mathrm{Im} d_0$ is a normal subgroup of $\mathrm{Ker} d_0$ one can form the quotient group $H_n(NG_\bullet, d_0)$. One has [19, Theorem 8.3.2]

$$\pi_n(G_\bullet) = H_n(NG_\bullet, d_0).$$

In case $G_\bullet$ is abelian, the normalized chain complex is a subcomplex of a complex, naturally associated to the simplicial set, and denoted by the same letter:

$$G_\bullet = \{\cdots G_p \xrightarrow{\partial_p} G_{p-1} \xrightarrow{\partial_{p-1}} G_{p-2} \cdots \xrightarrow{\partial_1} G_0 .\}$$

where $\partial_p = \sum_{q=0}^{p}(-1)^q G(i_q)$. The inclusion of $NG_\bullet \subset G_\bullet$ is a chain homotopy [19, Theorem 8.5.1], and so, for *Abelian* groups

$$H_n(G_\bullet, \partial) \cong H_n(NG_\bullet, d_0) \cong \pi_n(G_\bullet).$$

This is useful if one wants to define long exact sequences of homotopy groups.

Lemma 21 *Let $H_\bullet \subset G_\bullet$ be an inclusion of simplicial abelian groups. The quotients G_n/H_n then form a simplicial abelian group $(G/H)_\bullet$ and there is a long exact sequence*

$$\cdots \pi_p(H_\bullet) \to \pi_p(G_\bullet) \to \pi_p((G/H)_\bullet) \to \pi_{p-1}(H_\bullet) \to \cdots$$

Proof: The exact sequence of simplicial abelian groups

$$0 \to H_\bullet \to G_\bullet \to (G/H)_\bullet \to 0$$

induces a short exact sequence for the normalized chain complexes. The long exact sequence in homology then gives the result. □

Example 22 To any simplicial set $K_\bullet$ one associates the simplicial Abelian group $\mathbb{Z}K_\bullet$, obtained upon replacing K_n by the free abelian group on K_n (and the naturally induced face and degeneracy maps). This makes it possible to define *homology groups* for simplicial sets:

$$H_n(K_\bullet) = H_n(\mathbb{Z}K_\bullet).$$

Clearly, the complex of singular homology on a space X is just the complex defined by the simplicial abelian group associated the simplicial set $S_\bullet(X)$ of singular simplices. So its homology is singular homology of X. One can further show that the adjunction morphism $K_\bullet \to S_\bullet|K|$ and its inverse $|S_\bullet X| \to X$ induce homology isomorphisms ([19, Theorem 8.5.5]) so that there is no ambiguity when speaking of homology of simplicial sets and topological spaces.

Let $\mathfrak{C}$ be any category. Its *classifying space* is the simplicial space given by

$$B\mathfrak{C}_n = \text{set of strings of morphisms}\{a_0 \to a_1 \to \cdots a_n\}$$

with face map d_i defined by leaving out a_i, replacing $a_{i-1} \to a_{i+1}$ by the composition of the arrows $a_{i-1} \to a_i$ and $a_i \to a_{i+1}$; the degeneracy map s_j is defined by inserting a_j and id_{a_j} between a_j and a_{j+1}. See [18].

This can be applied to obtain the classifying space for any group G by regarding it as a category with one object $*$ and whose morphisms are given by the group elements, multiplication defining composition of morphisms. This yields BG where

$$(BG)_p = \underbrace{G \times \cdots \times G}_{p \text{ times}}$$

and for any non-decreasing $f : [q] \to [p]$ one has

$$BG(f)(g_1, \ldots, g_p) = (h_1, \ldots, h_q), \quad h_i = \prod_{j=f(i-1)+1}^{f(i)} g_j \quad (= e \text{ if } f(i-1) = f(i)).$$

In terms of face and degeneracy maps this comes indeed down to

$$\begin{array}{rcl} d_i(g_1, \ldots, g_p) & = & (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_p) \\ s_i(g_1, \ldots, g_p) & = & (g_1, \ldots, g_{i-1}, e, g_{i+1}, \ldots, g_{p-1}). \end{array}$$

This pointed simplicial set has indeed the desired property that $\pi_1(BG) = G$ and $\pi_i(BG) = 0$ as in the case of the ordinary classifying space.

If instead we take a simplicial group we have to modify this as follows:

$$\begin{array}{rcl} (BG)_p & = & G_{p-1} \times \cdots \times G_0, \quad p \geq 1 \\ (BG)_0 & = & *, \quad \text{one point.} \end{array}$$

We leave the determination of face and degeneracy operators as an exercise for the reader.

Note that the notation $(BG)_\bullet$ is consistent in that it gives back the old construction for a simplicial group associated to a group. It can be proved that this simplicial set plays the role of the classifying space in the set-up of Kan complexes. See [16, §21].

Note also that the simplicial complexes $K_\bullet = (BG)_\bullet$ thus obtained are *reduced*, meaning that K_0 is a single point. There is an adjoint functor from reduced complexes $K_\bullet$ to group complexes which plays the role of the loop-space functor and which is defined as follows. One sets

$$\begin{array}{rcl} (\Omega K)_n & = & (\text{free group on } K_{n+1}) \,/ \sim, \\ s_0 x & \sim & 1, \ \forall x \in K_n \end{array}$$

and one defines face and degeneracy maps on the generators as follows (here $[-]$ denotes the class of $- \in K_{n+1}$ in $(\Omega K)_n$ and $x \in K_{n+1}$)

$$d_0[x] \quad = \quad [d_0 x]^{-1}[d_1 x]$$

$$\begin{aligned} d_i[x] &= [d_{i+1}x], \quad i \geq 1 \\ s_i[x] &= [s_{i+1}x], \quad i \geq 0. \end{aligned}$$

Of course $(\Omega K)_n$ is a free group, and the maps above extend uniquely to group homomorphisms, making $(\Omega K)_\bullet$ into a simplicial group.

See [16, §27] for a proof that Ω and B are adjoint functors. Kan has shown (see [12]):

Lemma 23 *The adjunction morphism*

$$\Psi(G_\bullet) : \Omega BG_\bullet \to G_\bullet.$$

is a homotopy equivalence.

Note also that the B-construction makes sense for any simplicial monoid $M_\bullet$, since one only needs multiplication and a unity for the formulas to make sense. In the sequel, we'll assume that the monoid-law is *abelian* and we'll write it additively.

Definition 24 Let $M_\bullet$ be a simplicial abelian monoid. Its *homotopy theoretic group completion* is $(\Omega BM)_\bullet$. Its *naïve group completion* $M_\bullet^+$ is built from the naïve group completions M_n^+ of the constituent monoids M_n.

The naïve group completion imitates the construction from $\mathbb{Z}$ out of the natural numbers. So M_n^+ consists of pairs $(x, y) \in M_n^+$ modulo the equivalence relation $(x, y) \sim (x', y')$ if $x + y' = x' + y$. Clearly, the monoid operation induces one on M_n^+, making it into an abelian group. Also, face and degeneracy maps uniquely extend to give $M_\bullet^+$ the structure of a simplicial abelian group. Moreover, the natural injective monoid morphisms $M_n \to M_n^+$ given by $x \mapsto (x, 0)$ extend to

$$i : M_\bullet \to M_\bullet^+, \quad (\text{the plus morphism}).$$

The plus morphism induces a natural homomorphism of simplicial abelian groups

$$u : \Omega BM_\bullet \xrightarrow{\Omega B(i)} \Omega BM_\bullet^+ \xrightarrow{\Psi} M_\bullet^+.$$

Quillen has shown (see [9, Appendix Q]) that the map

$$BM_\bullet \xrightarrow{B(i)} B(M_\bullet^+)$$

is a homotopy equivalence. Since $\Omega B(i)$ is then also a homotopy equivalence, and one knows already that Ψ is a homotopy equivalence, this holds likewise for the composition u. So in homotopy theory there is no difference between the naïve group completion and the homotopic theoretic group completion. Summarizing, one has

Proposition 25 *Let $M_\bullet$ be a simplicial abelian monoid. There is natural homotopy equivalence $\Omega BM_\bullet \to M_\bullet^+$ between the the homotopy theoretic group completion and the naïve group completion. This holds in particular for abelian monoids themselves and for the simplicial monoid of singular simplices $S_\bullet X$ of a topological space X.*

3.2 Base systems

The fundamental idea behind this is that one wants to glue together the components of a topological Abelian monoid M by choosing base points in each connected component and using the addition for the gluing procedure. Formally, a *base system* for M is a pair (I, b) consisting of a set I and a map $b : I \to M$ such that every connected component of M contains at least one point in the image of b. The free group $F(I)$ has a natural partial ordering given by $\lambda \leq \mu$ if $\exists \nu \in F(I)$ with $\mu = \lambda + \nu$. This ordering can be used to define the *associated directed monoid* $\overrightarrow{M}_b$ by taking one copy $M = M_\lambda$ for each element $\lambda \in F(I)$ and by defining $b_{\lambda\mu} : M_\lambda \to M_\mu$ by $x \mapsto x + b(\nu)$. This map is a continuous base point preserving map (but it does not preserve the addition). The topological space

$$\varinjlim_{b} \overrightarrow{M}_b$$

is defined to be the infinite mapping telescope obtained by taking the disjoint union of the $M_\lambda \times I$ and identifying $(x, 1)$ with $(b_{\lambda\mu}(x), 0)$. It can be viewed as the limit space associated to the directed monoid.

Identifying M with M_0, where $0 \in F(I)$ is the zero element, there is a natural map

$$i_b : M \to \overrightarrow{M}_b$$

which induces a map to the mapping telescope and which will be denoted by the same letter.

Example 26 Let X be a connected topological space and let $X^{(d)}$ be its d-th symmetric power. Let $*$ be a base point in X and use $*^d$ as the base point in $X^{(d)}$. One takes $I = \mathbb{N}$ and one defines $b(d) = *^d$. Using the inclusions $X^{(e)} \hookrightarrow X^{(d+e)}$ defined by $[x] \mapsto [(x, *^d)]$ one builds an inductive system whose limit is $X^{(\infty)}$. The disjoint union

$$X^{[\infty]} := \coprod_{d \geq 0} X^{(d)}$$

is an abelian monoid and the choice of base points induces the structure of a directed monoid $\overrightarrow{X}_*$ whose limit is exactly $X^{(\infty)}$.

The classical Dold-Thom theorem [3] states:

Theorem 27 *Let X be a CW-complex X. There are natural isomorphisms*

$$\pi_k(X^{(\infty)}) \xrightarrow{\sim} H_k(X)$$

such that the Hurewicz map $\pi_k(X) \to H_k(X)$ is obtained after composing this isomorphism with the homomorphism $\pi_k(X) \to \pi_k(X^{(\infty)})$ induced by the obvious inclusion $X \to X^{(\infty)}$.

Let us next define the *homotopy groups of a directed monoid* $\overrightarrow{M}_b$ by first applying the homotopy functor to the directed monoid and then taking the direct limit of the associated direct system.

$$\pi_k(\overrightarrow{M}_b) := \varinjlim_{\lambda} \pi_k(M_\lambda).$$

Of course this equals the homotopy group $\pi_k(\varinjlim_{b} \overrightarrow{M}_b)$ of the corresponding mapping telescope, but one rarely uses this.

Let us next compare $\overrightarrow{M}_b$ with the associated singular simplicial directed set $\overrightarrow{S_\bullet M}_b$. Again, it is clear that the functor S applied to the directed monoid yields a direct system of simplicial sets and one may form

$$\varinjlim_{b} \overrightarrow{S_\bullet M}_b,$$

its direct limit. Also, the map $i_b : M \to \overrightarrow{M}_b$ induces the monoid homomorphism

$$S(i_b) : S_\bullet(M) \to \varinjlim_{b} \overrightarrow{S_\bullet M}_b$$

and the result from [9, §2.7]states that this in fact up to homotopy is the +-construction (naïve group completion). Note that the monoid structure on M induces one on $S_\bullet M$ and it is this monoid structure that is meant to be completed. To define the comparison map, let $b_n(\lambda) \in S_n(M)$, $\lambda \in F(I)$ be the totally degenerate n-simplex at the point $b(\lambda)$. Then define

$$\begin{array}{rcl} S_n(M_\lambda) & \to & (S_n M)^+ \\ s & \mapsto & s - b_n(\lambda). \end{array}$$

This induces

$$h : \varinjlim_{b} \overrightarrow{S_\bullet M}_b \to (S_\bullet M)^+$$

and one has

Proposition 28 *In following commutative diagram*

$$\begin{array}{ccc} S_\bullet M & \xrightarrow{\quad S(i_b) \quad} & \varinjlim_b \overrightarrow{S_\bullet M_b} \\ & \searrow \qquad \swarrow{\scriptstyle h} & \\ & (S_\bullet M)^+ & \end{array}$$

the map h is a homotopy-equivalence.

The proof of this result uses the full strength of Quillen's result reproduced in [9, Appendix Q].

3.3 Tractable monoids

One final comparison has to be made; one needs to know as to what sense the plus construction and the simplicial functor commute. To explain this, one has to remark that the naïve group completion of an abelian monoid M is the quotient of $M \times M$ under the diagonal action of M and there is a natural map

$$S_M : S_\bullet(M \times M)/S_\bullet M \to S_\bullet((M \times M)/M).$$

Suppose that one can prove that this is a homotopy-equivalence. Then one can combine it with the next easy Lemma to conclude that the natural map $(S_\bullet M)^+ \to S_\bullet(M^+)$ is a homotopy equivalence as well.

Lemma 29 *$S_\bullet(M \times M)/S_\bullet(M)$ is naturally isomorphic to $(S_\bullet M)^+$.*

Proof of the Lemma: Let us map $S_\bullet(M \times M)$ to $S_\bullet M \times S_\bullet M$ by sending $u \in S_n(M \times M)$ to $(p_1 u, p_2 u) \in S_n(M) \times S_n(M)$, where p_1 and p_2 are induced by the projections. Since the diagonal action of $S_\bullet M$ on both sides is equivariant with respect to this map, there is an induced map between the quotients. This is a homomorphism between simplicial groups. The inverse is induced from the map $S_\bullet M \to S_\bullet(M \times M)$ which sends s to $s \times 0$. This map can be extended in a natural way to a map $(S_\bullet M)^+ \to S_\bullet(M \times M)/S_\bullet M$ by sending (s,t) to $s \times t$. That this is the sought after inverse follows from the observation that, letting 0_n, the totally degenerate n-simplex at $0 \in M$ act on a singular n-simplex $s : \Delta_n \to M \times M$, yields the (degenerate) $2n$-simplex $\pi_1 s \times \pi_2 t : \Delta_{2n} \to M \times M$. □

So, more generally, for a monoid M acting on a topological space T, one needs to compare the singular simplicial set of a quotient space T/M and the quotient $S_\bullet T/S_\bullet M$. The notion of *tractable monoid-action* of M on a topological space T is conceived just so that the natural map

$$S_\bullet T/S_\bullet M \to S_\bullet(T/M)$$

induces isomorphisms on homotopy groups. To explain tractability we'll recall an auxiliary notion, that of a particular kind of inclusion, a *cofibration* $i : A \hookrightarrow X$. This means by definition that, given a map $\tilde{f} : X \to Y$ whose restriction $f = \tilde{f}|A$ fits into a homotopy $H : A \times I \to Y$ (i.e. $f(x) = H(x, 0)$), fits itself in a homotopy $\tilde{H} : X \times I \to Y$ extending H.

Definition-Lemma 30 An abelian monoid M *acts tractably* on a topological space T, if there is a filtration $T_0 \subset T_1 \cdots \subset T_n \subset$ whose topological union is T and such that there are cofibrations $R_n \subset S_n$ such that for $n \geq 1$ the inclusions $T_{n-1} \subset T_n$ fit in a pushout square (or fiber co-product)

$$\begin{array}{ccc} R_n \times M & \hookrightarrow & S_n \times M \\ \downarrow & & \downarrow \\ T_{n-1} & \hookrightarrow & T_n. \end{array}$$

If the cofibrations $R_n \subset S_n$ are relative CW-complexes, we say that (T, M) is a *tractable CW-space.* For these T/M is a CW-complex.

If the diagonal action of M on $M \times M$ is tractable, one says that *the monoid M is tractable.*

If M acts tractably on T and if M has the "cancellation property" ($sx = sy$ implies $x = y$), the natural map

$$S_\bullet T/S_\bullet M \to S_\bullet(T/M)$$

induces isomorphisms on homotopy, and so since a a tractable monoid M has the cancellation property, the natural map

$$(S_\bullet M)^+ \to S_\bullet(M^+)$$

induces isomorphisms on homotopy.

The proof of this is not hard. It can be found in [7, Theorem 1.4]. See also [6, Lemma 1.3].

Remark. The terminology tractable monoids is just another way to call certain properties of monoids introduced in [15].

Example 31 Let X be a projective variety and $M = \mathcal{C}_m(X)$ be the monoid of effective algebraic m-cycles with the Chow topology. Let us verify that this is a tractable monoid. So one considers $T = M \times M$ with the diagonal action of M. One takes M_d to be the degree d cycles (with respect to a fixed projective embedding). Now fix some bijection $\nu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and set

$$\begin{array}{rc} T_n := & \left[\bigcup_{\nu(a,b)\leq n} M_a \times M_b\right] \cdot M \\ S_n := & M_{a_n} \times M_{b_n}, \quad \nu(a_n, b_n) = n \\ R_n := & \mathrm{Im}\left[\bigcup_{c>0} M_{a_n-c} \times M_{b_n-c} \times M_c \to M_{a_n} \times M_{b_n}\right]. \end{array}$$

It is easily verified that these fit into a push-out diagram as above. One can now inductively provide S_n with a semi-algebraic triangulation so that $R_n \subset S_n$ is a subcomplex. Here one uses that any projective algebraic variety can be triangulated by semi-algebraic simplices in such a way that any given finite union of semi-algebraic closed subsets figures as a subcomplex (see [11]) together with the fact that the image of a semi-algebraic map under a continuous algebraic map (such as the multiplication maps which define R_n) stays semi-algebraic. This also shows that $R_n \subset S_n$ is relative CW-complex and so $\mathcal{Z}_m(Y)$ is a tractable CW-space.

3.4 Application to Lawson homology

Let us apply the results of the previous two sections to the topological monoid $\mathcal{C}_m(X)$ of algebraic m-cycles on a projective variety X, where one puts the Chow topology on $\mathcal{C}_m(X)$. One chooses any base system $b : I \to \mathcal{C}_m(X)$ (see §3.2).

Collecting the results from the previous sections, one gets

Theorem 32

1) *There is a natural homotopy equivalence*

$$\Omega B\mathcal{C}_m X \to \mathcal{Z}_m X$$

from the homotopy theoretic group completion $\Omega B\mathcal{C}_m X$ *of* $\mathcal{C}_m X$ *to the group* $\mathcal{Z}_m X$ *of algebraic* m*-cycles with the Chow topology.*

2) *There are natural homotopy equivalences*

$$\varinjlim_b \overrightarrow{S_\bullet \mathcal{C}_m X}_b \to (S_\bullet(\mathcal{C}_m X))^+ \leftarrow S_\bullet(\mathcal{Z}_m X).$$

3) *There are natural isomorphisms*

$$\pi_k \overrightarrow{S_\bullet(\mathcal{C}_m X)}_b \xrightarrow{\sim} \pi_k \left(S_\bullet(\mathcal{C}_m X)\right)^+ .$$

and hence natural isomorphisms

$$\pi_k\overrightarrow{\mathcal{C}_m X}_b \xrightarrow{\sim} \pi_k\left(S_\bullet(\mathcal{C}_m X)\right)^+ \xleftarrow{\sim} \pi_k(\mathcal{Z}_m X) \xleftarrow{\sim} \pi_k(\Omega B\mathcal{C}_m X).$$

Proof:

1. This follows directly from Proposition 25. It is stated explicitly in [15].

2. The first homotopy equivalence is Proposition 28. The second homotopy equivalence follows from the fact that $\mathcal{C}_m X$ is a tractable monoid (Example 31) and from Definition-Lemma 30.

3. This follows from the fact that direct limits commute with homotopy groups.

The last assertions follows from the previous assertions together with the fact that the homotopy groups of a topological space are isomorphic to those of the associated simplicial set of singular simplices (Lemma 20). □

Example 33 Continuing with Example 26, let us look at a complex projective variety X. The group of zero-cycles $\mathcal{Z}_0(X)$ is the naïve group completion of the abelian monoid $X^{[\infty]}$ of effective zero-cycles. Since $\pi_k(\mathcal{Z}_0(X)) \cong \pi_k(\overrightarrow{\mathcal{C}_0 X}_b) \cong \pi_k(X^{(\infty)})$, the classical Dold-Thom theorem can thus be reinterpreted as the existence of a canonical isomorphism

$$\pi_k(\mathcal{Z}_0(X)) \xrightarrow{\sim} H_k(X).$$

References

[1] Almgren, F.: Homotopy groups of the integral cycle groups, Topology **1** (1962), 257–299.

[2] Barlet, D.: Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complex de dimension finie, in *Fonctions de Plusieurs Variables Complexes II*, Lect. Notes in Math **482** (1975), 1–158.

[3] Dold, A., R. Thom: Quasifaserungen und unendliche symmetrische Produkte, Ann. Math. **67** (1958), 239–281.

[4] Federer and W. Fleming, Normal and integral currents, Ann. of Math. **72** (1960), 458–520.

[5] Friedlander, E.: Algebraic cycles, Chow varieties, and Lawson homology, Comp. Math. **77** (1991), 55–93.

[6] Friedlander, E.: Algebraic cocycles on normal, quasi-projective varieties, Comp. Math. **110** (1998), 127–162.

[7] Friedlander, E., O. Gabber: Cycle spaces and intersection theory, Topological Methods in Modern Mathematics (Austin, Texas), Publish or Perish (1993), 325–370.

[8] Friedlander, E., B. Lawson: A theory of algebraic cocycles, Ann. Math **136**(1992), 361–428.

[9] Friedlander, E., B. Mazur: Filtration on the homology of algebraic varieties, Memoirs of the A.M.S. **529**(1994).

[10] Fulton, W.: *Intersection theory*, Springer Verlag (1984).

[11] Hironaka, H.: Triangulations of algebraic sets, Proc. Symp. Pure Math., A.M.S. **29** (1976), 165–18.

[12] Kan, D.: On homotopy theory and c.s.s. groups, Ann. Math. **68** (1958), 282–312.

[13] Lawson, B.: Algebraic Cycles and Homotopy Theory, Ann. Math. **129** (1989), 253–291.

[14] Lawson, B.: Spaces of algebraic cycles, in *Surveys in Differential Geometry* **2**, International Press (1995), 137–213.

[15] Lima-Filho, P.: Completions and fibrations for topological monoids, Trans. Amer. Math. Soc. **340** (1993), 127–147.

[16] May, P.: Simplicial objects in algebraic topology, Van Nostrand (1967).

[17] Munkres, E.: *Elementary differential topology*, Princeton University Press, Princeton, Ann. Math. Studies **54** (1963).

[18] Segal, G.: Classifying spaces and spectral sequences, Publ. Math., Inst. Hautes Étud. Sci. **34** (1968), 105–112.

[19] Selick, P.: *Introduction to homotopy theory*, Fields Inst. Monographs, A.M.S, Providence R.I. (1997).

[20] Spanier, E.: *Algebraic Topology*, Springer Verlag (1966).