Hensel's p-adic Numbers: early history

Fernando Q. Gouvêa

Notes for a talk at the AMS regional meeting in Providence, RI, October 2-3, 1999. Please note that this is still a preliminary report. Corrections, comments, and suggestions are welcome. I'm especially interested in finding references to p-adic numbers in papers published before ca. 1935.

Fernando Q. Gouvêa
Dept. of Mathematics
Colby College
Waterville, ME 04901
fqgouvea@colby.edu

Introduction: why I became interested in the history of p-adic numbers.

- Request for an article from Abe Shenitzer. I'm somewhat embarrassed that I didn't put more historical information in my book!
- What the history books say is not very satisfying.
- A Kronecker student creating a new number system?
- Why did it take so long to get accepted?

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What are they?

Let p be a prime number.

Any positive integer can be written "in base p" as

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n,$$

with $0 \leq a_i \leq p - 1$.

Idea: generalize by allowing infinite expansions!

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The p-adic numbers are the set of all "Laurent expansions in p", i.e., things that look like

$$a_{-r} p^{-r} + \cdots + a_{-1} p^{-1} + a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n + \ldots$$

again with $0 \leq a_i \leq p - 1$.

They form a field called $\mathbb{Q}_p$.

More generally, one can consider similar expansions in fields of algebraic numbers.

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Pre-history

- Kummer's "ideal prime divisors" as a way to do arithmetic over cyclotomic fields.
- Kummer's use of "p-adic methods."
- Dedekind generalizes the method to algebraic number fields.
- Dedekind and Weber show that the same general approach can be applied to function fields in one variable.

Besides emphasizing the analogy here (as Hensel himself does), I should also mention the theorems about how a prime splits in a field $\mathbb{Q}(\theta)$. One considers the minimal polynomial $p(X)$ of θ and its reduction modulo powers of p. There is an R such that factoring $p(X)$ modulo p^R and higher powers yields the factorization of p in $\mathbb{Q}(\theta)$. Thinking in these terms suggests that we write the integers which are the coefficients of $p(X)$ *in base* p, which certainly starts our thoughts in the right direction. This is in Hensel too, in the 1897 paper.

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A "Rosetta Stone"

Geometry	Function Fields	Number Theory
"Riemann sphere"	$\mathbb{C}(X)$	$\mathbb{Q}$
complex plane	$\mathbb{C}[X]$	$\mathbb{Z}$
point $\alpha \in \mathbb{C}$	irreducible $X - \alpha$	prime p
Riemann surface	finite extension $K/\mathbb{C}(X)$	finite extension $K/\mathbb{Q}$
"local" behavior	Laurent series around α	!??

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Hensel's Idea

To study behavior of a function "near α", use Taylor or Laurent series in powers of $(X - \alpha)$ to supplement the algebraic approach of Dedekind and Weber.

To study an algebraic number, use an expansion in terms of powers of a prime number p.

Note: in each case, may need *rational* powers.

(First publication 1897, many subsequent papers. Books in 1908 and 1913)

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Birth: Hensel's Theory

The idea for $\alpha \in \mathbb{Q}$:

- Consider first $\alpha = \frac{a}{b}$ with b not divisible by p.
- Since b is then invertible modulo p, $\alpha \equiv a_0 \pmod{p}$ for some $0 \leq a_0 \leq p - 1$.
- Write $\alpha = a_0 + p\alpha_1$.
- Repeat with α_1.
- If p does divide b, work with $p^r\alpha$, then divide by p^r.

For algebraic numbers, slightly more delicate.

- p-adic expansion of a given algebraic number
- first formally, then some attempt at defining convergence
- later focuses on the field of all possible expansions

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At first, Hensel talked of finding the p-adic expansion of a given algebraic number.

Then, he started considering the set of all possible expansions as an object of independent interest.

At first, basically a formal construction.

Later, various attempts to justify the "convergence" of p-adic series.

Hensel's attempts at dealing with the issue of convergence are mostly pretty confused (and confusing!):

- "Mass number"
- Finding series that converge both p-adically and in $\mathbb{C}$ to a given number.

On the other hand, from the beginning he introduced the concept of the "order" of a number (what we would call the p-adic valuation).

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Immediate Applications

- Factors of the discriminant of $K/\mathbb{Q}$ (1897).
- Factorization of p in $K = \mathbb{Q}(\theta)$ in terms of p-adic factorization of minimal polynomial for θ (1904ff).
- Simplification (?) of basic algebraic number theory (1904ff).

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Difficulties

- Hensel's "proof" that e is transcendental (1905). (See Peter Ullrich's papers.)
- Hilbert's opinion. (?)
- Questions of "style." Explicit methods versus abstract methods.
- Did Hensel push too hard?

Hensel's incorrect proof goes like this. Start from the equation

$$e^p = \sum_{n=0}^{\infty} \frac{p^n}{n!}.$$

Hensel checks that this series converges in $\mathbb{Q}_p$ and concludes that it satisfies an equation of the form $y^p = 1 + p\varepsilon$ with ε a p-adic unit. If e is algebraic, it follows that $(\mathbb{Q}(e) : \mathbb{Q}) \geq p$. But p was arbitrary, so we have a contradiction, and e is transcendental. The problem is that the fact that the series converges in $\mathbb{Q}_p$ does not mean that it converges to e^p (even if e is assumed algebraic).

The difficulty seems to be that topological concepts are still not clear in anyone's mind, and the argument that the equality in $\mathbb{R}$ allows us to rearrange the left hand side to get a p-adic expansion is superficially plausible.

The role of this "proof" in the early history is emphasized by Peter Ullrich in his "On the Origins of p-adic Analysis" (Symposia Gaussiana, 1995) and "The Genesis of Hensel's p-adic numbers" (in *Charlemagne and his Heritage: 1200 Years of Civilization and Science in Europe*, volume 2 (Mathematical Arts), date?).

Weil, writing in intro to Kummer's collected papers:

> Hilbert dominated German mathematics for many years after Kummer's death. More than half of his famous *Zahlbericht* is little more than an account of Kummer's number-theoretical work, with inessential improvements; but his lack of sympathy for his predecessor's mathematical style, and more specifically for his brilliant use of p-adic analysis, shows clearly through many of the somewhat grudging references to Kummer in that volume.

F. Lemmermeyer and N. Schappacher, writing in the introduction to the recent translation of Hilbert's *Zahlbericht*:

> Readers of the first generation, echoing Hilbert's announcement, have praised the *Zahlbericht* in particular for having simplified Kummer. Thus, Hasse in 1932 mentions Kummer's "complicated and less than transparent proofs," which Hilbert replaced by new ones. In 1951, however, Hasse added an afterthought describing the two rival traditions in the history of number theory: on the one hand, there is the Gauss-Kummer tradition which aims at explicit, constructive control of the objects studied, and which was carried on in particular by Kronecker and Hensel. The Dedekind-Hilbert tradition, on the other hand, aims above all at conceptual understanding.

From Bourbaki's "Éléments d'Histoire":

> But for a long time it seems as if the p-adic numbers inspired a great distrust in contemporary mathematicians; an attitude current without doubt towards ideas that were too "abstract," but that the slightly excessive enthusiasm of their author (so frequent in mathematics amongst the zealots of new theories) was not without justifying in part.

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Algebra, Topology

1910: Steinitz publishes his fundamental paper on the abstract theory of fields, probably the first paper on “abstract algebra.” The p-adic numbers are cited as a major motivation for the new theory.

ca. 1910: Topological ideas, especially due to Fréchet and Riesz, become clearer, allowing a new understanding of the p-adic numbers.

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The p-adics as a foundational tool

Hensel showed, in several papers and in his books, that one could give a simpler account of the theory of divisibility in algebraic number fields if one used his p-adic methods.

This seems to have eventually been understood. See, for example, “A New Development of the Theory of Algebraic Numbers,” by G. E. Wahlin (Transactions AMS, 1915) and the 1923 *Algebraic Numbers* report by Dickson, Mitchell, Vandiver, Wahlin.

After Steinitz, the p-adics also start showing up as a standard example in papers on algebra.

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p-adic Analysis

- Hensel already considered elementary functions defined by power series, from 1907 on.
- Kúrschak introduces valuations (1912). The p-adic numbers are interpreted in terms of the topology of metric spaces.
- Ostrowski's theorem on valuations of $\mathbb{Q}$ (1917).
- 1920–1935: complete theory of valuations (Deuring, Schmidt, Krull, etc.).
- Strassman studies functions defined by power series (1926).

Note that one of the difficulties here is that one does not have the Cauchy Integral Theorem. But Weierstrass had already worked out how to get much of function theory directly in terms of power series, so Hensel and others follow his lead.

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Rediscovery: Hasse and the Local-Global Principle

- A chance encounter in a used book shop (1920).
- Hasse's thesis: the local-global principle (1922).

Hasse's bookshop story:

> [After Hecke left Göttingen] I decided to continue my studies under Kurt Hensel in Marburg … What prompted my decision was this. In 1913 Hensel published a book on number theory. I found a copy of this book at an antiquarian's in Göttingen and bought it. I found his completely new methods fascinating and worthy of thorough study.(…)

(From "Kurt Hensels entscheidener Anstoss zur Entdeckung des Lokal-Global-Prinzips," Crelle, (reference?).)

The theorem on binary quadratic forms now known as the Hasse–Minkowski Theorem was the crucial example that showed that the application on p-adic methods could "burst open" a problem. In particular, the idea of checking a property "everywhere locally" became quickly a central idea in number theory.

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Despite all this, Hermann Weyl's 1938 book still needs to argue the case.

"The ultimate verdict may be that the one outstanding way for any deeper penetration into the subject is the Kummer-Hensel p-adic theory." (Weyl, *Algebraic Theory of Numbers*, 1940, based on lectures given in 1938–39)

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Some comments

- p-adic *methods* have a long history (even before Kummer).
- p-adic *numbers* were created as a result of an extended analogy between two fields of mathematics.
- p-adic numbers came first, the formal theory (fields, valuations) came later.
- The theory interested algebraists and people interested in foundational issues first.
- Acceptance of the theory was affected by the social dynamics of the mathematical community.