Generalized Weil Intermediate Jacobians
Talk in Torino, March 23 2012

C. Peters; report of joint work with Müller-Stach and V. Srinivas[1]

[1] See http://arxiv.org/abs/1105.4108

Situation

(X, g) compact, oriented Riemannian manifold, $\dim X = 2n$.

Notation

- $H^k(X) := H^k(X; \mathbb{R})$;
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Metric data

$$* : H^k(X) \to H^{2n-k}(X) \quad \text{Hodge star-operator, } *^2 = (-1)^k.$$

Example 1

$n = 1 \implies X$ is a topological oriented compact surface, say of genus g. We have

$$\begin{array}{ccccc} \mathrm{Conf}(X) = \left\{ \begin{array}{c} \text{conf. eq. classes} \\ \text{of metrics} \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{c} \text{oriented compl.} \\ \text{structures} \end{array} \right\} \\ \in & & \in \\ [g] & \mapsto & X_g . \end{array}$$

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One then has

- jacobian of $X_g := H^1(X)/H^1(X)_{\mathbb{Z}}$ as real torus;
- $*$ (depends on g) gives the complex structure (note $*^2 = -\mathrm{id}$);
- the intersection pairing $\langle , \rangle$ gives the (principal) polarization.

Example 2

Now assume $\dim X = n = 2m + 1$ and set

$$J^n(X, g) = \{H^n(X)/H^n(X)_{\mathbb{Z}}, *, \langle , \rangle\}.$$

This is Lazzeri's jacobian, again a principally polarized abelian variety.

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Extend it to other odd cohomology groups? Needs to take $H^k(X) \oplus H^{2n-k}(X)$ which is preserved by $*$. On this space $*^2 = -\mathrm{id}$. Also the intersection-pairing is non-degenerate and skew on $H^k(X)_{\mathbb{Z}} \oplus H^{2n-k}(X)_{\mathbb{Z}}$ and so we can introduce the principally polarized abelian variety:

$$J^k(X, g) := \{H^k(X) \oplus H^{2n-k}(X)/(H^k(X) \oplus H^{2n-k}(X))_{\mathbb{Z}}, \langle , \rangle\}$$
$$k \text{ odd and } k \neq n.$$

Remarks

Let (X, λ) be a a projective polarized manifold. Then $J^k(X)$ is NOT Weil jacobian J_W^k. For example, if $n = 3$ and $k = 1$ the Weil jacobian is

$$\begin{aligned} J_W^1 &:= \{H^1(X)/H^1(X)_{\mathbb{Z}}, C, B\}; \\ C|H^{p,1-p} &= \mathrm{i}^{2p-1}, p = 0, 1 \text{ (Weil operator)}; \\ B(x, y) &= \langle L^2 x, y\rangle, L = c_1(\lambda) \text{ (Riemann form)}. \end{aligned}$$

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and C and $*$ can be compared as follows:

$$\begin{array}{ccc} H^1(X) & \xrightarrow[*]{\simeq} & H^5(X) \\ \downarrow C & & \| \\ H^1(X) & \xrightarrow[-\frac{1}{2}L^2]{\simeq} & H^5(X). \end{array}$$

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We see that:

$$J^1(X, g) \overset{\text{(isogeneous)}}{\sim} \{\text{Weil Jacobian for } H^1\} \times \{\text{dual jacobian}\}.$$

Moduli aspect

One has a “period map”

$$\begin{array}{ccc} \mathrm{Conf}(X) & \xrightarrow[p]{\hspace{2cm}} & \mathbb{H}_{b_k} \\ \in & & \in \\ [g] & \longmapsto & J^k(X_g). \end{array} \qquad b_k = \dim H^k(X)$$

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Example: X a genus g curve. $\mathrm{Diff}(X)^0$ (diffeo's isotopic to id_X) acts on $\mathrm{Conf}(X) \implies \mathcal{T}_g = \mathrm{Conf}(X)/\mathrm{Diff}(X)^0$, Teichmüller space. The Teichmüller group Γ_g acts on $\mathcal{T}_g \implies M_g = \Gamma_g \backslash \mathcal{T}_g$, moduli space M_g:

$$\begin{array}{ccc} \mathrm{Conf}(X) & \xrightarrow[p]{\hspace{2cm}} & \mathbb{H}_g \\ \downarrow & & \| \\ \mathcal{T}_g = \mathrm{Conf}(X)/\mathrm{Diff}(X)^0 & \xrightarrow[p]{\hspace{2cm}} & \mathbb{H}_g \\ \downarrow & & \downarrow \\ M_g = \Gamma_g \backslash \mathcal{T}_g & \hookrightarrow & \mathrm{Sp}(g)_{\mathbb{Z}} \backslash \mathbb{H}_g . \end{array}$$

Even cohomology

Now $\dim X = 2n$ with $n \equiv 1 \mod 2$, say $n = 2m+1$.

Modified topological data One has an involution ι on *even cohomology* $H^+(X)_{\mathbb{Z}}$:

$$\iota|H^{4*} = \mathrm{id}, \quad \iota|H^{4*+2} = -\mathrm{id}$$

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Modified complex structure On $H^{2k}(X) \oplus H^{2n-2k}(X)$ one takes $J(\alpha, \beta) = (*\alpha, -*\beta)$. Then $J^2 = -\mathrm{id}$. We get the principally polarized abelian variety (PPAV):

$$J^{2k}(X, g) := H^{2k}X) \oplus H^{2n-2k}(X)/(H^{2k}(X) \oplus H^{2n-2k}(X))_\mathbb{Z}$$

polarization: ω^+.

Generalization

Proposition

Let V *an* $\mathbb{R}$*-space with metric* b *and symplectic form* ω*. There is a unique complex structure* J *with*

1. $b(Jx, Jy) = b(x, y)$ *for all* $x, y \in V$*;*
2. $\omega(Jx, Jy) = \omega(x, y)$ *for all* $x, y \in V$*;*
3. *the form* $b_{\omega,J}$ *defined by* $b_{\omega,J}(x, y) := \omega(x, Jy)$ *is a definite) metric.*

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Remark

In general $b_{\omega,J} \neq b$.

Terminology $b_{\omega,J} = c \cdot b, c \in \mathbb{R}_+ \implies (b, \omega)$ a coherent pair.

Associated PPAV

Remark

$$V = V_{\mathbb{Z}} \otimes \mathbb{R} \qquad \text{and } \omega | V_{\mathbb{Z}} \times V_Z \to \mathbb{Z} \text{ unimodular}$$

$$\implies J(V, b, \omega) := (V/V_Z, \omega|V_Z) \text{ is a PPAV.}$$

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Example (X, g) as before; the pair

$$b = b^{(g)}(\alpha, \beta) = \langle \alpha, *\beta \rangle \text{ (Hodge metric)} ; \omega = \omega^+.$$

is coherent. In fact $J(H^+, b^{(g)}, \omega^+) = \prod J^{2p}(X, g)$.

Twists by isomorphisms

$$H^+(X)\xrightarrow[\tau]{\sim}H^+(X) \quad ; \quad H^+(X)_{\mathbb{Q}}\xrightarrow[\gamma]{\sim}H^+(X)_{\mathbb{Q}}$$
$$b^{(g)}_\tau(\alpha,\beta) = b^{(g)}(\tau\alpha,\tau\beta) \quad ; \quad \omega^+_\gamma(\alpha,\beta) = \omega^+(\gamma\alpha,\gamma\beta).$$

1) The pair $(b^{(g)}_\tau, \omega^+_\gamma)$ need not be coherent.
2) The resulting jacobian need not be principally polarized.

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1) The pair $(b^{(g)}_\tau, \omega^+_\gamma)$ need not be coherent.
2) The resulting jacobian need not be principally polarized.
Example (Coherent pair, PAV): Take a unit $\mathbf{a} = \mathbf{b}^2 \in H^{4*}(X)_{\mathbb{Q}}$ and set

$$
\begin{aligned}
\omega^+_{\mathbf{a}}(\alpha,\beta) &:= \langle \mathbf{a}, \alpha\cdot\iota(\beta)\rangle = \omega^+(\mathbf{b}\alpha,\mathbf{b}\beta) \\
b^{(g)}_{\mathbf{b}}(\alpha,\beta) &:= b^{(g)}(\mathbf{b}\alpha,\mathbf{b}\beta).
\end{aligned}
$$

Put $\mathbf{b} = \sqrt{\mathbf{a}} \implies J(H^+(X), b^{(g)}_{\sqrt{\mathbf{a}}}, \omega^+_{\mathbf{a}})$ is a PAV but PPAV?

K-groups: basics

For any "good" topological space X the K-group of virtual vector bundles on X is:

$$\begin{array}{rcl} K(X): & = & \{\mathbb{Z}[\text{iso-classes of } \mathbb{C}\text{-vect. bundles on } X]/\sim\} \\ \text{where} & & [E \oplus F] \sim [E] + [F]. \end{array}$$

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For a complex vector bundle F define the Chern character $\mathrm{ch}(F) \in H^{2*}(X; \mathbb{Q})$ by

$$\mathrm{ch}(F) = \sum e^{\gamma_i} = m + c_1(F) + \tfrac{1}{2}(c_1^2(F) - c_2(F)) + \cdots$$

where

$$1 + c_1(F)x + \cdots c_m(F)x^m = (1 + \gamma_1 x) \cdots (1 + \gamma_m x)$$

Jacobians for K-groups

Theorem (Atiyah-Hirzebruch)

Assume X is torsion free ($H^*(X;\mathbb{Z})$ no torsion).
(1) The Chern character gives an injection

$$ch : K(X) \longrightarrow H^+(X), \quad \Lambda(X) := \mathrm{Im}(ch) \text{ a lattice.}$$

(2) There exist multipliers $\mathbf{a} = 1 + \mathbf{a}_2 + \cdots \in \Lambda(X)$, i.e with $\langle \mathbf{a}\alpha, \beta\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Lambda(X)$;
(3) If $\mathbf{a}$, multiplier with $\mathbf{a} \in H^{4*}(X) \implies \omega^+_{\mathbf{a}}$ unimodular.

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As a consequence, for a multiplier $\mathbf{a}$ we have a PPAV

$$(H^+(X)/\Lambda(X), b^{(g)}_{\sqrt{\mathbf{a}}}, \omega^+_{\mathbf{a}})$$

Jacobians for K-groups

S. Müller-Stach, C. Peters, V. Srinivas

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As a consequence, for a multiplier $\mathbf{a}$ we have a PPAV

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Note also that there is a complex conjugation on $K(X)$ and

$$\omega_{\mathbf{a}}^+(\mathrm{ch}(\alpha), \mathrm{ch}(\beta)) = \int_X \mathbf{a} \cup \mathrm{ch}(\alpha \otimes \bar{\beta}).$$

Enter: Dirac-operators

Again: (X, g) a compact oriented Riemannian manifold $\dim = 2n$, n odd. Assume: X has no torsion and is spin.
Example X a compact Riemann surface, an abelian variety, K3-surface, a Calabi-Yau.

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Spin $\implies \exists$ a Dirac-operator $\not{D}_e$, $e \in K(X)$. Also, one has a certain characteristic class, Hirzebruch's A-roof genus

$$\hat{A}(X) = 1 + \frac{1}{24} c_1^2(X) + \cdots \in H^{4*}(X)$$

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which enters in the famous

Theorem (Atiyah-Singer index theorem)

Let (X, g) be a compact Riemannian manifold with a spin structure, and let $e \in K(X)$. Then

$$\mathrm{ind}(\not{D}_e) = \int_X \hat{A}(X)\mathrm{ch}(e).$$

Jacobians for spin manifolds

Corollary (Witten, Moore-Witten)

The A-roof genus is a multiplier; indeed

$$\omega^{+}_{\hat{A}}(\mathrm{ch}(\alpha), \mathrm{ch}(\beta)) = \mathrm{ind}(\not{D}_e), \quad e = \alpha \otimes \bar{\beta}.$$

In particular this defines a PPAV

$$(H^{+}(X)/\Lambda(X), b^{(g)}_{\sqrt{\hat{A}}}, \omega^{+}_{\hat{A}})$$

canonically associated to any (torsion free) compact spin manifold X of dimension $2n$, n odd.

Jacobians of even weight polarized Hodge structures

Data: **1)** (W, Q): integrally polarized Hodge structure, weight 2ℓ. **2)** Weil-operator C_W (i.e. $C_W | H^{p,q} = \mathrm{i}^{p-q}$).
3) $p = \sum_j h^{2\ell-2j,2j}, q = k - p \implies \mathrm{SO}(W, Q) \simeq \mathrm{SO}(p, q)$.

Griffiths domain: $\mathrm{SO}(W, Q)/\,\mathrm{U}(2h^{2\ell,0}) \times \cdots \mathrm{U}(2h^{\ell+1,\ell-1}) \times \mathrm{SO}(h^{1,1})$.

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Construction Set $V = W \oplus W^\vee(-k)$. Then

$$J(x+\hat{Q}y) := \hat{Q}C_W(x) - C_W(y) \implies J^2 = -\mathrm{id}_W\,.$$

Then $\exists$ polarized weight 1 Hodge structure (V,q) with $C_V = J$ and

$$q(x_1+\hat{Q}y_1, x_2+\hat{Q}y_2) = -Q(x_1,y_2) + Q(y_1,x_2).$$

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Note: polarized weight 1 Hodgestructures on V, $\dim V = 2g$ are classified by $\mathrm{Sp}(g)/\,\mathrm{U}(g)$ parametrizing PPAV $\implies$

Lemma

$J(W, Q) := (V/V_{\mathbb{Z}}, q)$ is a PAV.

The connection with cohomology-jacobians

Situation (X, ω) compact Kähler $\dim_{\mathbb{C}} X = n$ with g_ω the corresponding Riemannian metric. Recall that primitive cohomology

$$W_\ell := H^{2\ell}_{\rm prim}(X)_{\mathbb{R}},$$

is polarized by the Riemann form

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Proposition (Lefschetz decomposition)

Suppose ω is integral (e.g. (X,ω) a polarized projective manifold). Suppose n is odd. On $H^{2*}(X)$ define $\tilde{\omega}^+(x,y) = (-1)^{\ell+1}\int_X \omega^{n-\ell} x\cdot\iota(y)$. Then, for $k \le n$ we have:

$$J^{2k}(X, b^{g_\omega}, \tilde{\omega}^+) \overset{\text{(isogeneous)}}{\sim} \prod_{\ell=0}^{k} J(W_\ell, Q_\ell).$$

Group-theoretic explanation

Proposition

Introduce the (well-defined) homomorphism

$$\begin{array}{rcl} \psi : \mathrm{SO}(W, Q) & \to & \mathrm{Sp}(W \oplus W^\vee(-k), q) \\ f & \mapsto & \psi(f), \qquad \psi(f)(x + \hat{Q}y) = f(x) + \hat{Q}(f(y)). \end{array}$$

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With $\tilde{\psi}$ the induced map, one has a commutative diagram

$$D(W) := \mathrm{SO}(W,Q)/H \xrightarrow{\tilde{\psi}} \mathrm{Sp}(g)/\mathrm{U}(g)$$

$$\mathrm{SO}(W,Q)/H \xrightarrow{\pi} \mathrm{SO}(W,Q)/K \longrightarrow \mathrm{Sp}(g)/\mathrm{U}(g)$$

where $K \subset \mathrm{SO}(W,Q)$ is the unique maximal compact subgroup containing H.

An example: back to moduli

Let $E = E_\tau$ be the elliptic curve $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ and let α, β be the two cycles coming from the two lattice generators $\{1, \tau\}$. Set $W = H^1(E) \otimes H^1(E)$ a natural polarizated weight 2 Hodge structure. There is a period map $p : \mathfrak{h} \to D(W)$.

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Define 1)

$$M := \begin{pmatrix} \tau^2 & \tau & \tau & 1 \\ \bar{\tau}^2 & \bar{\tau} & \bar{\tau} & 1 \end{pmatrix}, N := \begin{pmatrix} |\tau|^2 & -(\tau+\bar{\tau}) & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

2) An involution ι on (4×2)-matrices: exchange column 1 and 4 as well as column 2 and 3.

An example: back to moduli

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2) An involution ι on (4×2)-matrices: exchange column 1 and 4 as well as column 2 and 3.
3) The period map p composed with $\tilde{\psi}$ is described by

$$B := \begin{pmatrix} M & \mathrm{i}\iota(M) \\ N & -\mathrm{i}\iota(N) \end{pmatrix}.$$

An example: continuation

One views this a a map into a Grasmannian and one calculates two Plücker-coordinates:

1) $\det \begin{pmatrix} M \\ N \end{pmatrix} = (\tau - \bar{\tau})(\tau^2 + 6|\tau|^2 + \bar{\tau}^2)$

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2) Replace the first column by the last column of the block matrix gives new determinant $-\mathrm{i}(\tau - \bar{\tau})(\tau + \bar{\tau})^2$.

3) Form the quotient $-\mathrm{i}\dfrac{\tau^2 + 6|\tau|^2 + \bar{\tau}^2}{(\tau + \bar{\tau})^2}$ which is non-constant but neither holomorphic nor anti-holomorphic $\implies \tilde{\psi} \circ p$ neither holomorphic nor anti-holomorphic.