The Quarterly Journal of Mathematics
Quart. J. Math. 00 (2018), 1–34; doi:10.1093/qmath/hay038

ON COMPLETE INTERSECTIONS IN VARIETIES WITH FINITE-DIMENSIONAL MOTIVE

by ROBERT LATERVEER[†]

(*Institut de Recherche Mathématique Avancée, Université de Strasbourg, 7 Rue René Descartes, 67084 Strasbourg CEDEX, France*)

JAN NAGEL[‡]

(*Institut de Mathématiques de Bourgogne, 9 avenue Alain Savary, 21000 DIJON CEDEX, France*)

and CHRIS PETERS[§]

(*Discrete Mathematics, Technische Universiteit Eindhoven, Postbus 513, 5600 MB Eindhoven, Netherlands*)

[Received 25 February 2018. Revised 14 May 2018]

Abstract

Let X be a complete intersection inside a variety M with finite-dimensional motive and for which the Lefschetz-type conjecture $B(M)$ holds. We show how conditions on the niveau filtration on the homology of X influence directly the niveau on the level of Chow groups. This leads to a generalization of Voisin's result. The latter states that *if M has trivial Chow groups and if X has non-trivial variable cohomology parametrized by c-dimensional algebraic cycles, then the cycle class maps $A_k(X) \to H_{2k}(X)$ are injective for $k < c$*. We give variants involving group actions, which lead to several new examples with finite-dimensional Chow motives.

1. Introduction

1.1. *Background*

Let X be a smooth complex projective variety of dimension d. While the cohomology ring (see the conventions about the notation at the end of the introduction) $H^*(X)$ is well understood, this is far from true for the Chow ring $A^*(X)$, the ring of algebraic cycles on X modulo rational equivalence. The two are linked through the *cycle class map*

$$A^*(X) \to H^{2*}(X), \quad \gamma \mapsto [\gamma].$$

If this map is injective, we say that X *has trivial Chow groups*. If this is not the case, the kernel $A^*_{\mathrm{hom}}(X)$, the 'homologically trivial' cycles, can then be investigated through the *Abel–Jacobi map*

[†]Corresponding author. E-mail: robert.laterveer@math.unistra.fr

[‡]E-mail: johannes.Nagel@u-bourgogne.fr

[§]E-mail: c.a.m.peters@tue.nl

$$A^*_{\mathrm{hom}}(X) \to J^*(X)$$

with kernel $A^*_{\mathrm{AJ}}(X)$, the 'Abel–Jacobi trivial' cycles. If X is a curve, Abel's theorem tells us that $A^1_{\mathrm{AJ}}(X) = 0$.

The interplay between Hodge theoretic aspects of cohomology and cycles became apparent through the fundamental work of Bloch and Srinivas [7] as complemented by [20, 32]. They investigate the consequences for the Chow groups and cohomology groups of X if the class $\delta \in A^d(X \times X)$ of the diagonal $\Delta \subset X \times X$ admits a decomposition into summands having support on lower dimensional varieties. This clarifies the role of the so-called *coniveau filtration* $N^\bullet H^*(X)$ in cohomology which takes care of cycle classes supported on varieties of varying dimensions. Vial [41] discovered a variant which works better in homology which he called the *niveau filtration* $\widetilde{N}^\bullet H_*(X)$. We introduce a *refined niveau filtration* on homology $\widehat{N}^\bullet H_*(X)$ which is compatible with polarizations. The precise definitions are given below in Section 2.4. It suffices to say that we have inclusions $\widehat{N}^\bullet H_*(X) \subseteq \widetilde{N}^\bullet H_*(X) \subseteq N^\bullet H_*(X)$ with equality everywhere if the Lefschetz conjecture B is true for all varieties. Conjecture B is recalled below in Section 2.2.

Note that the Künneth formula $\delta = \sum_{k=0}^{2d} \pi_k$, with $\pi_k \in H^{2d-k}(X) \otimes H^k(X) = H^k(X)^* \otimes H^k(X)$, can be interpreted as an identity inside the ring of endomorphisms of $H^*(X)$. Since $\delta \in H^{2d}(X \times X)$ acts as the identity on $H^*(X)$, in End $H^*(X)$ one thus obtains the (cohomological) *Künneth decomposition*

$$\mathrm{id} = \sum_{k=1}^{2d} \pi_k, \quad \pi_k \in \mathrm{End}\, H^*(X) \text{ a projector with } \pi_k|_{H^j(X)} = \delta_{jk} \cdot \mathrm{id}.$$

The projectors are mutually orthogonal, that is, $\pi_j \circ \pi_k = 0$ if $j \neq k$. Moreover, the Künneth decomposition is by construction compatible with Poincaré duality and so is called *self-dual*; in other words, π_k is the transpose of π_{2d-k} for all $k < d$.

Even if the *Künneth components* π_k are classes of algebraic cycles, their sum need not give a decomposition of the diagonal. If this is the case, and if, moreover, these give a self-dual decomposition of the identity in End $A^*(X)$ by mutually orthogonal projectors, one speaks of a (self-dual) *Chow–Künneth decomposition*, abbreviated as 'CK-decomposition'. Its existence has been conjectured by Murre [28], and it has been established in low dimensions and a few other cases.

One would like to have a refined CK-decomposition which takes into account the coniveau filtration or the (refined) niveau filtration, since then the conclusions of Bloch and Srinivas [7] can be applied. This is related to the validity of the standard conjecture $B(X)$ as reviewed in Section 2.2.

1.2. *Set up and results*

Following Voisin [45, 46], we consider complete intersections X of dimension d inside a given smooth complex projective variety M and we ask about the relations between the Chow groups of M and X. On the level of cohomology, this is a consequence of the classical Lefschetz theorems: apart from the 'middle' cohomology $H^d(X)$, the cohomology of X is completely determined by $H^*(M)$, while for the middle cohomology, one has a direct sum splitting

$$H^d(X) = H^d_{\mathrm{fix}}(X) \oplus H^d_{\mathrm{var}}(X)$$

into fixed cohomology $H^d_{\mathrm{fix}}(X) = i^*H^d(M)$ and its orthogonal complement $H^d_{\mathrm{var}}(X)$ under the cup product pairing. Here $i\colon X \hookrightarrow M$ is the inclusion, and $i^*\colon H^d(M) \to H^d(X)$ is injective.

For this to have consequences on the level of Chow groups, it seems natural to assume that M has trivial Chow groups. This is the point of view of Voisin in [**46**]. Her main result uses the notion of a subspace $H \subset H^k(X)$ 'being parametrized by c-dimensional algebraic cycles' [**46**, Def. 0.3] which is slightly stronger than demanding that $H \subset \widehat{N}^c H^k(X)$, where $\widehat{N}$ is our refined version of Vial's filtration. A comparison of our filtration with Vial's is given in Section 3.2. See in particular Remark 4.7. We can now state Voisin's main result from [**46**]:

THEOREM 1.1 *Assume that M has trivial Chow groups and that X has non-trivial variable cohomology parametrized by c-dimensional algebraic cycles. Then the cycle class maps $A_k(X) \to H_{2k}(X)$ are injective for $k < c$.*

Our idea is to replace the condition of M having trivial Chow groups by finite dimensionality of the motive of M—which conjecturally is true for all varieties. (See [**29**] for background on Chow motives.) The main idea which makes this operational is the following nilpotency result (= Proposition 2.9): if r is the codimension of X in M, a degree r correspondence which restricts to a cohomologically trivial degree zero correspondence on X is nilpotent as a correspondence on X.

The second ingredient is due to Voisin [**45**, Proposition 1.6]: a degree d *cohomogically* trivial relative correspondence can be modified in a controlled way such that the new relative correspondence is fiberwise *rationally* equivalent to zero.

Given these inputs, the argument leading to our results now runs as follows. First we make use of the refined niveau filtration by way of Propositions 4.5 and 4.8 to find relative correspondences that decompose the diagonal in *homology* in the way we want. To the difference we apply the Voisin result. This provides first of all information on the level of the *Chow groups of the fibers* and, secondly, allows us to apply the nilpotency result. Writing this out gives strong variants of the above theorem of Voisin. These have been phrased in homology rather than cohomology because, as mentioned before, Vial's filtration and ours behave better in the homological setting. One of our main results can be paraphrased as follows.

THEOREM 1.2 (= Theorem 6.6). *Suppose that $B(M)$ holds, that the Chow motive of M is finite-dimensional and that $H_k(M) = N^{[\frac{k+1}{2}]}H_k(M)$ for $k \leq d$. Suppose $H_d^{\mathrm{var}}(X) \neq 0$, and that for some positive integer c, we have $H_d^{\mathrm{var}}(X) \subset \widehat{N}^c H_d(X)$. Then $A_k^{\mathrm{hom}}(X) = 0$ if $k < c$ or $k > d - c$.*

Voisin's result is a direct consequence: by [**39**, Theorem 5] varieties with trivial Chow groups have finite-dimensional motive and conjecture B holds for them as well and the condition $H_k(M) = N^{[\frac{k+1}{2}]}H_k(M)$ holds since M has trivial Chow groups. Surprisingly, if we apply Vial's result [**38**], we find that if the condition in the above theorem holds for $c = [\frac{d}{2}]$, then $h(X)$ itself also has finite dimension and up to motives of curves and Tate twists is a direct factor of $h(M)$ (Corollary 6.7).

The known examples of finite-dimensional motives are all directly related to curves, which very much limits the search for examples. However, inside the realm of motives, we can use other projectors besides the identity, namely those that come from group actions. In Section 7, we have formulated variants of the main result involving actions of a finite abelian group, say G. Then, even if the level of the Hodge-niveau filtration on variable cohomology is too big to apply our main theorems, there might be a G-character space which has the correct Hodge-level. Provided the (generalized) Hodge conjecture holds, which is automatically the case in dimensions ≤ 2, this then ensures the desired condition on the niveau filtration. In Section 8, we construct examples where this is the case and for which one of the group variants of the main theorem can be successfully applied. These examples all yield new finite-dimensional motives because of the abovementioned result of Vial.

We thus obtain several new examples of finite-dimensional motives:

- hypersurfaces in abelian 3-folds, including the Burniat–Inoue surfaces,
- hypersurfaces in a product of a hyperelliptic curve and certain types of K3 surfaces,
- hypersurfaces in 3-folds that are products of three curves, one of which is hyperelliptic,
- odd-dimensional complete intersections of four quadrics—generalizing the Bardelli example [3].

The surface examples are all of general type.

For simplicity, we have only considered involutions since then all invariants can easily be calculated, but it will be clear that the method of construction allows for many more examples of varieties admitting all kinds of finite abelian groups of automorphisms.

NOTATION 1.3 *Varieties will be defined over* $\mathbf{C}$. *We use* H^*, H_* *for the (co)homology groups with* $\mathbf{Q}$*-coefficients and likewise we write* A^*, A_* *for the Chow groups with* $\mathbf{Q}$*-coefficients.*

The category of Chow motives (over a field k) is denoted by $\mathrm{Mot}_{\mathrm{rat}}(k)$, the category of *covariant* homological motives by $\mathrm{Mot}_{\mathrm{hom}}(k)$ and the category of numerical motives $\mathrm{Mot}_{\mathrm{num}}(k)$. For a smooth projective manifold X, we let $h(X) \in \mathrm{Mot}_{\mathrm{rat}}(k)$ be its Chow motive.

We denote the integer part of a rational number a by $[a]$.

2. Preliminaries

2.1. *Correspondences*

If X and Y are projective varieties with X irreducible of dimension d_X, a correspondence of degree p is an element of

$$\mathrm{Corr}_p(X, Y) := A_{d_X+p}(X \times Y).$$

A degree p correspondence γ induces maps

$$\gamma_*\colon A_k(X) \to A_{k+p}(Y), \quad \gamma_*\colon H_k(X) \to H_{k+2p}(Y).$$

If, moreover, X and Y are smooth projective, we have correspondences of cohomological degree p, that is, elements

$$\gamma \in \mathrm{Corr}^p(Y, X) := A^{d_Y+p}(Y \times X),$$

which induce

$$\gamma^*\colon A^k(Y) \to A^{k+p}(X), \quad \gamma^*\colon H^k(Y) \to H^{k+2p}(X).$$

DEFINITION 2.1 Let $\gamma \in \mathrm{Corr}_p(X, X) = A_{d+p}(X \times X)$ be a self-correspondence of degree p where $d = d_X$.

- Let Z be smooth and equi-dimensional. We say that γ *factors through Z with shift i* if there exist correspondences $\alpha \in \mathrm{Corr}_i(Z, X)$ and $\beta \in \mathrm{Corr}_{-j}(X, Z)$ $(i - j = p)$ such that $\gamma = \alpha \circ \beta$ and $d - (i + j) = \dim Z$.
- We say that γ is *supported on $V \times W$* if

$$\gamma \in \mathrm{Im}\,(A_{d+p}(V \times W) \xrightarrow{(i \times j)_*} A_{d+p}(X \times X))$$

  where $i\colon V \to X$ and $j\colon W \to X$ are inclusions of subvarieties of X.

The usefulness of these concepts follows from the following evident results.

Lemma 2.2

- *If a correspondence $\gamma \in \mathrm{Corr}_0(X, X)$ factors through Z with shift c, then γ and ${}^t\gamma$ act trivially on $A_j(X)$ for $j < c$ or $j > d - c$.*
- *If a correspondence $\gamma \in \mathrm{Corr}_0(X, X)$ is supported on $V \times W \subset X \times X$, then γ acts trivially on $A_j(X)$ for $j < \mathrm{codim}\, V$ or $j > \dim W$ and ${}^t\gamma$ acts trivially on $A_j(X)$ for $j < \mathrm{codim}\, W$ or $j > \dim V$.*

2.2. *Standard conjecture $B(X)$*

Let X be a smooth complex projective variety of dimension d, and $h \in H^2(X)$ the class of an ample line bundle. The hard Lefschetz theorem asserts that the map

$$L_X^{d-k}\colon H_{2d-k}(X) \to H_k(X)$$

obtained by cap product with h^{d-k} is an isomorphism for all $k < d$. One of the standard conjectures asserts that the inverse isomorphism is algebraic:

Definition 2.3 Given a variety X, we say that $B_k(X)$ *holds* if the isomorphism

$$\Lambda^{d-k} = (L^{d-k})^{-1}\colon H_k(X) \xrightarrow{\cong} H_{2d-k}(X)$$

is induced by a correspondence. We say that the *Lefschetz standard conjecture $B(X)$ holds* if $B_k(X)$ holds for all $k < d$.

Remark 2.4 The Lefschetz (1,1) theorem implies that $B_k(X)$ holds if $k \leq 1$ and hence it holds for curves and surfaces. It is stable under products and hyperplane sections [17, 18] and so, in particular, it is true for complete intersections in products of projective spaces. It is known that $B(X)$, moreover, holds for the following varieties:

- abelian varieties [17, 18];
- 3-folds not of general type [37];
- hyperkähler varieties of $K3^{[n]}$-type [10];
- Fano varieties of lines on cubic hypersurfaces [25, Corollary 6];
- d-dimensional varieties X which have $A_k(X)$ supported on a subvariety of dimension $k + 2$ for all $k \leq \frac{d-3}{2}$ [38, Theorem 7.1];
- d-dimensional varieties X which have $H_k(X) = N^{[\frac{k}{2}]} H_k(X)$ for all $k > d$ [39, Theorem 4.2].

Below we shall use the following well known implication of $B(X)$.

PROPOSITION 2.5 ([17, Theorem 2.9]). *Suppose that $B(X)$ holds. Then the Künneth projectors are algebraic, that is, there exist correspondences $\pi_k \in \mathrm{Corr}_0(X, X)$ such that $\pi_{k*}|_{H_j(X)} = \delta_{kj} \cdot \mathrm{id}$ and $\Delta_X \sim_{\mathrm{hom}} \sum_k \pi_k$.*

2.3. *Finite-dimensional motives and nilpotence*

We refer to [1, 13, 16, 29] for the definition of a Chow motive and its dimension. We also need the concept of a *motive of abelian type*; by definition this is a Chow motive M for which some twist $M(n)$ is a direct summand of the motive of a product of curves.

A crucial property of varieties with finite-dimensional motive is the nilpotence theorem.

THEOREM 2.6 (Kimura [16]). *Let X be a smooth projective variety with finite-dimensional motive. Let $\Gamma \in \mathrm{Corr}_0(X, X)$ be a correspondence which is numerically trivial. Then there exists a non-negative integer N such that $\Gamma^{\circ N} = 0$ in $\mathrm{Corr}_0(X, X)$.*

Actually, the nilpotence property (for all powers of X) could serve as an alternative definition of finite-dimensional motive, as shown by a result of Jannsen [15, Corollary 3.9]. Conjecturally, any variety has finite-dimensional motive [16]. We are still far from knowing this, but at least there are quite a few non-trivial examples:

REMARK 2.7 The following varieties are known to have a finite-dimensional motive:

- varieties dominated by products of curves [16] as well as varieties of dimension ≤ 3 rationally dominated by products of curves [40, Example 3.15];
- $K3$ surfaces with Picard number 19 or 20 [34];
- surfaces not of general type with vanishing geometric genus [12, Theorem 2.11] as well as many examples of surfaces of general type with $p_g = 0$ [33, 47];
- Hilbert schemes of surfaces known to have finite-dimensional motive [8];
- Fano varieties of lines in smooth cubic 3-folds, and Fano varieties of lines in smooth cubic 5-folds [24];
- generalized Kummer varieties [49, Remark 2.9(ii)];
- 3-folds with nef tangent bundle [40, Example 3.16]), as well as certain 3-folds of general type [42, Section 8];
- varieties X with Abel–Jacobi trivial Chow groups (that is, $A^k_{AJ}X = 0$ for all k) [39, Theorem 4];
- products of varieties with finite-dimensional motive [16].

REMARK 2.8 It is worth pointing out that, up till now, all examples of finite-dimensional Chow motives happen to be of abelian type. On the other hand, ‘many’ motives are known to lie outside this subcategory, for example the motive of a general hypersurface in $\mathbf{P}^3$ [2, Remark 2.34].

The following result is a kind of ‘weak nilpotence’ for subvarieties of a variety M with finite-dimensional motive; any correspondence that comes from M and is numerically trivial turns out to be nilpotent.

PROPOSITION 2.9 *Let M be a smooth projective variety with finite-dimensional Chow motive and let $X \subset M$ be a smooth projective subvariety of codimension r. For any correspondence $\Gamma \in \mathrm{Corr}_r(M, M)$ with the property that the restriction*

$$\Gamma|_X \in \mathrm{Corr}_0(X, X)$$

is homologically trivial, there exists a non-negative integer N such that

$$(\Gamma|_X)^{\circ N} = 0 \text{ in } \mathrm{Corr}_0(X, X).$$

Proof. Put $L = i_* \circ i^* \in \mathrm{Corr}_{-r}(M, M)$ and $T = \Gamma \circ L \in \mathrm{Corr}_0(M, M)$. We have

$$\Gamma|_X = (i \times i)^*(\Gamma) = i^* \circ \Gamma \circ i_*.$$

By induction on k one shows that

$$\Gamma|_X^{k+1} = i^* \circ T^k \circ \Gamma \circ i_* \tag{2.1}$$

for all $k \geq 0$. As

$$\begin{aligned} T^2 &= \Gamma \circ i_* \circ i^* \circ \Gamma \circ i_* \circ i^* \\ &= \Gamma \circ i_* \circ \Gamma|_X \circ i^*, \end{aligned}$$

T^2 is homologically trivial. Hence, T^2 is nilpotent by [16], say $T^{2\ell} = 0$. Hence, Γ_X is nilpotent of index $N = 2\ell + 1$ by (2.1). □

2.4. *Coniveau and niveau filtration*

DEFINITION 2.10 (Coniveau filtration [6]). Let X be a smooth projective variety of dimension d. The jth level of the *coniveau filtration* on cohomology (with **Q**-coefficients) is defined as the subspace generated by the classes supported on subvarieties Z of dimension $\leq d - j$:

$$N^j H^k(X) = \sum_Z \mathrm{Im}\,(i_*\colon H^k_Z(X) \to H^k(X)).$$

This gives a decreasing filtration on $H^k(X)$. We may instead use smooth varieties Y of dimension exactly $d - j$ provided we use degree j correspondences from Y to X: such a correspondence sends Y to a cycle Z of dimension $\leq d - j$ in X and all cycles can be obtained in this way. When we rewrite this in terms of homology, we get

$$N^j H_k(X) = \sum_{Y,\gamma} \mathrm{Im}\,(\gamma_*\colon H_k(Y) \to H_k(X)),$$

where Y is smooth projective of dimension $k - j$ and $\gamma \in \mathrm{Corr}_0(Y, X)$.

Since the jth level of the filtration consists of the classes supported on varieties of dimension $k-j$, the filtration stops beyond $k/2$: a variety of dimension $<k/2$ has no homology in degrees $\geq k$:

$$0 = N^{[\frac{k}{2}]+1}H_k(X) \subset N^{[\frac{k}{2}]}H_k(X) \subset \cdots \subset N^1 H_k(X) \subset N^0 H_k(X) = H_k(X).$$

REMARK 2.11 Under Poincaré duality, one has an identification $N^j H^k(X) = N^{d-k+j}H_{2d-k}(X)$.

Vial [41] introduced the following variant of the coniveau filtration:

DEFINITION 2.12 (Niveau filtration). Let X be a smooth projective variety. The *niveau filtration* on homology is defined as

$$\widetilde{N}^j H_k(X) = \sum \mathrm{Im}\,(\gamma_*\colon H_{k-2j}(Z) \to H_k(X)),$$

where the sum is taken over all smooth projective varieties Z of dimension $k-2j$, and all correspondences $\gamma \in \mathrm{Corr}_j(Z, X)$.

REMARK 2.13 The idea behind this definition is that one should be able to lower the dimension of the variety Y appearing in Definition 2.10 using the Lefschetz standard conjecture. By Hard Lefschetz, we have an isomorphism $\Lambda_Y^j\colon H_{k-2j}(Y) \xrightarrow{\cong} H_k(Y)$ and by the Lefschetz hyperplane theorem a surjection $\iota_*\colon H_{k-2j}(Z) \to H_{k-2j}(Y)$ with $Z = Y \cap H_1 \cap \ldots \cap H_j$ a complete intersection of Y with j general hyperplanes. Hence, there is a surjective map $\iota_* \circ \Lambda_Y^j\colon H_{k-2j}(Z) \to H_k(Y)$ which is algebraic if $B_{k-2j}(Y)$ holds and thus $N^j H_k(X) = \widetilde{N}^j H_k(X)$.

This discussion also shows that

- $\widetilde{N}^j H_k(X) \subset N^j H_k(X)$
- $\widetilde{N}^j H_k(X) = N^j H_k(X)$ if $k-2j \leq 1$.

2.5. *On variable and fixed cohomology*

Let M be a smooth projective variety of dimension $d+r$ and $i\colon X \hookrightarrow M$ a smooth complete intersection of dimension d. Let us assume $B(M)$ so that the operator Λ^r on $H_*(M)$ is induced by an algebraic cycle Λ_M^r on $M \times M$. Set

$$\pi^{\mathrm{fix}}(X) := i^* \Lambda_M^r i_*, \qquad \pi^{\mathrm{var}}(X) = \Delta - \pi^{\mathrm{fix}}(X).$$

Recall that setting

$$H_d^{\mathrm{fix}}(X) = \mathrm{Im}(i^*\colon H_{d+2r}(M) \to H_d(X)),$$
$$H_d^{\mathrm{var}}(X) = \ker(i_*\colon H_d(X) \to H_d(M)),$$

one has a direct sum decomposition

$$H_d(X) = H_d^{\mathrm{fix}}(X) \oplus H_d^{\mathrm{var}}(X),$$

which is orthogonal with respect to the intersection product. We claim the following result.

LEMMA 2.14 *The operators* $\pi^{\text{fix}}(X)$ *and* $\pi^{\text{var}}(X)$ *are homological projectors which give the projection of the total cohomology onto* $H^{\text{fix}}(X)$, $H^{\text{var}}(X) = H_d^{\text{var}}(X)$, respectively.

Proof. We first observe that i_*: $H_*(X) \to L^r H_*(M)$ since $i_* H^{\text{fix}}(X) = i_{*\circ} i^* H(M) = L^r H(M)$. On the image of L the two operators L and Λ are inverses. So, since $i^*{}_\circ i_* = L^r$, (in fact this is only true up to a multiplicative constant but changing Λ^r accordingly corrects this i) we find

$$\begin{aligned}(i^* \circ \Lambda^r \circ i_*)^2 &= i^* \circ \Lambda^r \circ i_* i^* \circ \Lambda^r \circ i_* \\ &= i^* \circ \Lambda^r \circ L^r \Lambda^r \circ i_* \\ &= i^* \circ \Lambda^r \circ i_*,\end{aligned}$$

that is, π^{fix} is indeed a projector, and so is π^{var}. These projectors define a splitting on cohomology given by

$$z = i^* \Lambda^r i_* z + (z - i^* \Lambda^r i_* z).$$

On the image of i_* the two operators L and Λ commute and are each other's inverse and so

$$\begin{aligned}i_*(z - i^* \Lambda^r i_* z) &= i_* z - L^r \Lambda^r i_* z \\ &= i_* z - i_* z = 0,\end{aligned}$$

which shows that π^{var} indeed gives the projection onto variable homology and so π^{fix} projects onto the fixed cohomology. □

REMARK 2.15 The degree zero correspondences π^{fix} and π^{var} are not necessarily projectors on the level of Chow groups, although one can show that finite-dimensionality of $h(M)$ and $B(M)$ can be used to modify these correspondences in such a way that they become projectors. For what follows we do not need this.

3. Niveau filtrations and polarizations

3.1. *Polarizations*

Recall that for $k \leq d = \dim X$, we have the Lefschetz decomposition

$$H^k(X) = \oplus_r L^r H_{\text{pr}}^{k-2r}(X).$$

Following [48, p. 77], we define a polarization Q_X on $H^k(X)$ as follows. Given $a, b \in H^k(X)$, write $a = \sum_r L^r a_r$, $b = \sum_r L^r b_r$ and define

$$Q_X(a, b) = \sum_r (-1)^{\frac{k(k-1)}{2}+r} \langle L^{d-k+2r} a_r, b_r \rangle$$

where

$$\langle\, ,\rangle \colon H^{2d-k+2r}(X) \otimes H^{k-2r}(X) \to H^{2d}(X) \cong \mathbf{Q}$$

denotes the cup product. As the Lefschetz decomposition is Q_X-orthogonal, we can rewrite this in the following form. Let $p_r\colon H^k(X) \to L^r H^{k-2r}_{\mathrm{pr}}(X)$ be the projection, and define

$$s_X = \sum_r (-1)^{\frac{k(k-1)}{2}+r} L^r \circ p_r.$$

Then $Q_X(a, b) = \langle L^{d-k}(a), s_X(b)\rangle$.

When we translate this to homology, we obtain a polarization Q_X on $H_k(X)$ ($k \leq d$) given by

$$Q_X(a, b) = \langle a, \Lambda^{d-k}(s_X(b))\rangle$$

where s_X is (up to sign) the alternating sum of the projections $p_r\colon H_k(X) \to L^r H^{\mathrm{pr}}_{k+2r}(X)$ to the primitive homology (dual to primitive cohomology).

Lemma 3.1 *If $B_\ell(X)$ holds for $\ell \leq 2\dim X - k - 2$ the operator $s_X \in \mathrm{End}(H_k(X))$ is algebraic.*

Proof. See [9, Lemma 7] or [41, Lemma 1.7] □

3.2. *Modified niveau filtration*

We start by a discussion of adjoint correspondences. This material is treated from a cohomological point of view in [11, Section 4.2].

Definition 3.2 Let X and Y be smooth projective varieties of dimension d_X, d_Y. Let $\gamma \in \mathrm{Corr}_j(X, Y)$.

(i) We say that γ *admits a k-adjoint* if there exists $\gamma^{\mathrm{adj}} \in \mathrm{Corr}_{-j}(Y, X)$ such that

$$Q_Y(\gamma_*(a), b) = Q_X(a, \gamma^{\mathrm{adj}}_*(b))$$

for all $a \in H_{k-2j}(X)$, $b \in H_k(Y)$.

(ii) We say that γ *admits an adjoint* if it admits a k-adjoint for all k.

Proposition 3.3 *If the standard conjectures $B(X)$ and $B(Y)$ hold, every correspondence $\gamma \in \mathrm{Corr}(X, Y)$ admits an adjoint.*

Proof. Let $\gamma \in \mathrm{Corr}_j(X, Y)$ and consider the map

$$\gamma_*\colon H_k(X) \to H_{k+2j}(Y).$$

As $B(X)$ and $B(Y)$ hold, the operators s_X and s_Y are algebraic by Lemma 3.1. As s_X and s_Y commute with the Lambda operator, we obtain

$$\begin{aligned} Q_Y(\gamma_*(a), b) &= \langle \gamma_*(a), \Lambda_Y^{d_Y-k-2j}(s_Y(b))\rangle \\ &= \langle a, {}^t\gamma_*(\Lambda_Y^{d_Y-k-2j}(s_Y(b)))\rangle \\ &= \langle a, s_X(\Lambda_X^{d_X-k}(s_X(L_X^{d_X-k}({}^t\gamma_*(\Lambda_Y^{d_Y-k-2j}(s_Y(b)))))))\rangle. \end{aligned}$$

Hence,

$$\gamma^{\mathrm{adj}} = s_X \circ L_X^{d_X-k} \circ {}^t\gamma \circ \Lambda_Y^{d_Y-k-2j} \circ s_Y$$

is an adjoint of γ. □

To use the existence of an adjoint, we need a linear algebra lemma (cf. [**43**, **41**, Lemma 5, Lemma 1.6]).

Lemma 3.4 *Let H and H' be finite-dimensional $\mathbf{Q}$-vector spaces equipped with nondegenerate bilinear forms $Q\colon H \times H \to \mathbf{Q}$ and $Q'\colon H' \times H' \to \mathbf{Q}$. Suppose that there exist linear maps*

$$\alpha\colon H' \to H, \quad \beta\colon H \to H'$$

such that

(a) *α is surjective*;
(b) *$Q'|_{\mathrm{Im}(\beta\times\beta)}$ is non-degenerate*;
(c) *$Q(\alpha(x), y) = Q'(x, \beta(y))$ for all $x \in H'$, $y \in H$.*

Then $\alpha\circ\beta\colon H \to H$ is an isomorphism.

Proof. As H is finite dimensional, it suffices to show that $\ker(\alpha\circ\beta) = 0$. Suppose that $y \in \ker(\alpha \circ \beta)$. Then $\beta(y) \in \ker(\alpha) \cap \mathrm{Im}(\beta)$. By (c), we have

$$0 = Q(\alpha(\beta(y)), z) = Q'(\beta(y), \beta(z))$$

for all $z \in H$, hence $\beta(y) = 0$ by condition (b). This gives

$$0 = Q'(x, \beta(y)) = Q(\alpha(x), y)$$

for all $x \in H'$ and since α is surjective we obtain $y = 0$. □

Corollary 3.5 *Suppose that γ: $\mathrm{Corr}_j(Y, X)$ admits an adjoint. Consider the map $\gamma_*\colon H_{k-2j}(Y) \to H_k(X)$. Then $\gamma_*\circ\gamma_*^{\mathrm{adj}}\colon H_k(X) \to H_k(X)$ induces an isomorphism*

$$\gamma_* \circ \gamma_*^{\mathrm{adj}}\colon \mathrm{Im}(\gamma_*) \xrightarrow{\sim} \mathrm{Im}(\gamma_*).$$

Proof. Apply the previous lemma with $H' = H_k(X)$, $\alpha = \gamma_*$, $\beta = \gamma_*^{\mathrm{adj}}$ and $H = \mathrm{Im}(\gamma_*) \subseteq H_k(X)$. Condition (a) is satisfied by construction, (b) by Hodge theory (Hodge–Riemann bilinear relations) and (c) by the adjoint condition. □

Definition 3.6 The *modified niveau filtration* $\widehat{N}^\bullet$ is defined by

$$\widehat{N}^j H_k(X) = \sum \mathrm{Im}(\gamma_*\colon H_{k-2j}(Z) \to H_k(X)),$$

where the sum runs over all pairs (Z, γ) such that Z is smooth projective of dimension $k - 2j$ and such that $\gamma \in \mathrm{Corr}_j(Z, X)$ admits a k-adjoint.

We have

$$\widehat{N}^j H_k(X) \subseteq \widetilde{N}^j H_k(X) \subseteq N^j H_k(X).$$

The filtrations $N^\bullet$ and $\widetilde{N}^\bullet$ are compatible with the action of correspondences. The filtration $\widehat{N}^\bullet$ is compatible with correspondences that admit an adjoint.

PROPOSITION 3.7 *Let* $\gamma \in \mathrm{Corr}_j(X, Y)$. *If* $B(X)$ *and* $B(Y)$ *hold then we have* $\gamma_* \widehat{N}^c H_k(X) \subseteq \widehat{N}^{c+j} H_{k+2j}(Y)$.

Proof. There exists a smooth projective variety Z and a correspondence $\lambda \in \mathrm{Corr}_c(Z, X)$ such that λ admits an adjoint and

$$\widehat{N}^c H_k(X) = \mathrm{Im}\, \lambda_* \colon H_{k-2c}(Z) \to H_k(X).$$

We have

$$\lambda_* \widehat{N}^c H_k(X) = \mathrm{Im}(\gamma \circ \lambda)_* \colon H_{k-2c}(Z) \to H_{k+2j}(Y)$$

The image is contained in $\widehat{N}^{c+j} H_{k+2j}(Y)$ since γ admits an adjoint by Proposition 3.3 and $(\gamma \circ \lambda)^{\mathrm{adj}} = \lambda^{\mathrm{adj}} \circ \gamma^{\mathrm{adj}}$. □

4. On Künneth decompositions

DEFINITION 4.1 Let X be a smooth projective variety.

- We say that X admits a *refined Künneth decomposition* if there exist mutually orthogonal correspondences $\pi_{i,j} \in \mathrm{Corr}_0(X, X)$ such that
  - $\Delta_X \sim_{\mathrm{hom}} \sum_{i,j} \pi_{i,j}$
  - $(\pi_{i,j})_*|_{\mathrm{Gr}^q_N H_p(X)} = \begin{Bmatrix} \mathrm{id} \text{ if } (p, q) = (i, j) \\ 0 \ \ (p, q) \neq (i, j). \end{Bmatrix}$
  - $\pi_{i,j} = 0$ if and only if $\mathrm{Gr}^j_N H_i(X) = 0$.
- We say that X admits a *refined Chow–Künneth decomposition* if in addition the $\pi_{i,j}$ are projectors and $\Delta_X \sim_{\mathrm{rat}} \sum_{i,j} \pi_{i,j}$.
- We say that X admits a refined Künneth (or Chow–Künneth) decomposition *in the strong sense* if $\pi_{i,j}$ factors with shift j through a smooth projective variety $Z_{i,j}$ of dimension $i - 2j$ for all i and j.

REMARK 4.2 By [**41**, Proposition 1.4] there exists a Q_X-orthogonal splitting

$$H^*(X) = \oplus_{i,j} \mathrm{Gr}^j_N H_i(X).$$

The variety X admits a refined Künneth decomposition if this decomposition lifts to the category $\mathrm{Mot}_{\mathrm{hom}}(k)$ of homological motives. It admits a refined Chow–Künneth decomposition if the decomposition lifts to the category $\mathrm{Mot}_{\mathrm{rat}}(k)$ of Chow motives.

In an analogous way, one can define refined Künneth (or Chow–Künneth) decompositions with respect to the filtrations $\widetilde{N}^\bullet$ and $\widehat{N}^\bullet$.

Proposition 4.3 *If $B(X)$ holds, there exists a refined Künneth decomposition in the strong sense with respect to the filtration $\widehat{N}^\bullet$.*

Proof. (This proof is a reformulation of the argument of [**41**, Theorem 1] in terms of the modified niveau filtration.) Conjecture $B(X)$ implies that the Künneth components are algebraic, that is, there exist correspondences $\pi_i \in \mathrm{Corr}_0(X, X)$ such that $(\pi_i)_*|_{H_j(X)} = \delta_{ij} \cdot \mathrm{id}$. By Proposition 3.7, the proof of [**41**, Proposition 1.4] goes through for the filtration $\widehat{N}^\bullet$, and we obtain a Q_X-orthogonal splitting

$$H^*(X) = \oplus_{i,j} \mathrm{Gr}^j_{\widehat{N}} H_i(X).$$

The aim is to construct correspondences $\pi_{i,j} \in \mathrm{Corr}_0(X, X)$ that induce this decomposition. This is done by descending induction on j. If $j > i/2$ we take $\pi_{i,j} = 0$. Suppose that the correspondences $\pi_{i,k}$ have been constructed for $k > j$. As before there exists Z, smooth of dimension $i - 2j$, and $\gamma \in \mathrm{Corr}_j(Z, X)$ such that

$$\widehat{N}^j H_i(X) = \mathrm{Im}(\gamma_*\colon H_{i-2j}(Z) \to H_i(X)).$$

By replacing γ with $\pi_i \circ \gamma$ if necessary, we may assume that $\gamma_*|_{H_\ell(Z)} = 0$ if $\ell \neq i - 2j$. The correspondence $\pi = \pi_i - \sum_{k>j} \pi_{i,k}$ induces the projection $\widehat{N}^j H_i(X) \to \mathrm{Gr}^j_{\widehat{N}} H_i(X)$. Put $\gamma' = \pi \circ \gamma$. By construction

$$\gamma'_*\colon H_{i-2j}(Z) \to \mathrm{Gr}^j_{\widehat{N}} H_i(X)$$

is surjective. As $B(X)$ holds, π admits an adjoint by Proposition 3.3. By definition, γ admits an adjoint, hence $\gamma' = \pi \circ \gamma$ admits an adjoint and the correspondence $T = \gamma' \circ (\gamma')^{\mathrm{adj}}$ induces an isomorphism

$$\varphi = T_*\colon \mathrm{Gr}^j_{\widehat{N}} H_i(X) \to \mathrm{Gr}^j_{\widehat{N}} H_i(X)$$

by Corollary 3.5. By the Cayley–Hamilton theorem, there exists a polynomial expression $\psi = P(\varphi)$ such that $\psi \circ \varphi = \mathrm{id}$. Put $U = \psi(T)$ and define $\pi_{i,j} = U \circ T$. As $T_* = \varphi$ and $U_* = \psi$ we have

$$(\pi_{i,j})_*|_{\mathrm{Gr}^j_{\widehat{N}} H_i(X)} = \mathrm{id}$$
$$(\pi_{i,j})_*|_{\mathrm{Gr}^q_{\widehat{N}} H_p(X)} = 0 \text{ if } (p, q) \neq (i, j).$$

By construction $\pi_{i,j}$ factors with shift j through a smooth projective variety of dimension $i - 2j$ and $\pi_{i,j} = 0$ if and only if $\mathrm{Gr}^j_{\widehat{N}} H_i(X) = 0$. □

Corollary 4.4 *If $B(X)$ holds and $H_k(X) \subseteq \widehat{N}^c H_k(X)$, then there exists $\pi'_k \in \mathrm{Corr}_0(X, X)$ such that $\pi_k \sim_{\mathrm{hom}} \pi'_k$ and such that π'_k factors with shift c through a smooth projective variety Z of dimension $k - 2c$ as in Definition 2.1.*

Proof. By Proposition 4.3, we obtain a decomposition

$$\pi_k = \sum_j \pi_{k,j}.$$

with respect to the filtration $\widehat{N}^\bullet$. As $H_k(X) \subseteq \widehat{N}^c H_k(X)$, we have $\pi_{k,j} = 0$ for all $j < c$, and the result follows. □

The corollary can be generalized to the following setting. Suppose that there exists $\pi_k \in \mathrm{Corr}_0(X, X)$ such that $(\pi_k)_*|_{H_\ell(X)} = \delta_{k\ell} \cdot \mathrm{id}$. If $\pi \in \mathrm{Corr}_0(X, X)$ satisfies

$$\begin{aligned} \pi \circ \pi &\sim_{\mathrm{hom}} \pi \\ \pi \circ \pi_k &\sim_{\mathrm{hom}} \pi_k \circ \pi \sim_{\mathrm{hom}} \pi \end{aligned}$$

the motive (X, π) is a direct factor of (X, π_k) in $\mathrm{Mot}_{\mathrm{hom}}(k)$.

Corollary 4.5 *Suppose that $B(X)$ holds and that $\pi \in \mathrm{Corr}_0(X, X)$ is a correspondence as above. Let $H_\pi = \mathrm{Im}(\pi) \subseteq H_k(X)$ be the sub-Hodge structure defined by π. If $H_\pi \subseteq \widehat{N}^c H_k(X)$, there exists a correspondence $\pi' \sim_{\mathrm{hom}} \pi$ such that π' factors with shift c through a smooth projective variety Z as in Definition 2.1.*

Proof. The proof of Proposition 4.3 shows that we have a decomposition $\pi_k = \sum_j \pi_{k,j}$ in $\mathrm{Mot}_{\mathrm{hom}}(k)$. Hence,

$$\pi = \pi_k \circ \pi = \sum_j \pi_{k,j} \circ \pi.$$

Suppose that there exists $j_0 < c$ such that $\pi_{k,j_0} \circ \pi \neq 0$. Then there exists $x \in H_k(X)$ such that $\pi_{k,j}(\pi(x)) \neq 0$. Hence $H_\pi \cap \mathrm{Im}(\pi_{k,j_0}) \neq 0$. This contradicts the hypothesis $H_\pi \subseteq \widehat{N}^c H_k(X)$ since $\pi_{k,j_0}|_{\widehat{N}^c H_k(X)} = 0$. □

This result implies a modification of [**21**, Corollary 3.4, Lemma 3.5] that we need later on.

Corollary 4.6 *We make the same assumptions about M and X. Suppose that $H_d^{\mathrm{var}}(X) \subset \widehat{N}^c H_d(X)$. Then $\pi^{\mathrm{var}} \sim_{\mathrm{hom}} \widetilde{\pi}^{\mathrm{var}}$ where $\widetilde{\pi}^{\mathrm{var}} \in \mathrm{Corr}^0(X, X)$ factors through a smooth projective variety Z with shift c in the sense of Definition 2.1.*

Remark 4.7 The condition $H_d(X) \subset \widehat{N}^c H_d(X)$ may be replaced by Voisin's condition of 'being parametrized by algebraic cycles of codimension c' [**46**, Definition 0.3]. Voisin's condition implies that

$$\gamma_* \circ {}^t\gamma_*\colon H_d(X) \to H_d(X)$$

is a multiple of the identity. Our condition implies that there exists an adjoint γ^{adj} such that $\gamma_* \circ \gamma_*^{\mathrm{adj}}$ is an isomorphism with an algebraic inverse (see Corollary 3.5 and the proof of Proposition 4.5). This weaker result suffices for our purposes.

Proposition 4.8 *Suppose that $B(X)$ holds and that for every smooth projective variety Z of dimension $k - 2j$ the condition $B_\ell(Z)$ holds if $\ell \leq k - 2j - 2$. Then $\widetilde{N}^j H_k(X) = \widehat{N}^j H_k(X)$.*

Proof. It suffices to show that for every pair (Z, γ) as in Definition 3.6, γ admits a k-adjoint. This follows directly from Lemma 3.1. □

Corollary 4.9 *We have* $\widetilde{N}^j H_k(X) = \widehat{N}^j H_k(X)$ if $k - 2j \leq 3$. *In particular, if* $H_k(X) = N^{[\frac{k}{2}]} H_k(X)$ *the filtrations* $\widetilde{N}$ *and* $\widehat{N}$ *on* $H_k(X)$ *coincide with the coniveau filtration. This is true unconditionally on* $H_k(X)$, $k \leq 3$. *If the conjecture* $B(M)$ *holds, all three filtrations are equal on* $H_k(X)$ *for* $k \leq 4$.

Remark 4.10 The condition $B_\ell(Z)$ in Proposition 4.8 is needed to obtain an algebraic correspondence that induces s_Z. If $H \subset H_d(X)$ is a sub-Hodge structure such that there exists a smooth projective variety Z of dimension $d - 2c$ such that $H^{\mathrm{pr}}_{d-2c}(Z) \to H$ is surjective then this condition is not needed and we have $H \subset \widehat{N}^c H_d(X)$. We present an example below.

Example 4.11 Let $X \subset \mathbf{P}^{d+1}$ be a smooth hypersurface of degree $d + 1$. Let $Z = F_1(X)$ be the Fano variety of lines contained in X. If X is general then Z is smooth of dimension $d - 2$ and the incidence correspondence induces a surjective map (cylinder homomorphism)

$$\gamma_*\colon H^{\mathrm{pr}}_{d-2}(Z) \to H^{\mathrm{pr}}_d(X);$$

see [27, Theorem (5.34)]. Hence, $H^{\mathrm{pr}}_d(X) \subset \widehat{N}^1 H_d(X)$ by the previous remark.

Concerning the existence of a refined Chow–Künneth decomposition (in the strong sense) for the filtrations $N^\bullet$, $\widetilde{N}^\bullet$ and $\widehat{N}^\bullet$ we have the following.

Proposition 4.12 *Let X be a smooth projective variety over* $\mathbf{C}$ *such that $B(X)$ holds and $h(X)$ is finite dimensional. Then the following hold.*

(i) *There exists a refined Chow–Künneth decomposition in the strong sense for the filtration* $\widehat{N}^\bullet$.
(ii) *There exists a refined Chow–Künneth decomposition in the strong sense for*
- $\widetilde{N}^\bullet$ if $\dim X \leq 5$,
- $N^\bullet$ if $\dim X \leq 3$.

Proof. By Proposition 4.3, there exists a refined Künneth decomposition in the strong sense for the filtration $\widehat{N}^\bullet$. If $h(X)$ is finite dimensional the ideal

$$\ker A_d(X \times X) \to H_{2d}(X \times X)$$

is nilpotent, and the refined Künneth decomposition lifts to $\mathrm{Mot}_{\mathrm{rat}}(k)$ by a lemma of Jannsen [14]. This proves part (i). Part (ii) follows from the comparison between the filtrations: $\widetilde{N}^j H_i(X) = \widehat{N}^j H_i(X)$ if $i - 2j \leq 3$ (Corollary 4.9) and $N^j H_i(X) = \widetilde{N}^j H_i(X)$ if $i - 2j \leq 1$. □

Remark 4.13 Part (ii) is due to Vial [41]. The assumption $\dim X \leq 5$ can be replaced by the conditions of Proposition 4.8.

Remark 4.14 Using Proposition 4.12, the main result of [22] can be extended to arbitrary dimension, provided one replaces Vial's filtration $\widetilde{N}^\bullet$ in the statement of [22, Theorem 3] by the filtration $\widehat{N}^\bullet$.

5. A variant of Voisin's arguments

PROPOSITION 5.1 *Let Γ be a codimension-k cycle on $\mathcal{X}\times_B \mathcal{X}$ and suppose that for $b\in B$ very general,*

$$\Gamma|_{X_b\times X_b} \text{ in } H^{2k}(X_b\times X_b)$$

is supported on $V_b\times W_b$, with $V_b, W_b\subset X_b$ closed of codimension c_1, respectively *c_2. Then there exist closed $\mathcal{V},\mathcal{W}\subset\mathcal{X}$ of codimension c_1, respectively c_2, and a codimension-k cycle Γ' on $\mathcal{X}\times_B \mathcal{X}$ supported on $\mathcal{V}\times_B \mathcal{W}$ and such that*

$$\Gamma'|_{X_b\times X_b} = \Gamma|_{X_b\times X_b} \text{ in } H^{2k}(X_b\times X_b)$$

for all $b\in B$.

Proof. Use the same Hilbert schemes argument as in [45, Proposition 3.7], which is the case $V_b = W_b$. □

PROPOSITION 5.2 *Suppose that $H_k(X_b) = \widehat{N}^c H_k(X_b)$ for all $k\in\{e+1,\dots,d\}$ and all $b\in B$. Then there exist families $\mathcal{Z}_k\to B$ of relative dimension $k-2c$ and relative degree zero correspondences $\Pi'_k\in \mathrm{Corr}_B(\mathcal{X},\mathcal{X})$ such that*

(a) Π'_k *factors through* $\mathcal{Z}_k$;
(b) $\Pi'_k|_{X_b\times X_b}$ *is homologous to the kth Künneth projector $\pi_k(X_b)$ for $k=e+1,\dots,d$.*

Proof. Using the assumptions and a Hilbert scheme argument as in [46], there exist a Zariski open subset $U\subset B$, a finite étale covering $\pi\colon V\to U$, a family $\mathcal{Z}_k\to V$ of relative dimension $i-2c$ and relative correspondences $\Gamma\in \mathrm{Corr}_V(\mathcal{Z}_k,\mathcal{X})$, $\Gamma'\in \mathrm{Corr}_V(\mathcal{X},\mathcal{Z}_k)$ such that

$$(*)\quad Q(\Gamma_v(x), y) = Q'(x, \Gamma'_v(y))$$

for all $x\in H_k(X_{\pi(v)})$, $y\in H_{k-2c}(Z_{\pi(v)})$ and $v\in V$. We now consider Γ and Γ' as relative cycles over U. Let $u\in U$. If $\pi^{-1}(u)=\{v_1,\dots,v_N\}$, we have $\Gamma_u=\sum_j \Gamma_{v_j}$, $\Gamma'_u=\sum_j\Gamma'_{v_j}$. As condition (*) holds for all v_j, we obtain

$$(*)\quad Q(\Gamma_u(x), y) = Q'(x, \Gamma'_u(y)).$$

We can extend $\mathcal{Z}$ to B by relative projective completion and desingularization, and extend Γ and Γ' to relative correspondences over B by taking their Zariski closure.

As before, let $H_k^{\mathrm{fix}}(X_b)$ be the image of the restriction map $H_{k+2r}(M)\to H_k(X_b)$. As $B(M)$ holds, there exists an algebraic cycle β_{d+r-k} that induces the operator Λ^{d+r-k}. Set $R_k=\beta_{d+r-k}\circ L^{d-k}\circ\pi_{k+2r}(M)$. If we pull back these cycles to $M\times M\times B$ and then to $\mathcal{X}\times_B\mathcal{X}$, we obtain relative correspondences $\Pi_k\in \mathrm{Corr}_B(\mathcal{X},\mathcal{X})$ such that $\Pi_k|_{H_k^{\mathrm{fix}}(X_b)}$ is the identity for all k (see for example [21, Lemmas 3.2 and 3.3]). Note that, by construction, R_k factors through a subvariety of dimension $r+k$ of M and $\Pi_k|_{X_b\times X_b}$ factors through a subvariety $Y_b\subset X_b$ of dimension k, that is, $\Pi_k|_{X_b\times X_b}\in \mathrm{Im}\, A_d(Y_b\times X_b)\to A_d(X_b\times X_b)$.

Write $\mathcal{T} = \Gamma \circ \Gamma' \in \mathrm{Corr}_B(\mathcal{X}, \mathcal{X})$. Replacing $\mathcal{T}$ by $\Pi_k \circ \mathcal{T}$ if necessary, we may assume that $\mathcal{T}|_{X_b \times X_b}$ acts as zero on $H_j(X_b)$ for all $j \neq k$. By construction, $(\mathcal{T}_b)_*: H_k(X_b) \to H_k(X_b)$ is an isomorphism, hence it has an algebraic inverse by the Cayley–Hamilton theorem, as we saw in the proof of Proposition 4.3. We want to perform a relative version of this construction. To this end, note that since $f: \mathcal{X} \to B$ is a smooth morphism, the sheaf $R_k f_* \mathbf{Q}$ is locally constant. Hence, there exists an open covering $\{U_\alpha\}$ of B and isomorphisms f_α from $R_k f_* \mathbf{Q}|_{U_\alpha}$ to the constant sheaf with fiber $H_k(X_0)$ ($0 \in U_\alpha$ a base point). As $\mathcal{T}$ is a relative correspondence defined over B, the maps $(\mathcal{T}|U_\alpha)_*: R_k f_* \mathbf{Q}|_{U_\alpha} \to R_k f_* \mathbf{Q}|_{U_\alpha}$ induce automorphisms

$$T_\alpha: H_k(X_0) \to H_k(X_0)$$

that commute with the transition functions $f_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$:

$$T_\alpha = f_{\alpha\beta} \circ T_\beta \circ f_{\alpha\beta}^{-1}.$$

Hence, the characteristic polynomial of T_α does not depend on α. This implies that there exists a polynomial $P(\lambda)$ such that

$$P(\mathcal{T}_b)_* = (\mathcal{T}_b)_*^{-1}$$

for all $b \in B$. Define $\mathcal{U} = P(\mathcal{T}) \in \mathrm{Corr}_B(\mathcal{X}, \mathcal{X})$ and set $\Pi'_k = \mathcal{U} \circ \mathcal{T}$. □

Corollary 5.3 *There exist relative correspondences Π_{left}, Π_{mid} and Π_{right} and families $\mathcal{Y} \to B$ of relative dimension d, $\mathcal{Z} \to B$ of relative dimension $d - 2c$, such that*

- Π_{left} *is supported on $\mathcal{Y} \times_B \mathcal{X}$ and Π_{right} is supported on $\mathcal{X} \times_B \mathcal{Y}$;*
- Π_{mid} *factors through $\mathcal{Z}$;*
- *the restriction of*

$$\Delta_{\mathcal{X}/B} - \Pi_{\mathrm{left}} - \Pi_{\mathrm{mid}} - \Pi_{\mathrm{right}}$$

to $X_b \times X_b$ is homologous to zero for all $b \in B$.

Proof. Define $\Pi_{\mathrm{left}} = \sum_{k=0}^{e} \Pi_k$, $\Pi_{\mathrm{mid}} = \Pi'_d + \sum_{k=e+1}^{d-1} (\Pi'_k + {}^t\Pi'_k)$ and $\Pi_{\mathrm{right}} = {}^t\Pi_\ell$. For the support condition on Π_ℓ and Π_r use Proposition 5.1. □

6. The main results

The setup that we consider in this section is the following. Let M be a smooth projective variety of dimension $d + r$. Let $L_1, \ldots, L_r$ be very ample line bundles on M, and let $f: \mathcal{X} \to B$ denote the family of all smooth complete intersections of dimension d defined by sections of $E = L_1 \oplus \ldots \oplus L_r$. We write $X_b = f^{-1}(b)$. The next result plays a major role in deriving the main results. It uses the assumption that the L_j are very ample in a crucial way.

Proposition 6.1 (Voisin [46]). *Suppose that for general $b \in B$ one has that X_b has non-trivial variable homology in degree d. Let $\mathcal{D}$ be a codimension-d cycle on $\mathcal{X} \times_B \mathcal{X}$ with the property that*

$$\mathcal{D}|_{X_b \times X_b} = 0 \quad in\ H_{2d}(X_b \times X_b).$$

Then there exists a codimension-d cycle γ *on* $M \times M$ *such that*

$$\mathcal{D}|_{X_b \times X_b} - \gamma|_{X_b \times X_b} = 0 \quad in\ A_d(X_b \times X_b)$$

for all $b \in B$.

Proof. As we will show, the argument is really the same as that of Voisin's original result [46, Proposition 1.6] (where it is assumed that M has trivial Chow groups).

Consider the blow-up $\widetilde{M \times M}$ of the diagonal and the natural quotient map μ: $\widetilde{M \times M} \to M^{[2]}$ to the Hilbert scheme of zero-dimensional subschemes of M of length 2. Set $\mathbf{P} = \mathbf{P}H^0(X, E)$ and as in [46, Lemma 1.3] introduce

$$I_2(E) := \{(s, y) \in \mathbf{P} \times \widetilde{M \times M} | s|_{\mu(y)} = 0\}.$$

Next, consider the blow-up of $\mathcal{X} \times_B \mathcal{X}$ along the relative diagonal:

$$p\colon \widetilde{\mathcal{X} \times_B \mathcal{X}} \to \mathcal{X} \times_B \mathcal{X}.$$

Observe that $\widetilde{\mathcal{X} \times_B \mathcal{X}}$ is Zariski open in $I_2(E)$ and so it makes sense to restrict cycles on $I_2(E)$ to the fibers $\widetilde{X_b \times X_b}$ of $\widetilde{\mathcal{X} \times_B \mathcal{X}} \to B$. Very ampleness of the L_j implies that $I_2(E) \to \widetilde{M \times M}$ is a projective bundle and hence its cohomology can be expressed in terms of cohomology coming from $\widetilde{M \times M}$ and a tautological class. Assume now that

$$\exists\ R \in A^d(I_2(E)) \text{ with } R|_{\widetilde{X_b \times X_b}} \sim_{\text{hom}} 0.$$

Voisin shows that this implies the existence of a codimension-d cycle γ on $M \times M$ and an integer k such that

$$(p_b)_*(R|_{\widetilde{X_b \times X_b}}) = k\Delta_{X_b \times X_b} + \gamma|_{X_b \times X_b} \quad \text{in } A_d(X_b \times X_b).$$

The first summand acts on all of homology, while the second summand, by construction, acts only on the fixed homology. So the assumption that there is some variable homology implies that $k = 0$ and so the cycle γ is homologous to zero. To prove the above variation, suppose we are given $\mathcal{D}$ of codimension d on $\mathcal{X} \times_B \mathcal{X}$ as above. As $\widetilde{\mathcal{X} \times_B \mathcal{X}} \subset I_2(E)$ is Zariski open, there exists a codimension-d cycle R on $I_2(E)$ such that $R|_{\widetilde{\mathcal{X} \times_B \mathcal{X}}} = p^*\mathcal{D}$. Then we have

$$R|_{\widetilde{X_b \times X_b}} = p^*\mathcal{D}|_{\widetilde{X_b \times X_b}} = (p_b)^*(\mathcal{D}|_{X_b \times X_b}) = 0 \text{ in } H_{2d}(\widetilde{X_b \times X_b})$$

for all $b \in B$, where p_b: $\widetilde{X_b \times X_b} \to X_b \times X_b$ denotes the blow-up of the diagonal. Hence, if we apply Voisin's original proposition to this cycle R, we get the desired conclusion. □

THEOREM 6.2 *Notation as above. Suppose that* $B(M)$ *holds and the Chow motive of* M *is finite-dimensional. Assume that for a general* $b \in B$ *the fiber* X_b *has non-trivial variable homology*:

$$H_d(X_b)^{\mathrm{var}} \neq 0,$$

and that for some non-negative integers c, e, *with* $e < d$, *we have*

$$H_k(X_b) = \widehat{N}^c H_k(X_b) \quad \textit{for all } k \in \{e+1,\ldots,d\}.$$

Then, for any $b \in B$, *we have*

$$\textit{Niveau } (A_k(X_b)) \le e - k \quad \textit{for all } k < \min\{d-e, c\},$$

that is, there exists a subvariety $Y_b \subset X_b$ *of dimension* e *such that* $A_k(Y_b) \to A_k(X_b)$ *is surjective.*

Proof. *Step* 1. We first construct a homological decomposition of the diagonal of X_b

$$\Delta_{X_b} \sim_{\mathrm{hom}} \Delta_{\mathrm{left}} + \Delta_{\mathrm{mid}} + \Delta_{\mathrm{right}} \quad \text{in } H_{2d}(X_b \times X_b),$$

where the right-hand side consists of self-correspondences of X of degree 0, $\Delta_{\mathrm{right}} = {}^t\Delta_{\mathrm{left}}$ and Δ_{mid} factors with shift c through a smooth variety Z.

This is done as follows. As conjecture B is stable by hyperplane sections (see Remark 2.4), the complete intersections X_b satisfy $B(X_b)$, and hence, by Proposition 2.5, there are correspondences $\pi_j \in \mathrm{Corr}^0(X_b, X_b)$, $j = 0,\ldots,2d$ inducing the corresponding homological Künneth projectors. By Proposition 4.5, for $k \in \{e+1,\ldots,d\}$, we have that $\pi_k(X_b) \sim_{\mathrm{hom}} \pi_k'(X_b)$, a projector that factors through a variety of dimension $k-2c$ with shift c as in Definition 2.1. Now set

$$\Delta_{\mathrm{left}} = \sum_{k \le e} \pi_k(X_b)$$
$$\Delta_{\mathrm{right}} = {}^t\Delta_{\mathrm{left}}$$
$$\Delta_{\mathrm{mid}} = \sum_{k=e+1}^{2d-e-1} \pi_k'(X_b).$$

Step 2. We spread out the fiberwise correspondences $\Delta_{\mathrm{left}}, \Delta_{\mathrm{right}}, \Delta_{\mathrm{mid}}$ to the family of hypersurfaces

$$\mathcal{X} \to B,$$

using Voisin's argument in the form of Propositions 5.1 and 5.2. This gives a homological decomposition of the relative diagonal, in the sense that there exists $\mathcal{Y} \subset \mathcal{X}$ of relative dimension d and a family $\mathcal{Z} \to B$ of relative dimension $d-2c$, and codimension-d cycles

$$\Pi_{\mathrm{left}}, \quad \Pi_{\mathrm{right}}, \quad \Pi_{\mathrm{mid}}$$

on $\mathcal{X} \times_B \mathcal{X}$ such that $\Pi_{\mathrm{left}}, \Pi_{\mathrm{right}}$ have support on $\mathcal{Y} \times_B \mathcal{X}$, respectively on $\mathcal{X} \times_B \mathcal{Y}$, and Π_{mid} factors through $\mathcal{Z} \to B$ such that, for any $b \in B$, restriction gives back the diagonal:

$$(\Pi_{\text{left}} + \Pi_{\text{mid}} + \Pi_{\text{right}})|_{X_b\times X_b} = \Delta_{X_b} \quad \text{in } H_{2d}(X_b \times X_b).$$

Step 3. We upgrade this to rational equivalence using properties of M. So we consider the difference

$$\mathcal{D} := \Delta_{\mathcal{X}} - \Pi_{\text{left}} - \Pi_{\text{mid}} - \Pi_{\text{right}},$$

a relative correspondence with the property that

$$\mathcal{D}|_{X_b\times X_b} = 0 \quad \text{in } H_{2d}(X_b \times X_b),$$

for all $b \in B$. We now apply the key Proposition 6.1 to $\mathcal{D}$. We find a codimension-d cycle γ on $M \times M$ such that

$$\mathcal{D}|_{X_b\times X_b} - \gamma|_{X_b\times X_b} = 0 \quad \text{in } \mathrm{Corr}_0(X_b \times X_b),$$

for all $b \in B$. The crucial point is that the restriction $\gamma|_{X_b\times X_b} \in \mathrm{Corr}_0(X_b \times X_b)$ is homologically trivial, and so, by Proposition 2.9, is nilpotent.

Step 4. We can now finish the proof. Observe that a specialization argument reduces the proof to showing it for a general $b \in B$ (cf. [**45**, Theorem 1.7] and [**46**, Theorem 0.6]). For general b, the fiber X_b will be in general position with respect to $\mathcal{Y}$ and $\mathcal{Z}$ so that

$$\Gamma_{\text{left}} := \Pi_{\text{left}}|_{X_b\times X_b}$$

will be supported on $Y_b \times X_b$ with Y_b of dimension c, and likewise

$$\Gamma_{\text{mid}} := \Pi_{\text{mid}}|_{X_b\times X_b} \tag{6.1}$$

will factor with a shift c. Let Γ_{right} be the transpose of Γ_{left}. For some $N \gg 0$, we have

$$(\Delta_{X_b} - \Gamma_{\text{left}} - \Gamma_{\text{mid}} - \Gamma_{\text{right}})^{\circ N} = 0 \quad \text{in } \mathrm{Corr}_0(X_b \times X_b), \tag{6.2}$$

where $\Gamma_{\text{left}}, \Gamma_{\text{right}}$ is supported on $Y_b \times X_b$, respectively on $X_b \times Y_b$, and Γ_{mid} factors through Z_b with shift c as in equation (6.1).

Since Γ_{left} is supported on $Y_b \times X_b$, Lemma 2.2 implies that its action on $A_k(X_b)$ is trivial for $k < \operatorname{codim} Y = d - e$. The correspondence Γ_{mid} by construction factors through Z_b with shift c and so (by the same lemma) its action on $A_k(X_b)$ is trivial, since $k < c$. Now expand the expression (6.2) to conclude that

$$(\Delta_{X_b})_* = (\text{polynomial in } \Gamma_{\text{right}})_*\colon\ A_k(X_b) \to A_k(X_b).$$

Since Δ_{X_b} acts as the identity on $A_k(X_b)$, this implies indeed that $A_k(X_b)$ is supported on Y_b, a variety of dimension e. □

REMARK 6.3 It is possible to be more precise: in the situation of Theorem 6.2, we even have that

$$\cdot L^{d-e}\colon A^{e-k}(X_b) \to A^{d-k}(X_b)$$

is surjective in the range $k < \min\{d-e, c\}$, so the k-cycles of X_b are supported on a dimension e complete intersection. To obtain this, we remark that the Γ_{right} in the above proof can be expressed in terms of L^{d-e}, just as in the proof of [23].

Recall that for curves $A_0^{\mathrm{AJ}} = 0$ and so, if $A_0(X_b)$ is supported on a curve, we have $A_0^{\mathrm{AJ}}(X_b) = 0$. We thus deduce that for $c = 1$, $e = 1$, we get the following special case:

COROLLARY 6.4 *Let M be a smooth $(d+1)$-dimensional projective variety for which $B(M)$ holds and whose (Chow) motive is finite dimensional.*

Let X_b, $b \in B$ be the family of all smooth hypersurfaces in a very ample linear system and suppose that

$$H_d(X_b)^{\mathrm{var}} \neq 0 \quad \text{and} \quad H_k(X_b) = \widehat{N}^1 H_k(X_b), \quad k = 2, \ldots, d$$

for general $b \in B$. Then

$$A_0^{AJ}(X_b) = 0$$

for all $b \in B$.

REMARK 6.5 (1) In view of Corollary 4.9(1), for $n = 2$, the condition on the coniveau becomes $N^1 H_2(X_b) = H_2(X_b)$, that is, all cohomology is algebraic. For $n = 3$, we should have in addition that $N^1 H_3(X_b) = H_3(X_b)$, that is, $h^{3,0}(X_b) = 0$, as well as the generalized Hodge conjecture for $H^3(X_b)$.

(2) Note that in Corollary 6.4 there is no condition on $H_{d+1}(M)$, so $p_g(M)$ could be non-zero. In this case, nothing is known about the Chow groups of M, so it is remarkable that one can at least control the image $\mathrm{Im}\,(A_1(M) \to A_0(X_b))$.

We now come to our second main theorem. It asserts that a 'short' niveau filtration on the variable cohomology already has strong implications for the Abel–Jacobi kernels.

THEOREM 6.6 *Let $i\colon X \hookrightarrow M$ be a complete intersection of dimension d. Suppose that*

- *$B(M)$ holds*;
- *the Chow motive of M is finite dimensional*;
- *$H_d^{\mathrm{var}}(X) \neq 0$ and for some positive integer c we have $H_d^{\mathrm{var}}(X) \subset \widehat{N}^c H_d(X)$.*

Then for $k < c$ or for $k > d - c$ we have

$$i^*\colon A_{k+r}(M) \twoheadrightarrow A_k(X), \quad i_*\colon A_k(X) \hookrightarrow A_k(M).$$

Moreover, in this range

$$A_k^{\mathrm{var}}(X) = \ker(A_k(X) \xrightarrow{i_*} A_k(M)) = 0,$$

If in addition

(a) $H_k(M) = N^{[\frac{k}{2}]}H_k(M)$ *for* $k \le d$, *then* $A_k^{\mathrm{AJ}}(X) = 0$ *if* $k < c$ *or* $k > d - c$;
(b) $H_k(M) = N^{[\frac{k+1}{2}]}H_k(M)$ *for* $k \le d$, *then* $A_k^{\mathrm{hom}}(X) = 0$ *if* $k < c$ or $k > d - c$.

Proof. Let X be a smooth complete intersection. In Section 2.5, we showed that there is a decomposition

$$\Delta_X = \pi^{\mathrm{fix}}(X) + \pi^{\mathrm{var}}(X)$$

which in cohomology induces projection onto fixed and variable cohomology, respectively. By Proposition 5.2, there exist relative codimension-d cycles Π' and Π^{var} on $\mathcal{X} \times_B \mathcal{X}$ such that Π' comes from $M \times M \times B$ and Π^{var} induces $\pi^{\mathrm{var}}(X)$. Moreover, the restriction of

$$R = \Delta_{\mathcal{X}/B} - \Pi' - \Pi_d^{\mathrm{var}}$$

to the general fiber is homologically trivial. By Proposition 6.1, there exists a codimension-d cycle γ on $M \times M$ such that

$$R|_{X_b \times X_b} - \gamma|_{X_b \times X_b}$$

is rationally equivalent to zero for $b \in B$ general. In particular, $\gamma|_{X \times X}$ is homologically trivial. Hence, $\gamma|_{X \times X}$ is nilpotent by Proposition 2.9. Let N be the index of nilpotency of $\gamma|_{X \times X}$. We obtain

$$0 = \gamma^{\circ N}|_{X \times X} = (\Delta_X - \pi^{\mathrm{fix}}(X) - \pi^{\mathrm{var}}(X))^{\circ N}.$$

By assumption (3) and Corollary 4.6, the correspondence $\pi^{\mathrm{var}}(X)$ factors through a correspondence of degree $-c$ over a variety of dimension $d - 2c$ and so acts trivially on $A_k(X)$ if $k < c$ or $k > d - c$. Setting $\psi = \pi^{\mathrm{fix}}(X)$, we find that for some polynomial P we have $P(\psi)_* \circ \psi_* = \psi_* \circ P(\psi)_* = \mathrm{id}$ on the Chow groups $A_k(X)$ with k in this range and the first assertion follows. For the second, observe that ψ acts as zero on $A_k^{\mathrm{var}}(X)$.

The assumption (a) above implies that $\pi^{\mathrm{fix}}(X)$ factors through a curve and so this summand acts trivially on $A_k^{\mathrm{AJ}}(X)$ for *all* k. So then the above argument indeed gives that $A_k^{\mathrm{AJ}}(X) = 0$ if $k < c$ or $k > d - c$. In case (b), $\pi^{\mathrm{fix}}(X)$ factors through a point and we obtain $A_k^{\mathrm{hom}}(X) = 0$ if $k < c$ or $k > d - c$. □

COROLLARY 6.7 *In the above situation, suppose that* $c = [\frac{d}{2}]$. *Then the motive* $h(X)$ *is finite-dimensional. Moreover, if for* M *we have* $A_k^{AJ}(M) = 0$ *for all* k, *then also* $A_k^{\mathrm{AJ}}(X) = 0$ *for all* k.

Proof. The assumptions imply surjectivity of i^*: $A_k^{AJ}(M, \mathrm{id}, r) \to A_k^{\mathrm{AJ}}(h(X), \mathrm{id}, 0)$ in the range $k = 0, \ldots, [\frac{d-2}{2}]$. We then apply Vial's result [**40**, Theorem 3.11] (NB: for an alternative proof of Vial's result in terms of birational motives, cf. [**26**, Appendix B]). □

7. Variants with group actions

Let M be a projective manifold of dimension $d + r$ and let $L_1, \dots, L_r$ be ample line bundles on M and, as before, set

$$E := L_1 \oplus \cdots \oplus L_r.$$

We assume that a finite group G acts on M and on the L_j, and that the linear systems $|L_j|^G$, $j = 1, \dots, r$, are base point free. The complete intersection in M corresponding to $s = (s_1, \dots, s_r) \in \mathbf{P}(H^0(M, E))$ is denoted X_s. We consider smooth complete intersections coming from G-invariant hypersurfaces and set accordingly

$$B := \{b \in \mathbf{P}(H^0(M, E)^G) | X_b \text{ is smooth}\}.$$

This is Zariski open in $\mathbf{P}(H^0(M, E)^G)$.

The graph of the action of $g \in G$ on M will be written $\Gamma_g \subset M \times M$. As before, we let $\widetilde{M \times M}$ be the blow-up of $M \times M$ in the diagonal and $M^{[2]}$ the Hilbert scheme of length 2 subschemes of M with the natural quotient morphism μ: $\widetilde{M \times M} \to M^{[2]}$. Consider the 'bad' locus

$$B_{E,\mu} = \{y \in \widetilde{M \times M} | \,\} \text{no } s \in H^0(M, E)^G \text{ separates the points of the length}-2 \text{ scheme } \mu(y).$$

Note that the G-invariant sections of E do not separate points in G-orbits. We demand instead that they separate entire G-orbits; in fact, we want something less stringent, as expressed by the following notion, involving the proper transforms $\widetilde{\Gamma}_g$ of Γ_g in $\widetilde{M \times M}$.

Definition 7.1 Assume (M, E) and G are as above. We say that $H^0(M, E)^G$ *almost separates* orbits if the 'bad' locus $B_{E,\mu}$ is contained in $\bigcup_{g \neq \mathrm{id}} \widetilde{\Gamma}_g \cup R_G$, where R_G is a (possibly empty) union of components of codimension $> \dim M = d + r$.

This demand ensures that $I_2(E) \to \widetilde{M \times M}$ is a repeated blow-up of a projective bundle so that its cohomology can be controlled. In order to have an analog of Proposition 6.1, we demand that for $g \in G$ the endomorphisms

$$\gamma_g^{\mathrm{var}} = [\Gamma_{g,b}]_*^{\mathrm{var}} \in \mathrm{End}\, H_d(X_b)^{\mathrm{var}}$$

should be independent. This can be tested using the following result.

Lemma 7.2 *Let ρ: $G \to \mathrm{GL}(V)$ be a representation of a finite group on a finite-dimensional $\mathbf{Q}$-vector space V. Then the endomorphisms $\{\rho_g, g \in G\}$ are independent in* End V *if G is abelian and every irreducible representation occurs in V.*

Proof. This is a consequence of elementary representation theory. We may work over $\mathbf{C}$. In the abelian case, the group ring $\mathbf{C}[G]$ is isomorphic to the regular representation of G and since the former has for its base the irreducible non-isomorphic characters, the elements g, $g \in G$, give a basis for $\mathbf{C}[G]$. The representation ρ induces an algebra homomorphism $\tilde{\rho}$: $\mathbf{C}[G] \to \mathrm{End}\, V$ which is injective if every irreducible representation occurs in V. So the images $\tilde{\rho}_g$, $g \in G$ form an independent set. □

Let us next introduce some notation. Suppose that $\chi: G \to \mathbf{Q}$ is a $\mathbf{Q}$-character defining an irreducible $\mathbf{Q}$-representation V_χ, that is, $\chi(g) = \mathrm{Tr}(g)|_{V_\chi}$ for all $g \in G$. The corresponding projector in the group ring of G is

$$\pi_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g) g \in \mathbf{Q}[G]$$

leading to

$$\Gamma_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g) \Gamma_{g,b} \in \mathrm{Corr}_0(X_b, X_b) \tag{7.1}$$

acting on the Chow group of M and on the homology groups of M as well as the homology of the complete intersections X_b. The latter action preserves the decomposition into variable and fixed homology. The jth Chow group of the motive (X, Γ_χ) is by definition

$$A_j(X, \Gamma_\chi) = \mathrm{Im}\,(\Gamma_\chi: A_j(X) \to A_j(X)) = A_j(X)^\chi,$$

where for any G-module V we set

$$V^\chi := \{v \in V | g(v) = \chi(g)v \text{ for all } g \in G\} = \{v \in V | (\Gamma_\chi)_* v = v\}.$$

Thus Γ_χ acts as the identity on V^χ.

We are now ready to formulate a variant of Proposition 6.1. Its validity is shown in the course of the proof of [**46**, Theorem 3.3].

Proposition 7.3 *Let (M, E), G and $B \subset \mathbf{P}(H^0(M, E)^G)$ be as above. Suppose that*

- *$H^0(M, E)^G$ almost separates orbits*;
- *the endomorphisms $\gamma_g^{\mathrm{var}} \in \mathrm{End}\, H_d(X_b)^{\mathrm{var}}$, $g \in G$, are linearly independent*;
- *for general $b \in B$ one has $H_d(X_b)^{\mathrm{var}} \neq 0$.*

Then for any $\mathcal{D} \in A^d(\mathcal{X} \times_B \mathcal{X})^\chi$ with the property that

$$\mathcal{D}|_{X_b \times X_b} = 0 \quad \text{in } H_{2d}(X_b \times X_b)^\chi,$$

there exists a codimension-d cycle γ on $M \times M$ such that

$$\mathcal{D}|_{X_b \times X_b} - \gamma|_{X_b \times X_b} = 0 \quad \text{in } A_d(X_b \times X_b)^\chi$$

for all $b \in B$.

Using this variant, the arguments we employed in Section 6 for Δ_X can thus be applied to Γ_χ provided we restrict to $H_*(X_b)^\chi$. Since Γ_χ acts as the identity on $A_j(X)^\chi$, the same conclusions as before can be drawn for these Chow groups and we obtain the following results.

THEOREM 7.4 *Let (M, E), G and $B \subset \mathbf{P}(H^0(M, E)^G)$ be as above. Moreover, let χ be a $\mathbf{Q}$-character for G and Γ_χ the associated projector (7.1). Suppose that*

- *$B(M)$ holds;*
- *$H^0(M, E)^G$ almost separates orbits;*
- *the endomorphisms $\gamma_g^{\mathrm{var}} \in \mathrm{End}\, H_d(X_b)^{\mathrm{var}}$, $g \in G$, are linearly independent;*
- *the Chow motive (M, Γ_χ) is finite dimensional.*

Assume, moreover, that for a general $b \in B$ one has $H_d(X_b)^{\mathrm{var}} \neq 0$ and that

$$H_k(X_b)^\chi \subset \widehat{N}^c H_k(X_b) \quad \text{for all } k \in \{e+1, \ldots, d\}.$$

Then, for any $b \in B$,

$$\text{Niveau } ((A_j(X_b))^\chi) \leq e - j \quad \text{for all } j < \min\{d-e, c\},$$

that is, there exists a subvariety $Z_b \subset X_b$ of dimension d such that $A_j(Z_b) \to A_j(X_b, \Gamma_\chi)$ is surjective if $j < \min\{d-e, c\}$.

THEOREM 7.5 *Notation as in the previous theorem. Let $X \subset M$ be a G-invariant complete intersection of dimension d. Suppose that*

- *$B(M)$ holds;*
- *$H^0(M, E)^G$ almost separates orbits;*
- *the endomorphisms γ_g^{var}, $g \in G$, are linearly independent in $\mathrm{End}(H_d(X)^{\mathrm{var}})$;*
- *the Chow motive (M, Γ_χ) is finite dimensional;*
- *$0 \neq H_n(X)^{\mathrm{var}}$ and for some positive integer c we have $H_d(X)^{\mathrm{var},\chi} \subset \widehat{N}^c H^d(X)$.*

Then, for $k < c$ or for $k > d - c$ we have

$$i^*\colon A_{k+r}(M)^\chi \twoheadrightarrow A_k(X)^\chi, \quad i_*\colon A_k(X)^\chi \hookrightarrow A_k(M)^\chi.$$

Moreover, in this range

$$A_k^{\mathrm{var}}(X)^\chi = \ker(A_k(X)^\chi \xrightarrow{i_*} A_k(M)^\chi) = 0.$$

If in addition $H_k(M)^\chi = N^{[\frac{k}{2}]} H_k(M)^\chi$ for $k \leq d$, then $A_k^{\mathrm{AJ}}(X)^\chi = 0$ if $k < c$ or $k > d - c$.

We also have the analog of Corollary 6.7:

COROLLARY 7.6 *In the above situation, suppose that $c = [\frac{d}{2}]$. Then the motive $h(X, \Gamma_\chi)$ is finite-dimensional.*

8. Examples

8.1. *Hypersurfaces in abelian 3-folds*

We let A be an abelian variety of dimension 3. Let $\iota = -1_A$ be the standard involution. Choose an irreducible principal polarization L that is preserved by ι. The following facts are well known (see for example [19]).

Facts:

- L is ample and sections of $L^{\otimes 2}$ correspond to even theta functions (and hence are invariant under the involution).
- $L^3 = 3! = 6$ and $\dim H^0(L^{\otimes 2}) = 8$.
- The linear system $|L^{\otimes 2}|$ defines a 2-to-1 morphism $\kappa\colon A \to \mathrm{Km}(A) \subset \mathbf{P}^7 = \mathbf{P}H^0(L^{\otimes 2})^*$, where $\mathrm{Km}(A)$ is the Kummer 3-fold associated to A, an algebraic 3-fold, smooth outside the images of the 2^6 two-torsion points of A.

We let $X = \{\theta_0 = 0\} \subset A$ be a general divisor in $|L^{\otimes 2}|$. This is a smooth surface invariant under ι and κ induces an étale double cover of surfaces $X \to Y := X/(\iota|_X) \subset \mathrm{Km}(A)$. The surfaces X and Y are of general type. The crucial properties of X are as follows. We use the standard notation for the character spaces for the action of $\mathbf{Z}/2\mathbf{Z} = \{\mathrm{id}, \iota\}$ on a vector space V:

$$V^{\pm} = \{v \in V | \iota(v) = \pm v\}.$$

Proposition 8.1
(1) *We have* $H_1(X)^+ = 0$.
(2) *The splitting*

$$H_2^{\mathrm{var}}(X) = H_2^{\mathrm{var},+}(X) \oplus H_2^{\mathrm{var},-}(X)$$

is non-trivial and $H_2^{\mathrm{var},+}(X) = N^1 H_2^{\mathrm{var},+}(X)$, that is, $H^{2,0}(X)^{\mathrm{var},+} = 0$.

Before giving the proof, we observe that Theorem 7.5 and Corollary 7.6 imply:

Corollary 8.2 *We have* $A_0^{\mathrm{var}}(X)^+ = 0$ *and the motive* $h(X)^+ = h(Y)$ *is finite dimensional (of abelian type).*

Proof of Proposition 8.1. (1) Since ι acts as $-\mathrm{id}$ on 1-forms, $b_1(Y) = b_1(X)^+ = 0$. (2) We consider cohomology instead of homology. Consider the Poincaré residue sequence

$$0 \to \Omega_A^3 \to \Omega_A^3(X) \xrightarrow{\mathrm{res}} \Omega_X^2 \to 0.$$

In cohomology, this gives

$$0 \to H^0(\Omega_A^3) \to H^0(\Omega_A^3(X)) \xrightarrow{\mathrm{res}} H^0(\Omega_X^2) \to H^1(\Omega_A^3) \to 0.$$

Since $H^0(\Omega^3_A(X)) = H^0(L^{\otimes 2})$, we deduce that

$$h^{0,2}_{\mathrm{var}}(X) = 7, \quad h^{0,2}_{\mathrm{fix}}(X) = 3.$$

By the residue sequence, variable holomorphic 2-forms are the Poincaré-residues along X of meromorphic 3-forms on A with at most a simple pole along $X = \{\theta_0 = 0\}$ and are given by expressions of the form

$$\frac{\theta}{\theta_0} dz_1 \wedge dz_2 \wedge dz_3$$

with θ being a theta-function on A corresponding to a section of $L^{\otimes 2}$, and where z_1, z_2, z_3 are holomorphic coordinates on $\mathbf{C}^3$. It follows that such forms are *anti-invariant* under ι and so $h^{2,0}_{\mathrm{var}}(X) = h^{2,0}_{\mathrm{var},-}(X) = 7$.

To complete the proof, we need to show that $H^{1,1}_{\mathrm{var},+}(X) = H^{1,1}_{\mathrm{var}}(Y)$ is non-trivial. This is a consequence of the following calculation. □

LEMMA 8.3 The invariants of X and Y are as follows.

Variety	b_1	$b_2^{\mathrm{var}} = (h^{2,0}_{\mathrm{var}}, h^{1,1}_{\mathrm{var}}, h^{0,2}_{\mathrm{var}})$	$b_2^{\mathrm{fix}} = (h^{2,0}_{\mathrm{fix}}, h^{1,1}_{\mathrm{fix}}, h^{0,2}_{\mathrm{fix}})$
X	6	43 = (7, 29, 7)	15 = (3, 9, 3)
Y	0	7 = (0, 7, 0)	15 = (3, 9, 3)

Proof. By Lefschetz' theorem $b_1(X) = b_1(A) = 6$. To calculate b_2 we observe that $c_1(X) = -2L|_X$ and $c_2(X) = 4L^2|_X$ so that

$$c_1^2(X) = c_2(X) = 4L^2|_X = 8L^3 = 48.$$

Since $c_2(X) = e(X) = 2 - 2b_1(X) + b_2(X) = 48$, it follows that $b_2(X) = 58$. Now $b_2^{\mathrm{fix},+}(X) = b_2(A) = 15$ and so $b_2^{\mathrm{var}}(X) = 43$. The 2-forms on X that are the restrictions of holomorphic 2-forms on A are clearly invariant and $h^{2,0}_{\mathrm{fix}}(X) = h^{2,0}_{\mathrm{fix},+}(X) = 3$. Since $h^{2,0}_{\mathrm{var}} = 7$, the invariants for X follow.

For $b_2(Y)$, we use that $\iota|X$ acts freely on the generic X and so $e(Y) = \frac{1}{2}e(X) = \frac{1}{2}c_2(X) = 24 = 2 + b_2(Y)$ implying that $b_2(Y) = 22$. Using Künneth, we find $b_2^{\mathrm{fix},+}(X) = b_2^+(A) = b_2(A) = 15$ and so $b_2^{\mathrm{fix},+}(X) = 15$ and $b_2^{\mathrm{var},+}(X) = 7$. Since $h^{2,0}_{\mathrm{var},+}(X) = 0$, this yields the invariants for Y. □

8.2. *Hypersurfaces in products of a hyperelliptic curve and a K3-surface*

Let C be a hyperelliptic curve with hyperelliptic involution ι_C, and let S be a K3-surface with $h(S)$ finite dimensional and which admits a fixed point free involution ι_2. Such surfaces exist; see for example the examples of Enriques surfaces in [**4**, Section 4] coming from a K3-surface with Picard number ≥ 19. By Remark 2.7, the motive of S (and hence of $M \coloneqq C \times S$) is finite dimensional. The involution $\iota = (\iota_1, \iota_2)$ acts without fixed points on M. We let L_1 be the hyperelliptic divisor on C and we pick a very ample divisor L_2 on S invariant under the Enriques involution ι_2 and we set $L = L_1 \boxtimes L_2$. Let

$$i \colon X \hookrightarrow M = C \times S$$

be a smooth hypersurface in $|L|$ invariant under ι. Since ι has no fixed points, $Y = X/\iota$ is a smooth surface. Since it has an étale cover with ample canonical divisor, Y is a surface of general type. The analogs of Proposition 8.1 and its corollary are valid here.

Proposition 8.4 *We have*

- $H_1(X)^+ = 0$;
- $H^{2,0}(X)^{\mathrm{fix},+} = 0$;
- *the splitting*

$$H_2^{\mathrm{var}}(X) = H_2^{\mathrm{var},+}(X) \oplus H_2^{\mathrm{var},-}(X)$$

  is non-trivial and $H^{2,0}(X)^{\mathrm{var},+} = 0$;
- $A_0(X)^{\mathrm{var},+} = 0$ *and the motive* $h(X)^+ = h(Y)$ *is finite dimensional of abelian type*.

Proof. To simplify notation, we write $2u = L_2^2$ with $u \in \mathbf{Z}$, which is possible since L_2^2 is even.
Step 1. *Calculation of the Betti numbers of* X and Y
We claim:

- $b_1(X) = 2g$ and $b_2(X) = 4g + 4(g+2)u + 46$,
- $b_1(Y) = 0$ and $b_2(Y) = g + (2g+2)u + 22$.

To show this, observe that the Künneth formula and the Lefschetz hyperplane theorem imply $b_1(X) = b_1(M) = b_1(C) = 2g$ and $b_1(Y) = b_1^+(X) = b_1^+(C) = 0$. To calculate $b_2(X)$, we calculate the Euler number $e(X) = c_2(X)$ from the Whitney product formula

$$\begin{aligned}&(1 + c_1(j^*L))(1 + c_1(X) + c_2(X))\\ &\quad = 1 + (2-2g)P_1 + 24P_2 + \cdots, \quad P_1 = i^*p_1^*[C], \ \ P_2 = i^*p_2^*[S]\end{aligned}$$

which gives $c_1(X) = (2-2g)P_1 - c_1(i^*L)$ and hence

$$\begin{aligned}c_2(X) &= 24P_2 - c_1(j^*L)c_1(X)\\ &= 24P_2 + (2g-2)P_1 \cdot c_1(i^*L) + c_1^2(i^*L)\\ &= 24P_2 + (2g-2)P_1 \cdot (2P_1 + \ell_2) + (2P_1 + \ell_2)^2, \quad \ell_2 = c_1(i^*p_2^*L_2).\end{aligned}$$

Identifying $H^4(X, \mathbf{Z})$ with the integers, we have

$$P_1^2 = 0, \quad P_2 = 2, \quad (P_1 \cdot \ell_2) = L_2^2 = 2u, \quad \ell_2^2 = 4u.$$

and so

$$c_2(X) = 48 + 4(g+2)u = 2 - 4g + b_2(X) \Longrightarrow b_2(X) = 46 + 4g + 4(g+2)u.$$

We calculate $b_2(Y)$ from the Euler number of Y as follows.

$$\begin{aligned} 2 + b_2(Y) = e(Y) &= \frac{1}{2}e(X) = \frac{1}{2}(2 - 2g + b_2(X)) \\ &\Longrightarrow b_2(Y) = \frac{1}{2}b_2(X) - (g+1) = 22 + g + 2(g+2)u. \end{aligned}$$

Step 2. *Variable and fixed homology*
Remarking that the fixed cohomology equals $\mathrm{Im}(i^*: H^2(M) \hookrightarrow H^2(X))$, we find $b_2^{\mathrm{fix}}(X) = b_2(M) = b_2(C) + b_2(S) = 23$. Since $M/\iota = \mathbf{P}^1 \times \{\text{Enriques surface}\}$, we find $b_2^{\mathrm{fix}}(X) = b_2(M/\iota) = 11$. We put the result in a table.

Variety	b_2^{var}	b_2^{fix}
X	$4g + 4(g+2)u + 23$	23
Y	$g + (2g+2)u + 22$	11

Step 3. *Hodge numbers of X*
As one readily verifies, the fixed cohomology has Hodge numbers

$$h^{2,0}_{\mathrm{fix}}(X) = 1, \quad h^{1,1}_{\mathrm{fix}}(X) = 21.$$

For the variable cohomology, we have

$$h^{2,0}_{\mathrm{var}}(X) = (g+1)\cdot u + g + 2, \quad h^{1,1}_{\mathrm{var}}(X) = 2(g+3)u + 2g - 2.$$

To see this, consider the Poincaré residue sequence in this situation.

$$0 \to \Omega^3_M \to \Omega^3_M(X) \xrightarrow{\mathrm{res}} \Omega^2_X \to 0.$$

From the long exact sequence in cohomology, we deduce that

$$\Omega^3_M(X) = p_1^*\omega_C(L_1) \wedge p_2^*\omega_S(L_2) \Longrightarrow h^{2,0}_{\mathrm{var}}(X) = h^0(C, \omega_C \otimes L_1)\cdot(h^0(S, L_2) - g). \tag{8.1}$$

By Riemann–Roch $h^0(C, \omega_C \otimes L_1) = h^1(L_1^*) = g + 1$ and $h^0(S, L_2) = u + 2$. The result for $h^{2,0}_{\mathrm{var}}(X)$ follows.
Step 4. *Hodge numbers of Y*
From the fact that M/ι is the product of $\mathbf{P}^1$ and an Enriques surface, we that find $h^{2,0}_{\mathrm{fix}+} = 0$ and $h^{1,1}_{\mathrm{fix}+} = 11$. To find the Hodge numbers for the variable cohomology, we use a basic observation. □

LEMMA 8.5 *We have $h^0(C, \omega_C \otimes L_1)^+ = 0$.*

Proof. Invariant meromorphic 1-forms on C having a pole at most in the hyperelliptic divisor correspond to meromorphic 1-forms on $\mathbf{P}^1$ with at most one pole. But there are no such forms. □

As a corollary, from (8.1), it then follows that $h^{0,2}_{\mathrm{var}}(X)^+ = 0$ and so $H^2(X)^+_{\mathrm{var}}$ is pure of type $(1, 1)$. We claim that $H^2(X)^+_{\mathrm{var}} \neq 0$. Indeed, our calculations lead to the following table.

Variety	$(h^{2,0}_{\mathrm{var}}, h^{1,1}_{\mathrm{var}}, h^{0,2}_{\mathrm{var}})$	$(h^{2,0}_{\mathrm{fix}}, h^{1,1}_{\mathrm{fix}}, h^{0,2}_{\mathrm{fix}})$
X	$((g+1)u + g + 2, 2(g+3)u + 2g + 21, g+1)u + g + 2)$	$(1, 21, 1)$
Y	$(0, 2(g+3)u + 2g + 10, 0)$	$(0, 11, 0)$

8.3. *Hypersurfaces in products of three curves*

Let $M = C_1 \times C_2 \times C_3$ where C_α are curves equipped with an involution ι_α. Assume that L_α is a very ample line bundle on C_α which is preserved by ι_α and such that the system $|L_\alpha|^{\iota_\alpha}$ gives a morphism. Put $\iota = (\iota_1, \iota_2, \iota_3)$ and let $X \subset M$ be a general member of the system $|L_1 \otimes L_2 \otimes L_3|^\iota$ where we identify L_α with its pull back to M. The group G generated by the three involutions ι_α acts on M. As in the previous subsections, one can calculate the various character spaces for the action of G on $H_2(X)^{\mathrm{var}}$. Suppose one factor, say C_1, is hyperelliptic. Using Lemma 8.5, one sees that this makes the niveau of $H_2(X)^{\iota_1,\mathrm{var}}$ equal to 1. Choosing the other factors suitably so that all character spaces appear in $H_2(X)^{\mathrm{var}}$ one finds (many) projectors π with $A_0^{\mathrm{AJ,var}}(X, \Gamma_\pi) = 0$. Let us give one concrete example.

We let C_1 be a genus g hyperelliptic curve, and C_2, C_3 genus 3 unramified double covers of some genus 2 curve. We take for L_1 the degree 2 hyperelliptic bundle and we take for L_α, $\alpha = 2, 3$, the degree 2 bundles for which the system $|L_\alpha|$ induces the unramified double cover of C_α onto the genus 2 curve. Note that ι acts without fixed points in this case. As before, we obtain a surface of general type $Y := X/\iota$. We find the following invariants.

Variety	b_1	$(h^{2,0}_{\mathrm{var}}, h^{1,1}_{\mathrm{var}}, h^{0,2}_{\mathrm{var}})$	$(h^{2,0}_{\mathrm{fix}}, h^{1,1}_{\mathrm{fix}}, h^{0,2}_{\mathrm{fix}})$
X	$2(g+6)$	$(7g + 16, 14g + 477g + 16)$	$(6g + 9, 12g + 21, 6g + 9)$
Y	8	$(0, 12g + 28, 0)$	$(4, 8, 4)$

Concluding, $H^2_{\mathrm{var},+}(X)$ is pure of type $(1, 1)$ and $H^2_{\mathrm{var}}(X)$ contains an invariant and anti-invariant part so that we can apply our considerations to the motive $(X, \frac{1}{2}(1 + \iota))$ and hence

$$A_0^{\mathrm{AJ,var}}(X)^+ = 0.$$

It follows, as before, that $h(Y) = h(X)^+$ is finite dimensional.

Remark 8.6 Using [22], we have that the map

$$A_1^{\mathrm{hom}}(Y) \otimes A_1^{\mathrm{hom}}(Y) \to A_0^{\mathrm{AJ}}(Y)$$

induced by intersection product is surjective, like in the case of an abelian surface. To see this, consider the commutative diagram

$$\begin{array}{ccc} H^{1,0}(Y) \otimes H^{1,0}(Y) & \xrightarrow{\quad \wedge \quad} & H^{2,0}(Y) \\ \downarrow \simeq & & \downarrow \simeq \\ \bigotimes^2 \left(H^{1,0}(C_2/\iota_2) \oplus H^{1,0}(C_3/\iota_3)\right) & \twoheadrightarrow & H^{1,0}(C_2/\iota_2) \otimes H^{1,0}(C_3/\iota_3), \end{array}$$

which shows that the top-line is a surjection.

8.4. *Odd-dimensional complete intersections of four quadrics*

The following example is due to Bardelli [3]. Let $\iota\colon \mathbf{P}^7 \to \mathbf{P}^7$ be the involution defined by

$$\iota(x_0\colon \dots\colon x_3\colon y_0\colon \dots\colon y_3) = (x_0\colon \dots\colon x_3\colon -y_0\colon \dots\colon -y_3).$$

Let $X = V(Q_0, \dots, Q_3)$ be the intersection of four ι-invariant quadrics. Then $H^{3,0}(X)^- = 0$, hence $H^3(X)^-$ is a Hodge structure of level 1. Bardelli showed that there exist a smooth curve C and a correspondence $\gamma \in \mathrm{Corr}_1(C, X)$ such that $\gamma_*\colon H_1(C) \to H_3(X)^-$ is surjective. Hence $H_3(X)^- \subseteq \widetilde{N}^1 H_3(X) = \widehat{N}^1 H_3(X)$. By Theorem 7.5, we get $A_0^{\mathrm{AJ}}(X)^- = 0$.

Consider the projector $p = \frac{1}{2}(\mathrm{id}_X - \iota^*)$. As $\iota_* = \iota^*$, we have ${}^t p = p$. Hence, the motive $N = (X, p)$ satisfies $N \cong N^\vee(3)$ and we can apply [40, Theorem 3.11] to the map $i^*\colon M = (\mathbf{P}^7, \frac{1}{2}(\mathrm{id}_{\mathbf{P}} - \iota^*)) \to N$. This shows that the motive $N = h(X)^-$ is finite dimensional; more precisely, it is a direct factor of $M' = M \oplus M^\vee(3) \oplus_i h(C_i)(i)$ for some curves C_i. As $A_i^{\mathrm{AJ}}(M') = 0$ for all i, we obtain that

$$A_i^{\mathrm{AJ}}(X)^- = 0$$

for all i. In other words, the quotient morphism $f\colon X \to Y := X/\iota$ induces an isomorphism

$$f^*\colon A^*_{\mathrm{AJ}}(Y) \xrightarrow{\cong} A^*_{\mathrm{AJ}}(X).$$

This example can be generalized to higher dimension.

Theorem 8.7 *Let ι be the involution on $\mathbf{P}^{2m+3}$ $(m \geq 2)$ defined by*

$$\iota(x_0\colon \cdots\colon x_{m+1}\colon y_0\colon \cdots\colon y_{m+1}) = (x_0\colon \cdots\colon x_{m+1}\colon -y_0\colon \cdots\colon -y_{m+1})$$

and let $X = V(Q_0, \dots, Q_3)$ be a complete intersection of four ι-invariant quadrics. Let $G = \{\mathrm{id}, \iota\}$ and let $\chi\colon G \to \{\pm 1\}$ be the character defined by $\chi(\iota) = (-1)^{m-1}$. Then $H^{2m-1}(X)^\chi$ is a Hodge structure of level 1, and there exist a smooth curve C and a correspondence $\gamma \in \mathrm{Corr}_{m-1}(C, X)$ such that $\gamma_*\colon H_1(C) \to H_{2m-1}(X)^\chi$ is surjective.

Proof. See [30, Chapter 3] *or* [31, Chapter 4]. □

COROLLARY 8.8 *The motive* $h(X)^\chi$ *is finite dimensional and* $A_i^{AJ}(X)^\chi = 0$ *for all* i.

REMARK 8.9 The same reasoning can be applied to the examples in [44] and the 3-fold studied in [35]. Also, as detailed in [26], our method gives an easy proof of the Bloch conjecture for Burniat–Inoue surfaces (this is first proven in [33, Theorem 9.1] and [5]). A more involved application of our method is given in [36].

Acknowledgements

We want to thank Claire Voisin, who kindly suggested the example of Section 8.1 to one of us. Thanks also to Claudio Pedrini, for helpful comments on an earlier version of this article.

References

1. Y. André, Motifs de dimension finie (d'après S.-I. Kimura, P. O'Sullivan,..), Séminaire Bourbaki, *Astérisque* **299** (2003/2004), Exp. No. 929, viii, 115–145.
2. J. Ayoub, Motives and algebraic cycles: a selection of conjectures and open questions, preprint available from user.math.uzh.ch/ayoub/.
3. F. Bardelli, On Grothendieck's generalized Hodge conjecture for a family of threefolds with trivial canonical bundle, *J. Reine Angew. Math.* **422** (1991), 165–200.
4. W. Barth and C. Peters, Automorphisms of Enriques surfaces, *Invent. Math.* **73** (1983), 383–411.
5. I. Bauer and D. Frapporti, Bloch's conjecture for generalized Burniat type surfaces with $p_g = 0$, *Rend. Circ. Mat. Palermo (2)* **64** (2015), 27–42.
6. S. Bloch and A. Ogus, Gersten's conjecture and the homology of schemes, *Ann. Sci. Ecole Norm. Sup* **4** (1974), 181–202.
7. S. Bloch and V. Srinivas, Remarks on correspondences and algebraic cycles, *Amer. J. Math.* **10** (1983), 1235–1253.
8. M. de Cataldo and L. Migliorini, The Chow groups and the motive of the Hilbert scheme of points on a surface, *J. Algebra* **251** (2002), 824–848.
9. F. Charles, Remarks on the Lefschetz standard conjecture and hyperkähler varieties, *Comment. Math. Helv.* **88** (2013), 449–468.
10. F. Charles and E. Markman, The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of $K3$ surfaces, *Comp. Math.* **149** (2013), 481–494.
11. L. Fu, On the coniveau of certain sub-Hodge structures, *Math. Res. Lett.* **19** (2012), 1097–1116.
12. V. Guletski and C. Pedrini, The Chow motive of the Godeaux surface, *Algebraic Geometry, A Volume in Memory of Paolo Francia* (Eds. M. C. Beltrametti, F. Catanese, C. Ciliberto, A. Lanteri and C. Pedrini), Walter de Gruyter, Berlin New York, 2002.
13. F. Ivorra, Finite dimensional motives and applications (following S.-I. Kimura, P. O'Sullivan and others), available from https://perso.univ-rennes1.fr/florian.ivorra/

14. U. Jannsen, Motivic sheaves and filtrations on Chow groups, in *Motives* (Eds. U. Jannsen, S. Kleiman and J.-P. Serre), Proceedings of Symposia in Pure Mathematics Vol. 55, Part 1, American Mathematical Society, Providence, RI, 1994, 245–302.

15. U. Jannsen, On finite-dimensional motives and Murre's conjecture, *Algebraic Cycles and Motives* (Eds. J. Nagel and C. Peters), Cambridge University Press, Cambridge, 2007.

16. S. Kimura, Chow groups are finite dimensional, in some sense, *Math. Ann.* **331** (2005), 173–201.

17. S. Kleiman, Algebraic cycles and the Weil conjectures, *Dix exposés sur la cohomologie des schémas*, North-Holland, Amsterdam, 1968, 359–386.

18. S. Kleiman, The standard conjectures, in *Motives* (Eds. U. Jannsen, S. Kleiman and J.-P. Serre), Proceedings of Symposia in Pure Mathematics Vol. 55, Part 1. American Mathematical Society, Providence, RI, 1994. (1994).

19. A. Lange and C. Birkenhake, *Complex Abelian Varieties*, Springer-Verlag, Berlin, 1992.

20. R. Laterveer, Algebraic varieties with small Chow groups, *J. Math. Kyoto Univ.* **38** (1998), 673–694.

21. R. Laterveer, Variations on a theorem of Voisin, submitted.

22. R. Laterveer, On a multiplicative version of Bloch's conjecture, *Beiträge Algebra Geom.* **57** (2016), 723–734.

23. R. Laterveer, A brief note concerning hard Lefschetz for Chow groups, *Can. Math. Bull.* **59** (2016), 144–158.

24. R. Laterveer, A remark on the motive of the Fano variety of lines of a cubic, *Ann. Math. Québec* **41** (2017), 141–154.

25. R. Laterveer, Algebraic cycles on Fano varieties of some cubics, *Results Math*. **72** (2017), 595–616.

26. R. Laterveer, J. Nagel and C. Peters, On complete intersections in varieties with finite-dimensional motive (longer version), arXiv:1709.10259.

27. J. Lewis, The cylinder correspondence for hypersurfaces of degree n in $\mathbf{P}^n$, *Amer. J. Math.* **110** (1988), 77–114.

28. J. Murre, On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II, *Indag. Math.* **4** (1993), 177–201.

29. J. Murre, J. Nagel and C. Peters, *Lectures on the Theory of Pure Motives,* University Lecture Series 61, American Mathematical Scoiety, Providence, RI (2013).

30. J. Nagel, *The Abel-Jacobi map for complete intersections*, Ph.D. Thesis, University of Leiden, 1997, http://nagel49.perso.math.cnrs.fr/thesis.pdf.

31. J. Nagel, *Cohomology of quadric bundles*, Habilitation Thesis, Université Lille 1, 2006, http://nagel49.perso.math.cnrs.fr/habilitation.pdf.

32. K. Paranjape, Cohomological and cycle-theoretic connectivity, *Ann. Math. (2)* **139** (1994), 641–660.

33. C. Pedrini and C. Weibel, Some surfaces of general type for which Bloch's conjecture holds, Recent Advances in Hodge Theory (Ed. M. Kerr), London Mathematical Society Lecture Note Series 427, Cambridge University Press, Cambridge, 2016, 308–329.

34. C. Pedrini, On the finite dimensionality of a $K3$ surface, *Manuscr. Math.* **138** (2012), 59–72.

35. C. Peters, Bloch-type conjectures and an example of a three-fold of general type, *Commun. Contemp. Math.* **12** (2010), 587–605.

36. C. Peters, A motivic study of generalized Burniat surfaces, arXiv:1710.02370.

37. S. Tankeev, On the standard conjecture of Lefschetz type for complex projective threefolds. II, *Izvestiya Math.* **75** (2011), 1047–1062.

38. C. Vial, Algebraic cycles and fibrations, *Documenta Math.* **18** (2013), 1521–1553.

39. C. Vial, Projectors on the intermediate algebraic Jacobians, *NY J. Math.* **19** (2013), 793–822.

40. C. Vial, Remarks on motives of abelian type, *Tohoku Math. J.* **69** (2017), 195–220.

41. C. Vial, Niveau and coniveau filtrations on cohomology groups and Chow groups, *Proc. Lond. Math. Soc.* **106** (2013), 410–444.

42. C. Vial, Chow–Künneth decomposition for 3- and 4-folds fibred by varieties with trivial Chow group of zero-cycles, *J. Alg. Geom*. **24** (2015), 51–80.

43. C. Voisin, Remarks on filtrations on Chow groups and the Bloch conjecture, *Ann. Mat. Pura Appl.* **183** (2004), 421–438.

44. C. Voisin, Sur les zéro-cycles de certaines hypersurfaces munies d'un automorphisme, *Ann. Scuola Norm. Sup. Pisa Cl. Sci* **19** (1992), 473–492.

45. C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, *Ann. Sci. École Norm. Sup*. **46** (2013), 449–475.

46. C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, II, *J. Math. Sci. Univ. Tokyo* **22** (2015), 491–517.

47. C. Voisin, Bloch's conjecture for Catanese and Barlow surfaces, *J. Differ. Geom.* **97** (2014), 149–175.

48. A. Weil, *Introduction à l'étude des variétés kählériennes*. Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Actualités Sci. Ind. 1267, Hermann, Paris, 1958.

49. Z. Xu, Algebraic cycles on a generalized Kummer variety, arXiv:1506.04297v1.