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A note on the primitive cohomology lattice of a projective surface

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Abstract: The isometry class of the intersection form of a compact complex surface can be easily determined from complex-analytic invariants. For projective surfaces the primitive lattice is another naturally occurring lattice. The goal of this note is to show that it can be determined from the intersection lattice and the self-intersection of a primitive ample class, at least when the primitive lattice is indefinite. Examples include the Godeaux surfaces, the Kunev surface and a specific Horikawa surface. There are also some results concerning (negative) definite primitive lattices, especially for canonically polarized surfaces of general type.

Keywords: COMPLEX PROJECTIVE SURFACES, PRIMITIVE INTERSECTION LATTICE.

1. Introduction

The intersection form of a compact connected orientable $4n$-dimensional manifold X is the bilinear, symmetric form on $\mathsf{H}_X = H^{2n}(X,\mathbb{Z})/(\text{torsion})$ given by cup product. By Poincaré duality this form is ***unimodular***, that is, its Gram matrix has determinant ± 1. The pair consisting of H_X and the intersection pairing is called the ***intersection lattice*** of X.

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If $X \subset \mathbb{P}^N$ is a smooth compact complex manifold with hyperplane section H, the orthogonal complement of the class of H^n in $H^{2n}(X, \mathbb{Z})$ is called the (middle) ***primitive cohomology***, denoted P_X. Precise knowledge of this lattice and its group of isometries turns out to be useful, especially for arithmetic questions. This motivates interest in the main result of this note which deals with the case of surfaces (=Theorem 4.1):

Theorem. *Let X be a complex projective surface with $p_g(X) \neq 0$ and let $c \in \mathsf{H}_X$ be a primitive representative of an ample divisor. Then the isometry class of the lattice $c^\perp$ is uniquely determined by the following data:*

1. *the triple $(b_1(X), c_1^2(X), c_2(X))$ of topological invariants,*
2. *whether or not c is characteristic*
3. *the self-intersection of c.*

The above result implies in particular that for a given surface X the primitive lattice does not not depend on the particular choice of the projective embedding of X, but only on the degree of X. The proof of the theorem uses firstly Nikulin's reformulation of the classical classification results on integral quadratic forms in terms of the discriminant quadratic form and, secondly, on a fine analysis of the type of intersection lattices occurring for projective surfaces based on the Enriques classification. This result is effective as illustrated for surfaces with small c_1^2, e.g. for some Horikawa surfaces. See Examples 4.2.

The assumption $p_g(X) \neq 0$ is equivalent to P_X being indefinite, a prerequisite for applying Nikulin's results. However in the definite situation one can in several instances still determine the isometry class of the primitive intersection lattice making use of a series of investigations by G. Watson [18, 19, 20, 21, 22, 23, 24, 25]. See Remark 4.3.

Remark 1.1. Primitive cohomology plays a central role in Hodge theory since the Hodge decomposition together with the intersection pairing gives P_X the structure of a polarized pure Hodge structure of weight $2n$. To explain why this is the case, consider an embedding $X \subset \mathbb{P}^N$. The Hodge structure on the middle primitive cohomology in smooth families $\{X_s\}_{s\in S}$ of smooth varieties embedded in the same $\mathbb{P}^N$ gives rise to a period map $S \to \Gamma\backslash D$ where D is a suitable period domain and where Γ, the (maximal) monodromy group, is the isometry group of the primitive lattice of a fibre X_s (all such groups are isomorphic). More precisely, since monodromy preserves the polarization, Γ is the subgroup of the isometry group of $P = \mathsf{P}_{X_s}$ inducing the identity on the discriminant group P^*/P.

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Conventions and Notations

- A lattice is a free $\mathbb{Z}$-module of finite rank equipped with a non-degenerate symmetric bilinear integral form which is denoted with a dot.
- A rank one lattice $\mathbb{Z}e$ with $e.e = a$ is denoted $\langle a \rangle$, orthogonal direct sums by $\oplus$. Other standard lattices are the hyperbolic plane U, and the root-lattices A_n, B_n $(n \geq 1)$, D_n, $n \geq 4$. and E_n, $n = 6, 7, 8$. Their p-adic localizations will be denoted by the same symbol. More details are given below in Section 2.
- If one replaces the form on the lattice L by m-times the form, $m \in \mathbb{Z}$, this scaled lattice is denoted $L(m)$.
- An inner product space over a field k is a k-vector space equipped with a non-degenerate symmetric bilinear form over k. It will likewise be denoted with a dot.
- The signature of a non-degenerate symmetric bilinear integral form b is denoted by (b^+, b^-) and the index by $\tau = b^+ - b^-$. The signature of the intersection lattice $\mathsf{H}_X = H^{2n}(X, \mathbb{Z})/\text{torsion}$, X a compact connected orientable $4n$-dimensional manifold, will be denoted by $\tau(X)$. If X is projective, its "primitive cohomology" is the integral primitive cohomology (classes of H_X orthogonal to an ample class) and is denoted by P_X.

2. On lattices

Unimodular lattices

As is well known (cf. [13, 14]) if a unimodular form is indefinite, its isometry class is uniquely determined by the signature and type of the form. The *type* of a bilinear symmetric form by definition is *even* or *odd*. Being even means that $x.x$ is even for all elements x of the lattice and odd otherwise. The results from loc. cit. state that odd unimodular forms are diagonalizable over the integers. This is evidently not the case for unimodular even forms. Instead these are orthogonal sums of three building blocks, the hyperbolic plane U, the positive definite root lattice E_8, and its negative $E_8(-1)$. The first has rank two and has a basis $\{e, f\}$ for which $e.e = f.f = 0$ and $e.f = 1$. The root lattice E_8 has rank

8 with form given in the basis by the Coxeter matrix for the root lattice E_8, that is by

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

It turns out that every indefinite even unimodular form is isometric to $\textcircled{\perp}^s U \textcircled{\perp} \textcircled{\perp}^t E_8$, a lattice of index $t \geq 0$, or to $\textcircled{\perp}^s U \textcircled{\perp} \textcircled{\perp}^t E_8(-1)$ if the index equals $-t < 0$.

For definite forms the situation is more complicated. The number of non-isometric lattices grows rapidly with the rank. See e.g. [16, Ch. IV § 2.3].

Characteristic elements

To test whether the form on a lattice L is even or odd, one makes use of a ***characteristic element*** $c \in L$. By definition it has the property that $c.x + x.x$ is even for all $x \in L$. Such characteristic elements exist if the discriminant of L is odd as one easily sees by reduction modulo 2. In fact, characteristic classes exist for inner product spaces over the field $\mathbb{F}_2$. Of course, if $c \in L$ is not isotropic and L is even, then $c^\perp$ is an even lattice, but this holds also if c is characteristic in an odd lattice L. For later use I set this apart:

Lemma 2.1. *If L is a lattice with odd discriminant and $c \in L$ not isotropic, i.e. $c \cdot c \neq 0$, then $c^\perp$ is an even lattice if and only if c is a characteristic element.*

Remark 2.2. An odd unimodular indefinite lattice being diagonalizable, the reader may be surprised that it can have unimodular even sublattices. That this is indeed the case can be illustrated with the lattice $L = \langle 1 \rangle \textcircled{\perp} \langle 1 \rangle \textcircled{\perp} \langle -1 \rangle$. The basic observation is that L is isometric to $\langle 1 \rangle \textcircled{\perp} U$. Explicitly, if $\{e_1, e_2, e_3\}$ is an orthogonal basis for L, then $c = 2c'$, $c' = e_1 + e_2 + e_3$ is a characteristic element with $c'.c' = 1$ and $c^\perp$ is the lattice with basis $\{e_1 + e_3, e_2 + e_3\}$ isometric to U.

Discriminant forms and the genus

Let L be a lattice. We recall the concept of discriminant group and discriminant form. Remark that the pairing on L extends to a $\mathbb{Q}$-bilinear pairing on $L \otimes \mathbb{Q}$

and induces the $\mathbb{Q}/\mathbb{Z}$-valued form on the discriminant group $A(L) = L^*/L$, $L^* = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ given by

$$b_L : A(L)\times A(L) \to \mathbb{Q}/\mathbb{Z}, \quad \bar{x}.\bar{y} \mapsto x.y \bmod \mathbb{Z}\,(\textbf{\textit{discriminant bilinear form}}).$$

Even lattices come with an integral quadratic form q given by $q(x) = \frac{1}{2}b(x,x)$ and for these one considers a finer invariant, the ***discriminant quadratic form***

$$q_L : A(L) \to \mathbb{Q}/\mathbb{Z}, \quad \bar{x} \mapsto q(x) \bmod \mathbb{Z}.$$

The discriminant form is completely local in the sense that it decomposes into p-primary forms where p is a prime dividing the discriminant. More precisely, it is the orthogonal sum of the discriminant forms of the localizations $L_p = L \otimes \mathbb{Q}_p$ and so it ties in with the ***genus*** of the lattice, i.e. the set of isometry classes $\{L_p\}_{p \text{ prime}}$ together with $L \otimes \mathbb{R}$. A celebrated result of V. Nikulin [15, Cor. 1.16.3] emphasizes the role of the discriminant form in determining the genus:

Theorem. *The genus of non-degenerate lattice is completely determined by its type, rank, index and the discriminant form.*

It is well known that the number of isometry classes in a genus is finite. It is also called the ***class number*** of the genus.

For applications in geometry it is important to have a criterion for class number 1 lattices. This is often the case in the indefinite situation as stated by another result due to V. Nikulin [15, 1.13.3 and 1.16.10] and M. Kneser [10]:

Theorem 2.3. *Let L be a non-degenerate **indefinite** lattice of rank r. Its class number is* 1 *in the following instances:*

1. *In case L is even and the discriminant group of L can be generated by $\le r-2$ elements. Hence, in this case L is uniquely determined by its rank, index and the discriminant quadratic form.*
2. *In case L is odd, and the discriminant group of L can be generated by $\le r-3$ elements. Hence, in this case L is uniquely determined by its rank, index and the discriminant bilinear form.*

These results will be in particular applied to primitive sublattices of L, i.e. sublattices S such that L/S is free of torsion. In case S is well understood, one can say much about its orthogonal complement:

Lemma 2.4. *Let S be a primitive non-degenerate sublattice of L and $T = S^\perp$ then* $\mathrm{disc}\,(S) = \pm\mathrm{disc}\,(S^\perp)$ *and* $(A(S), b_S)$ *is isometric to* $(A(T), -b_T)$.

For proofs, see e.g. [9, 11].

Intersection lattices

Lemma 2.4 has the following implication for intersection lattices:

Corollary 2.5. *Let X be a compact connected orientable $4n$-dimensional manifold X with indefinite intersection form and let $c \in \mathsf{H}_X$ be primitive with $c.c \neq 0$. If H_X is even assume that $b_n(X) \geq 4$ and if H_X is odd and c is not characteristic, assume that $b_n(X) \geq 5$. Then the isometry class of $c^\perp$ is uniquely determined by the signature (b^+, b^-) of H_X and the integer $c.c$.*

Proof. The discriminant form of $\mathbb{Z}.c$ equals $\langle 1/(c.c)\rangle$ and by Lemma 2.4 the discriminant form for $T := c^\perp$ equals $-\langle 1/(c.c)\rangle$ and, in particular, is a torsion form on a length one group. The assumptions imply that $1 \leq \operatorname{rank}(T) - 2$ in the even case, and $1 \leq \operatorname{rank}(T) - 3$ in the odd case. Since T is odd if and only if c is not characteristic, the statement follows. □

Assume now that X is a compact orientable 4-dimensional manifold with intersection lattice H_X. The second Stiefel–Whitney class w_2 is a characteristic class for the inner product space $H^2(X, \mathbb{F}_2)$. To pass to integral cohomology one uses the reduction mod 2 map, induced by the natural projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$:

$$\rho_2 : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}/2\mathbb{Z}). \tag{1}$$

Any lift of w_2 under ρ_2 is an integral characteristic element since the intersection pairing is compatible with reduction modulo 2. In the special case where X is a compact almost complex manifold of complex dimension 2, there is a canonical choice for a lift, namely the first Chern class c_1. We note a simple consequence:

Lemma 2.6. *The intersection pairing on a compact almost-complex surface X is even if $c_1(X)$ is divisible by 2 in integral cohomology. The converse is true if $H_1(X, \mathbb{Z})$ is free of 2-torsion.*

If $c_1^2(X) \neq 0$, and $c_1(X) = kc$ with c primitive, then the lattice $c_1(X)^\perp \subset H^2(X, \mathbb{Z})$ is a non-degenerate even lattice of discriminant $\pm c.c$.

Proof. The preceding remarks show that if $c_1(X)$ is divisible by 2 in cohomology, $x.x$ is even for all $x \in H^2(X, \mathbb{Z})$. For the converse, consider the long exact sequence associated to $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ and use that the intersection pairing on $H^2(X, \mathbb{F}_2)$ is non-degenerate. Here surjectivity of the map ρ_2 (cf. (1)) is used which follows since by Poincaré-duality, $H^3(X, \mathbb{Z}) \simeq H_1(X, \mathbb{Z})$ – which has no 2-torsion by assumption.

The penultimate assertion is also clear since $c_1.x + x.x = x.x$ is even for all $x \in c_1(X)^\perp$. The assertion about the discriminant is a special case of Lemma 2.4. □

3. Complex algebraic surfaces

Invariants

Let X be a compact complex projective surface. It is well known that the Chern numbers, a priori complex invariants, are in fact (oriented) topological invariants. This is clear for $c_2(X)$ since it can be identified with the Euler number $e(X)$. To see that $c_1^2(X)$ is a topological invariant, one invokes a deep theorem, the index theorem ([6, Thm. 8.2.2]):

Theorem 3.1 (Index theorem – special case). *For a compact differentiable* 4*-manifold X admitting a complex structure, the index $\tau(X)$ satisfies*

$$\tau(X) = \frac{1}{3}(c_1^2(X) - 2c_2(X)).$$

For algebraic surfaces the Hodge decomposition gives two more invariants for X, namely $q(X) = \frac{1}{2}b_1(X)$ and $p_g(X) = \dim H^{2,0}(X)$. In particular, q is a topological invariant. Because of ***Noether's formula*** [1, p. 26],

$$\chi(X) := 1 - q(X) + p_g(X) = \frac{1}{12}(c_1^2(X) + c_2(X)), \tag{2}$$

also p_g is a topological invariant.

Recalling that since c_1 is a characteristic element for the intersection lattice, these observations make it possible to determine the isometry class of H_X from the type of c_1 together with the integer invariants c_1^2 and c_2.

Example 3.2. A ***K3 surface*** by definition is a surface with $b_1 = 0$ and trivial canonical bundle and so $c_1 = 0$ and $p_g = 1$, $q = 0$ implying $2 = \frac{1}{12}c_2$. Hence $b_2 = 24 - 2 = 22$. The index theorem gives $\tau = \frac{1}{3}(-48) = -16$ and since the intersection lattice is even, it is isometric to $\perp^3 U \perp \perp^2 E_8(-1)$.

An ***Enriques surface*** has $p_g = q = 0$ while c_1 is 2-torsion. A similar reasoning shows that $U \perp E_8(-1)$ is its intersection lattice.

For algebraic surfaces (and more generally for compact Kähler surfaces) there is a characterization of the signature in terms of Hodge numbers:

Lemma 3.3 ([1, Thm. IV.2.6]). *Let X be a compact Kähler surface. Then the signature of X equals $(2p_g(X) + 1, h^{1,1}(X) - 1)$ where $h^{1,1}(X) = \dim H^{1,1}(X)$.*

Surface classification

I also make use of the Enriques classification of surfaces. The notion of a minimal surface plays an essential role. All surfaces are obtained from these by repeated blowing up in points. In the present context it is important to recall how the intersection lattice changes under a blow-up. Since blowing up X in a point does not affect H^i, $i \neq 2$ and replaces $H^2(X)$ by $H^2(X) \oplus \mathbb{Z}$, where the summand $\mathbb{Z}$ is generated by the exceptional curve which has self-intersection -1, one has:

Lemma 3.4. *Let X be a compact complex surface and let $\widetilde{X}$ be the surface obtained by blowing up X in a point. Then $\mathsf{H}_{\widetilde{X}} = \mathsf{H}_X \perp \langle -1 \rangle$. In particular, the intersection lattice of a non-minimal surface is odd.*

Moreover $c_1^2(\widetilde{X}) = c_1^2(X) - 1$, $c_2(\widetilde{X}) = c_2(X) + 1$ and $\tau(\widetilde{X}) = \tau(X) - 1$.

In the Enriques classification – besides the already mentioned classes (K3 surfaces, Enriques surfaces) – some other classes appear. Firstly the rational and ruled surfaces which by definition are obtained from the projective plane, respectively a minimal ruled surface by repeatedly blowing up and blowing down. Then there are the elliptic surfaces which by definition admit a holomorphic map onto a curve such that the general fibre is an elliptic curve. Among these are some ruled surfaces, the Enriques surfaces and some K3 surfaces. Next, there are so-called bi-elliptic or hyperelliptic surfaces and, finally, the large class of properly elliptic surfaces which by definition have Kodaira dimension 1. The surfaces with Kodaira dimension 2 are called "surfaces of general type". Together these exhaust the classification (see e.g. [2]). Summarizing, replete with invariants, one has:

Theorem 3.5 (Enriques classification)**.** *Every minimal complex projective surface belongs to exactly one of the following classes ordered according to their Kodaira dimension κ:*

κ	*Class*	b_1	p_g		c_1^2	c_2
$-\infty$	*minimal rational surfaces*	0	0		8 *or* 9	4 *or* 3
	ruled surfaces of genus > 0	2g	0		$8(1-g)$	$4(1-g)$
0	*Two-dimensional tori*	4	2		0	0
	K3 surfaces	0	1		0	24
	Enriques surfaces	0	0		0	12
	bielliptic surfaces	2	0		0	0
1	*minimal properly elliptic surfaces*				0	≥ 0
2	*minimal surfaces of general type*				> 0	> 0

In the next section one considers indefinite primitive lattices. Here I discuss the – rather small – list of surfaces having definite primitive lattices. First of all, these cannot be positive definite:

Lemma 3.6. *Let X be a complex projective surface. Then $H^2(X,\mathbb{R})$ is positive definite if and only if $b_2(X)=1$ and so $\mathsf{P}_X \neq 0$ cannot be positive definite.*

Proof. Lemma 3.3 implies that the signature of a primitive lattice is $(2p_g, h^{1,1}-1)$ which is positive definite precisely if $b_2 = \tau = 2p_g + 1$. Moreover, X is minimal of general type and one finds $c_1^2 = 10p_g - 8q + 9$ and $c_2 = 2p_g - 4q + 3$. Now invoke the Bogomolov–Miyaoka–Yau inequality (cf. [1, §VII.4]), stating

$$c_1^2 - 3c_2 \leq 0, \tag{3}$$

which gives $4p_g + 4q \leq 0$ and so $p_g = q = 0$. But then $b_2 = 1$ which forces $\mathsf{P}_X = 0$. □

Secondly, as to negative definite P_X, by Lemma 3.4 one may restrict to minimal surfaces and hence, inspecting the table from Theorem 3.5, one sees:

Lemma 3.7. *Let X be complex projective surface with $\mathsf{P}_X \neq 0$ and negative definite. Then X is either rational or ruled, a (possibly blown-up) Enriques surface, an elliptic surface with $p_g = 0$ or a surface of general type with $p_g = 0$.*

Remark 3.8. In the definite situation there might be more isometry classes in the genus. There are however instances where the class number is exactly one. For minimal surfaces that are canonically polarized and with $p_g = 0$ this can be used to determine the primitive cohomology. See the table in Remark 4.3.

In the next section one also needs the following result:

Lemma 3.9. *Let X be a complex projective surface with $b_2(X) \leq 4$ and $p_g(X) = 1$. Then X is a minimal algebraic surface satisfying $c_1^2(X) = 3c_2(X) = 18$, $q(X) = 0$ (and so $b_2(X) = 4$).*

Proof. Assume that X is minimal elliptic. Since $p_g = 1$ the surface is either K3 or properly elliptic. However, since $b_2 \leq 4$, the surface cannot be K3. So it is properly elliptic with invariants $c_1^2(X) = 0$ and $c_2(X) = 12(p_g(X) - q(X) + 1) = 12(2-q)$. On the other hand $c_2(X) = 2 - 4q(X) + b_2(X)$ and so $4 \geq b_2(X) \geq 2p_g + 1 = 3$ must be even and hence $b_2(X) = 4$, but then $c_2(X) = 12(2-q(X)) = 6 - 4q(X)$ which is impossible. If X is not minimal, for its minimal model we have $b_2 \leq 3$ and so it also does not exist

If X is of general type, then from $c_2(X) = 2 - 4q(X) + b_2(X) > 0$ one finds $q(X) = 0$. Since $b_2(X) = 3, 4$, from $24 = c_1^2(X) + c_2(X)$ one finds that either $(c_1(X), c_2(X)) = (19, 5)$ or $= (18, 6)$. The inequality (3) excludes the first possibility and then $b_2(X) = 4$. If X were not minimal and $\widetilde{X}$ its minimal model, then $b_2(\widetilde{X}) = 3$ which is excluded by the previous calculation. □

Remark 3.10. Since X satisfies $c_1^2(X) = 3c_2(X)$, by S.T. Yau's results [26], its universal cover is the unit ball. The existence of a surface with $p_g(X) = 1, q(X) = 0$ and $c_1^2(X) = 18$ is not known.[1] These are of course far from simply connected. For simply connected surfaces the maximum c_1^2 seems to be 12 (G. Urzua, unpublished).

4. On primitive intersection lattices of surfaces

The main result is as follows.

Theorem 4.1. *Let X be complex projective surface whose primitive lattice P_X is indefinite. Let $h \in \mathsf{H}_X$ be a primitive ample class. Then*

1. *If H_X is even, the isometry class of P_X is uniquely determined by the triple $(b_1(X), c_1^2(X), c_2(X))$ of topological invariants together with $h.h$.*
2. *In case H_X is odd, this depends in addition to h being characteristic or not: In case h is characteristic, P_X is even and otherwise it is odd. If the latter occurs, one assumes in addition that $b_2(X) \neq 4$.*

Proof. This is a direct consequence of Corollary 2.5. Indeed, since $\tau(X) = \frac{1}{3}(c_1^2(X) - c_2(X))$, the index of P_X equals $\tau(X) - 1$ and $\operatorname{rank}(\mathsf{P}_X) = b_2(X) - 1 = c_2(X) - 2b_1(X) - 1$. The result follows from Corollary 2.5 and Lemma 3.9. Indeed, the latter result implies that $b_2(X) \geq 4$. □

Examples 4.2. 1. For a complex projective surface X with $c_1^2(X) = 1$ and K_X ample and X embedded by a suitable multiple of K_X, one has $\mathsf{P}_X \simeq \textcircled{\perp}^s U \textcircled{\perp} \textcircled{\perp}^t E_8(-1)$ since the index is negative by the index formula (cf. Theorem 3.1). The Noether inequality [1, Theorem VII.3.1] stating that $p_g \leq \frac{1}{2}c_1^2 + 2$ implies that $p_g \leq 2$. Furthermore, in case $q > 0$, O. Debarre [4, 5] has show that $2p_g \leq c_1^2$ so that $p_g = 0$ in the present situation. From this and the Noether formula (2), one arrives at the following sets of possible invariants:

[1] Sai Kee Yeung explained to me that based on work of Cartwright–Steger (C.R. Math. Ac. Sc. Paris 348) and Prasad–Yeung (Inv. Math.186 & 182), one can construct unramified double covers of fake projective planes with these invariants.

χ	1	2	3
c_2	11	23	35
(p_g, q)	$(0,0)$	$(1,0)$	$(2,0)$
(s,t)	$(0,1)$	$(2,2)$	$(4,3)$

Here are some surfaces within this range of invariants (the list is far from complete!):

- The so-called Godeaux-type surfaces, i.e. those with $p_g = q = 0$ and $c_1^2 = 1$. For concrete examples, see e.g. [1, §VII.10]. Here P_X is unimodular and negative definite of rank 8. It is known that then $\mathsf{P}_X \simeq E_8(-1)$ (cf. Table 1).
- The Kynev surface from [12, 17] with $c_1^2 = 1$, $p_g = 1$ and $q = 0$ (so that $c_2 = 23$).
- E. Horikawa's (simply connected) surface from [7] with $c_1^2 = 1, c_2 = 35$ (so that $b_2(X) = 33$).

2. The simplest non-unimodular P_X are obtained for surfaces X with $c_1^2(X) = 2$ and K_X ample and X embedded by a suitable multiple of K_X. Here $\operatorname{disc}(\mathsf{P}_X) = \pm 2$. As before, using Noether's inequality, Debarre's inequality and the Noether formula, one arrives at the following sets of possible invariants:

χ	1	2	3	4
c_2	10	22	34	46
(p_g, q)	$(0,0), (1,1)$	$(1,0)$	$(2,0)$	$(3,0)$

These surfaces are known to exist. I give some examples:

- The numerical Campedelli surfaces, i.e., those with $p_g = q = 0$ and $c_1^2 = 2$. Again, for examples see e.g. [1, §VII.10] For these, P_X is negative definite of rank 2 and with $\operatorname{disc}(\mathsf{P}_X) = -2$. It is known that $\mathsf{P}_X \simeq E_7(-1)$. See Remark 4.3 and Table 1 below.
- The surfaces with $p_g = q = 1$ and $c_1^2 = 2$ have been completely classified. See [3]. Here P_X has signature $(2,9)$ and discriminant -2. Such a lattice is isometric to $\langle 2\rangle \perp U \perp E_8(-1)$. This follows from Theorem 2.3 since the given lattice has the correct signature and discriminant form.
- Horikawa's surface with $c_1^2 = 2, p_g = 3, q = 0$ (and $c_2 = 46$) from [8]. Here P_X has signature $(6,37)$ and discriminant -2. Such a lattice is isometric to $\langle 2\rangle \perp \perp^5 U \perp \perp^4 E_8(-1)$.

3. Let X be an Enriques surface. Then $\mathsf{H}_X \simeq U \oplus E_8(-1)$. Let c be a primitive vector in the U component, say $c = e+f$ where $\{e, f\}$ is the standard basis of U. Then $c^\perp \simeq \langle -2 \rangle \oplus E_8(-1)$. By the main theorem in [23] this lattice has class number 1. By loc. cit. for vectors of the form $c' = d.e+f$, $d \neq \pm 1$, the class number of the lattice $(c')^\perp$ is larger than 1.
   In fact, to interpret Watson's results, one has to be careful since his terminology differs form what is nowadays usual. First of all, Watson only considers quadratic forms and so the associated bilinear forms (the polar forms) are always even. His notation compares to the one used in this note as follows: $P = U$, $Q = \langle 2 \rangle$, $B = A_2$. $E = E_8$ so that the two forms of rank 9 having class number 1 are $F_9 = E_8 \oplus \langle 2 \rangle$ and G_9, an indecomposable form of discriminant 8 (in loc. cit. the discriminant of forms of odd rank have been divided by 2). The last form is not isometric to $E_8 \oplus \langle 8 \rangle$ since $(G_9)_2 = \oplus^3 U \oplus A_2 \oplus \langle -3.2^3 \rangle$.

Remark 4.3. If P_X is definite, Theorem 2.3 does not apply. However, there are lists of low rank definite lattices that have one isometry class in its genus. See e.g., [18, 19, 20, 21, 22, 23, 24, 25]. This leads to the following table.

Table 1: List of lattices P_X for X canonically polarized with $\chi(X) = 12$.

c_1^2, rank(P_X)	lattice	discrim. form
$(1, 8)$	$E_8(-1)$	0
$(2, 7)$	$E_7(-1)$	$\langle -1/2 \rangle$
$(3, 6)$	$E_6(-1)$	$\langle 1/3 \rangle$
$(4, 5)$	$D_5(-1)$	$\langle -1/4 \rangle$
$(5, 4)$	$A_4(-1)$	$\langle -4/5 \rangle$
$(6, 3)$	$A_2(-1) \oplus \langle -2 \rangle$	$\langle 1/3 \rangle \oplus \langle -1/2 \rangle$
$(7, 2)$	$\begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}$	$\langle 1/7 \rangle$
$(8, 1)$	$\langle -8 \rangle$	$\langle -1/8 \rangle$

That the given lattices of rank 8, 2 and 1 have class number 1 is trivial or else well known. For other ranks I refer to the cited articles by G. Watson. The lattices in the table indeed have rank $9-k$ and discriminant group $\mathbb{Z}/k\mathbb{Z}$, $k = c_1^2$ and so these match with those for which the results in loc. cit. show that the class number of the genus equals one.

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