Lecture notes, Marburg, 2012, "Ergodicity of quantum maps". (5 lectures of 50'.)

30/8/12

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Contents

I classical maps (dynamics of map $f : M \to M$)

ergodic map, mixing map, hyperbolic or Anosov map,

II Quantum maps (Weyl quantization of f if $M = \mathbb{T}^{2d} = (T^* \mathbb{R}^d)/\mathbb{Z}$)

"phase space" is a cotangent space

. → family of unitary operators: $\hat{f}_\hbar : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$,
with sequence of indices $\hbar = \hbar_N \xrightarrow[N \to \infty]{} 0$.

Hilbert space

. phase space representation of $\psi \in \mathcal{H}_\hbar$, semiclassical measure.

III Quantum ergodicity

. theorem of quantum ergodicity (Q.E.)

. Counter example Quantum Unique ergodicity (QUE) for Anosov maps

. Results on entropy of eigenfunctions for Anosov maps.

IV Geometric quantization (of symplectic map $f : (M,\omega) \to (M,\omega)$)

. (general) pre-quantization and quantization of map f.

. Special case of $M = \mathbb{T}^{2d} = (T^* \mathbb{R}^d)/\mathbb{Z}^{2d}$.

V Conclusion, Perspectives

General references (reviews): (see web pages).

Jens Marklof, Steph Nonnenmacher, Roman Schubert, Kelmer thesis

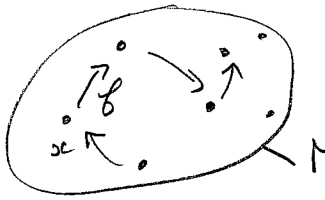
Steve Zelditch, Maciej Zworski, Gabriel Rivière, Nalini Anantharaman

Frederic Faure, Steve Rudnick

I Classical maps

1 ref: Books, • Brin, Stuck "Intro to dynamical systems"
• Katok, Hasselblatt " " "

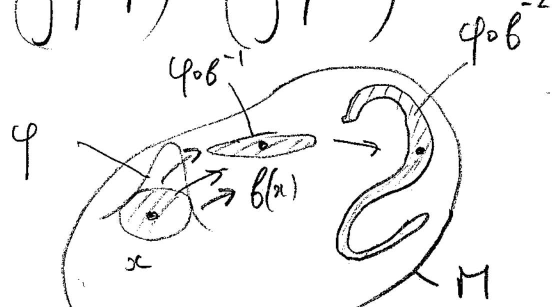
• **definition**: on a smooth compact manifold M,
2 let $f: M \to M$ be a smooth diffeomorphism.
The trajectory of $x \in M$, is the sequence
3 $\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x) := f(f(x)), \ldots, f^n(x), \ldots$



• **definition**: (Brin p. 75)
4 The map $f: M \to M$ is mixing if there exists a
5 measure $d\mu$ on M called "equilibrium measure" $\left(\int_M d\mu = 1\right)$
s.t. $\forall \varphi, \psi \in C^\infty(M)$,

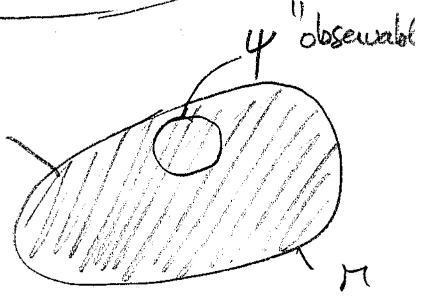
6 $$\underbrace{\int (\varphi \circ f^{-n}) \cdot \psi \, dx}_{\text{"correlation function" } C_{\varphi,\psi}(n)} \xrightarrow[n \to \infty]{} \left(\int \varphi \, d\mu\right) \cdot \left(\int \psi \, dx\right)$$

dx: any smooth measure



• rem: • mixing means "loss of information"
7 because for $n \to \infty$,
$\varphi \circ f^{-n}$ converges weakly towards "space average" $\int \varphi \, d\mu$ times measure dx.
• if f is non invertible,
8 replace $\varphi \circ f^{-n}$ by $\varphi \circ f^{n}$ or sum over pre-images.

$\varphi \circ f^{-n}$, $n \gg 1$



• f is exponentially mixing if $\exists \alpha > 0$, $\forall \varphi, \psi$,
9 $$\left| \int (\varphi \circ f^{-n}) \cdot \psi \, dx - \int \varphi \, d\mu \cdot \int \psi \, dx \right| = O\left(e^{-\alpha n}\right)$$

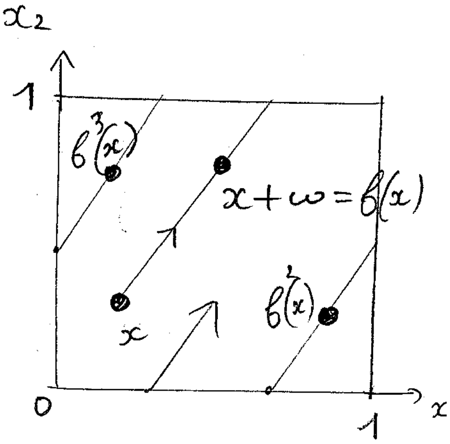
• definition (Brin p. 84)

$f : M \to M$ is ergodic if $\forall \varphi, \psi \in C^\infty(M)$,

1 $$\frac{1}{N} \sum_{n=0}^{N-1} \int \left(\varphi \circ f^{-n}\right) \cdot \psi \, dx \xrightarrow[N \to \infty]{} \int \varphi \, d\mu \cdot \int \psi \, dx$$

2 • rem : mixing $\Longrightarrow$ ergodic. (from Cesàro's th.)

• rem : ergodic means that "time average" of φ converges towards "space average" ($\int \varphi \, d\mu$) times measure dx.



• example : "Translation on torus"

let $M = \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$.

3 $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$.

4 map $f : \begin{cases} \mathbb{T}^d \longrightarrow \mathbb{T}^d \\ x \longmapsto x + \omega \pmod{\mathbb{Z}^d} \end{cases}$

then

5 1) f is ergodic with $d\mu = dx$, iff ω is "irrational"

i.e. if: $\forall k \in \mathbb{Z}^d$, $k \neq 0$,

6 then $k.\omega = k_1 \omega_1 + \dots + k_d \omega_d \notin \mathbb{N}$

7 (ie in dim $d = 1$, $\Leftrightarrow \omega \notin \mathbb{Q}$,

$d = 2$, $\Rightarrow \omega$ has an irrational slope $\frac{\omega_2}{\omega_1}$)

8 2) f is not mixing

9 3) if f is ergodic, it is uniquely ergodic,

i.e. dx is the only invariant measure.

<u>proof</u> : for $k \in \mathbb{Z}^d$, let

1 $$\varphi_k(x) := \exp(i\, 2\pi\, k \cdot x) \quad : \text{"Fourier Mode"}.$$

we will check (3-1) for Fourier modes which form a basis of $L^2(\mathbb{T}^d)$.

2 Then $$\int_{\mathbb{T}^d} (\varphi_k \circ f^{-n}) \cdot \varphi_l \, dx = \int \exp\left(i\, 2\pi\, k \cdot (x - n\omega) + i\, 2\pi\, l \cdot x\right) dx$$

3 $$= \exp\left(-i\, 2\pi\, n (k \cdot \omega)\right) \cdot \delta_{k=-l}$$

4 Then $$\frac{1}{N} \sum_{n=0}^{N-1} \int (\varphi_k \circ f^{-n}) \cdot \varphi_l \, dx \underset{\text{geometric series}}{=} \delta_{k=-l} \cdot \frac{1}{N} \frac{1-r^N}{1-r} \xrightarrow[N\to\infty]{} 0 \quad \text{if } r \neq 1$$

geometric series

$r = \exp(-i\, 2\pi (k \cdot \omega))$ $\Leftrightarrow (k \cdot \omega) \notin \mathbb{N}$

5 $$'' \quad = \delta_{k=-l} \quad \text{if } r = 1 \iff (k \cdot \omega) \in \mathbb{N}$$

for the right side of (3-1) :

6 $$\int \varphi_k \, dx \cdot \int \varphi_l \, dx = \delta_{k=0} \cdot \delta_{l=0}$$

One deduces (3-5) and (3-8). (from 4-3) with $k = -l$

(3-9) is also clear from its Fourier decomposition. ∎

ref wikipedia, used to detect tax evasion.

Corollary "<u>Benford law</u>" Consider the first digit k of the sequence $u_n = 2^n$ in basis 10 :

1, 2, 4, 8, 16, 32, 64, 128, 256, ...

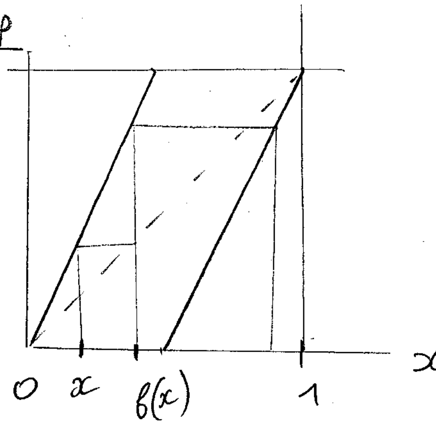
Then $1 \leq k \leq 9$ appears with probability $p_k = \dfrac{\log\left(1 + \frac{1}{k}\right)}{\log 10}$

$p_1 = 17\%$, $p_2 = 6\%$, ...

<u>proof</u> : $k \cdot 10^r \leq u_n < (k+1) \cdot 10^r$ with $r \in \mathbb{N}$, so

$D_k = [illegible] \left(\frac{\log(k+1)}{\log 10} - \frac{\log k}{\log 10}\right)$: because $v_n := \frac{\log u_n}{\log 10} = v_{n-1} + \frac{\log 2}{\log 10}$ is an irrat[illegible]

- <u>example</u> : "<u>linear expanding map</u>"



1 $$f : \begin{cases} \mathbb{T}^1 \longrightarrow \mathbb{T}^1 \\ x \longmapsto 2x \end{cases}$$

2 is <u>mixing</u>.

3 <u>rem</u> : . f is not invertible,

. f is even "super exponential mixing" i.e. (1-9)

4 holds for any $\alpha > 0$.

- <u>proof</u> : We check (2-6) for Fourier modes (4-1).

Let $k, l \in \mathbb{Z}$, then

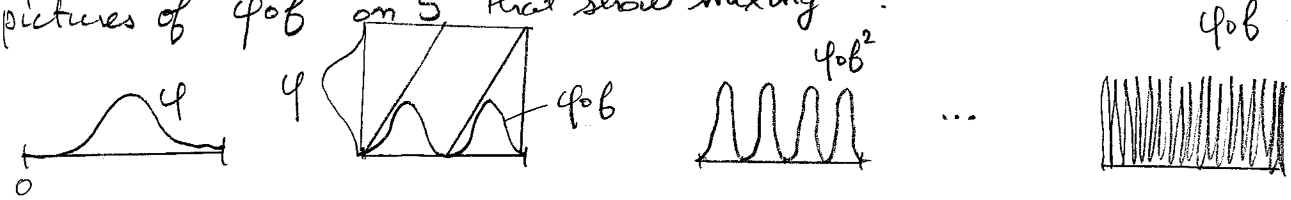
$$\int_{\mathbb{T}^1} (\varphi_k \circ f^n) \cdot \varphi_l \, dx = \int \exp\left(i\, 2\pi k\, 2^n x + i\, 2\pi\, l x\right) dx$$

$$= \delta_{2^n k = -l} \quad = 0 \quad \text{for } n \text{ large enough if } k \neq 0.$$

So this equals (4-6) for n large enough.

Finaly smooth functions φ, ψ have Fourier components which decay fast. We deduce (2-6). ∎

<u>rem</u> : pictures of $\varphi \circ f^n$ on S^1 that "shows mixing" :



. <u>ex</u> : show that μ in (2-6) and (3-1) is unique and is a measure.

- *example* : "Hyperbolic automorphism on torus".

1 is $f : \begin{cases} \mathbb{T}^d \longrightarrow \mathbb{T}^d \\ x \longmapsto Mx \mod \mathbb{Z}^d \end{cases}$

2 with $M \in SL_d(\mathbb{Z})$, hyperbolic matrix, i.e. eigenvalues λ satisfies $|\lambda| \neq 1$, $|\lambda| \neq 0$.

(*) ⟶

3 *rem* : . f is well defined because if $m \in \mathbb{Z}^d$, $M(x+m) = Mx + \underbrace{Mm}_{\in \mathbb{Z}^d} \equiv Mx \mod \mathbb{Z}$

4 . $f^{-1}(x) = M^{-1}x$, $M^{-1} \in SL_d(\mathbb{Z})$.

5 $\boxed{\text{• prop : } f \text{ is (super exp.) mixing}}$

6 • *rem* : if $M \in SL_d(\mathbb{Z})$ with no eigenvalue root of unity, then f is ergodic. (this is similar to 3-5).

proof : (similar to 5-2)

7 Let $k, l \in \mathbb{Z}^d$, we have

8 $$\int \left(\varphi_k \circ f^{-n}\right) \cdot \overline{\varphi_l}\, dx = \int \exp\left(i 2\pi \left(k \cdot M^{-n} x - l \cdot x\right)\right) dx$$
$$= \int \exp\left(i 2\pi \left({}^tM^{-n} k - l\right) x\right) dx = \delta_{{}^tM^{-n}k = l}$$

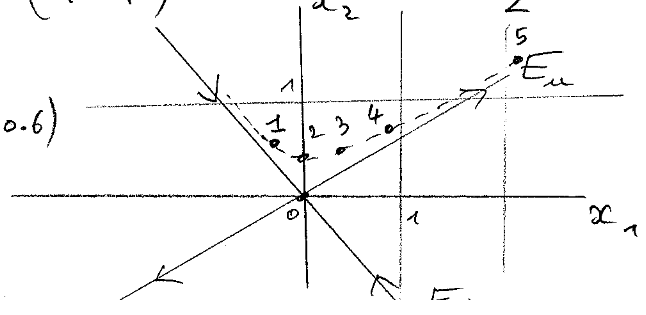
9 but if $k \neq 0$, then $\left|{}^tM^{-n} k\right| \xrightarrow[n \to \infty]{} \infty$ because M is hyperbolic.

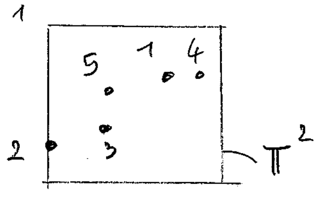
So (6-8) vanishes for n large enough. ∎

(*) simplest example is the "cat map", on $\mathbb{T}^2$,

10 $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\lambda = \lambda_u = \frac{3+\sqrt{5}}{2} \simeq 2.6 > 1$, $\lambda_s = \frac{1}{\lambda_u} < 1$ are eigenvalues

example : point 1 is $(-0.3, 0.6)$

11 



I.2 Hyperbolic (Anosov) map

- **déf**: a diffeomorphism $f : M \to M$ is <u>Anosov</u>, if there exists a Riem. metric g on M, an f-invariant continuous decomposition of TM,

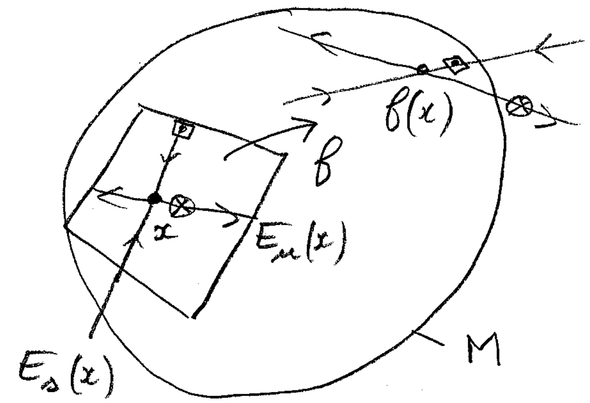
$$T_x M = E_u(x) \oplus E_s(x), \quad \forall x \in M,$$

a constant $\lambda > 1$, s.t. $\forall x \in M$,

$$|D_x f(v_s)|_g \leq \frac{1}{\lambda} |v_s|_g, \quad \forall v_s \in E_s(x),$$

$$|D_x f^{-1}(v_u)|_g \leq \frac{1}{\lambda} |v_u|_g, \quad \forall v_u \in E_u(x)$$

We call $\begin{cases} E_u(x) : \text{unstable direction} \\ E_s(x) : \text{stable direction} \end{cases}$



- example: <u>Hyperbolic automorphism on $\mathbb{T}^d$</u>, (6-1). because $Df = M$ is hyperbolic

- Rem: . in general, $x \in M \mapsto E_u(x), E_s(x)$ are not smooth but only Hölder Continuous with some exponent $0 < \beta \leq 1$.

  . It is conjectured that M is a infranil manifold (ex: torus $\mathbb{T}^d$)

Prop : "Structural stability". If $f : M \to M$ is Anosov, there exists $\varepsilon > 0$, for any $g : M \to M$ s.t.

1 $\|g - \mathrm{Id}\|_{C^1} \leq \varepsilon$, then

2 1) $(g \circ f)$ is also Anosov.

2) $\exists\, h : M \to M$ diffeom. Hölder continuous s.t. :

3
$$\begin{tikzcd} M \arrow[r, "g \circ f"] & M \\ M \arrow[u, "h"] \arrow[r, "f"] & M \arrow[u, "h"] \end{tikzcd}$$

: commutative diagram.

proof : for 1),

Th: (Anosov) If f is Anosov and preserves a smooth measure dx on M, then f is (exponentially) mixing.

proof: Faure-Roy-Sjöst. 2008 for a proof using semiclassical analysis.

Conjecture: Any smooth Anosov map $f : M \to M$ is mixing. (then expon. mixing).

Rem: below we will consider f Anosov and symplectic. Then f preserves ω (symplectic struct.) hence preserves volume form $dx = \frac{1}{d!}\omega^d$, with $2d = \dim M$.

hence f is mixing.

• Exercice: Consider the "linear cat map" $f_0 : \begin{cases} \mathbb{T}^2 \to \mathbb{T}^2 \\ x \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x \mod \mathbb{Z} \end{cases}$ (6-10)

For a given $n \geq 1$, find the set of periodic points of period n

i.e.: $P_n := \{ x \in \mathbb{T}^2,\ f_0^n(x) = x \}$

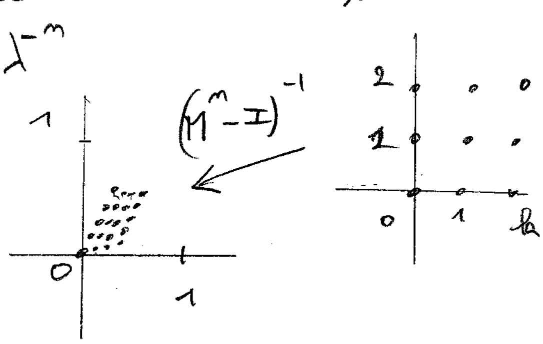
Show that $\# P_n = \lambda^n - 2 + \lambda^{-n}$ $(\sim \lambda^n$ for $n \gg 1)$

proof: $x \in P_n \iff M^n x = x + k, \quad k \in \mathbb{Z}^2.$

$\iff x = \underbrace{(M^n - I)^{-1}}_{\text{linear map}} k$. ← $k \in \mathbb{Z}^2$ lattice

We have

$|\det(M^n - I)| = |(\lambda^n - 1)(\lambda^{-n} - 1)| = \lambda^n - 2 + \lambda^{-n}$



So $(M^n - I)^{-1}$ is contractive for $n \geq 1$,

$P_n = \{ (M^n - I)^{-1} \mathbb{Z}^2 \} \cap [0,1[^2$,

$\# P_n = |\det(M^n - 1)| = \lambda^n - 2 + \lambda^{-n}$.

II Quantum dynamics (Weyl quantization)

rem : • In order to quantise a map $f : M \to M$, we have to assume that (M, ω) is symplectic manifold, and $f^*\omega = \omega$ (symplect. map).

(*) • "Weyl quantization" is specific to the cotangent symplectic space $T^*\mathbb{R}^d \cong \mathbb{R}^{2d}$.

So we will assume here that $M = \mathbb{T}^{2d} := \mathbb{R}^{2d}/\mathbb{Z}^{2d}$

• For simplicity we consider in this section:

$$M = \mathbb{T}^2 = (T^*\mathbb{R})/\mathbb{Z}^2 \cong \mathbb{R}^2/\mathbb{Z}^2, \quad \text{coordinates } (x, \xi),$$

$$\omega = dx \wedge d\xi \quad : \text{canonical symplect. structure on } M$$

$$f_0 \equiv \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : M \to M : \text{symplectic Anosov "cat map".}$$

* • A "quantization process",

associates a family of unitary operators

$$\hat{f}_\hbar : \mathcal{H}_\hbar \longrightarrow \mathcal{H}_\hbar$$

Hilbert space

with index $\hbar > 0$, called "quantum maps",

s.t. for $\hbar \to 0$, $\hat{f}_\hbar$ "mimics in a certain sense" the map f:

"wave packets $\varphi \in \mathcal{H}_\hbar$ evolve under $\hat{f}_\hbar$ as points $x \in M$ under f".

- This is inspired from physics.
- Some standard questions of semiclassical analysis are to relate properties of "classical map" f (integrability, mixing, ergodicity) to properties of "quantum maps" $\hat{f}_\hbar$, for $\hbar \to 0$, (spectrum, eigenvectors, ...)
- "geometric quantization" (section IV) allows to quantize $f : (M, \omega) \to (M, \omega)$: symplectic map

Rem :

- More generally, one could consider here:

$$M = \mathbb{T}^{2d} = \left(T^*\mathbb{R}^d\right)/\mathbb{Z}^{2d} \equiv \mathbb{R}^{2d}/\mathbb{Z}^{2d}, \text{ symplectic } \omega = \sum_{j=1}^{d} dx_i \wedge d\xi_i$$

$f_0 \in Sp(2d, \mathbb{Z})$

which defines a symplect. map $f_0 : M \to M$ (Anosov if f_0 is hyperbolic),

$f = g \circ f_0$ with $g : M \to M$ symplect. diffeom C^1-close to Id.

is also Anosov from (8-2).

II.1 Construction of the quantum Hilbert space $\mathcal{H}_\hbar$ associated to $\mathbb{T}^2 = (T^*\mathbb{R})/\mathbb{Z}^2$

Recall, the quantum Hilbert space associated to phase space $T^*\mathbb{R} \equiv \mathbb{R}$

1 is $L^2(\mathbb{R})$.

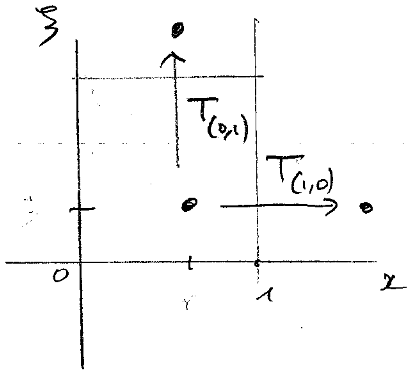
and we have $\hbar$-Fourier Transform; for $\hbar > 0$,

2
$$\tilde{\mathcal{F}}_\hbar : \begin{cases} \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \\ u(x) \longmapsto (\mathcal{F}\varphi)(\xi) := \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{\mathbb{R}} u(x)\, e^{-\frac{i x \xi}{\hbar}}\, dx \end{cases}$$

let also introduce:

3 $\hat{T}_{(1,0)} : u(x) \longmapsto u(x-1)$: "translation operator by 1 in x"

4 $\hat{T}_{(0,1)} : u \longmapsto \mathcal{F}^{-1}\, \hat{T}_{(1,0)}\, \tilde{\mathcal{F}}\, u$: " " by 1 in ξ"



5 These operators extend to $\mathcal{S}'(\mathbb{R})$.

Define

6
$$\boxed{\mathcal{H}_\hbar := \left\{ u \in \mathcal{S}'(\mathbb{R}) \text{ s.t. } \hat{T}_{(1,0)} u = u, \ \hat{T}_{(0,1)} u = u \right\}}$$

7 : "quantum Hilbert space associated to $\mathbb{T}^2 = (T^*\mathbb{R})/\mathbb{Z}^2$"

1 **Lemma:** if $\hbar = \dfrac{1}{2\pi N}$ with $N \in \mathbb{N} \setminus \{0\}$ then

$\dim_{\mathbb{C}} \mathcal{H}_\hbar = N$, otherwise $\mathcal{H}_\hbar = \{0\}$

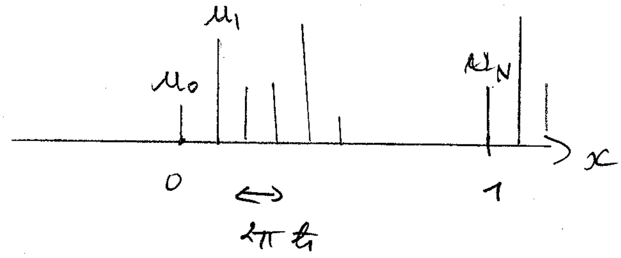
proof: Suppose $u \in \mathcal{H}_\hbar$. Periodicity in Fourier space $T_{(0,1)} u = u$ implies that u is a Dirac comb:

2 $$u(x) = \sum_{n \in \mathbb{Z}} u_n \cdot \delta(x - 2\pi\hbar n), \quad u_n \in \mathbb{C}.$$

then

3 $T_{(1,0)} u = u$

4 imposes that $N = \dfrac{1}{2\pi\hbar} \in \mathbb{N}$,



and $u_{n+N} = u_n$.

5 So $\mathcal{H}_\hbar \equiv \mathbb{C}^N \ni (u_0, u_1, \dots, u_{N-1})$. ∎

- Scalar product on $\mathcal{H}_\hbar$:

6 $$\langle u, v \rangle := \sum_{n=0}^{N-1} \overline{u_n}\, v_n$$

- rem: the Fourier transform $(\mathcal{F}u)$ of $u \in \mathcal{H}_\hbar$ is also

7 a Dirac comb.

- Let

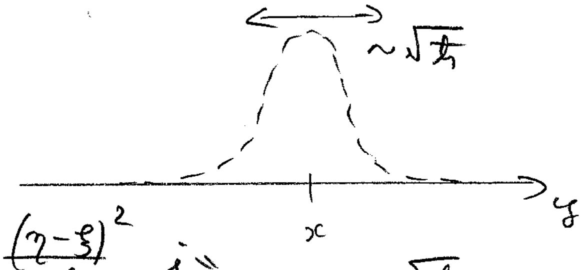
8 $$\boxed{\hat{P} : \begin{cases} \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{H}_\hbar \\ u \longmapsto \displaystyle\sum_{(m_1,m_2) \in \mathbb{Z}^2} \hat{T}_{(1,0)}^{m_1} \hat{T}_{(0,1)}^{m_2} u \end{cases}}$$

which "periodise u".

9 It is a surjective operator.

II.2 Phase space representation of $u \in \mathcal{H}_\hbar$ on $\mathbb{T}^2$

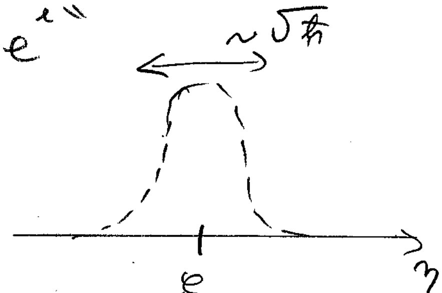
1 • **def** for $(x,\xi) \in \mathbb{R}^2$, a <u>wave packet</u> $\varphi_{(x,\xi)} \in \mathcal{S}(\mathbb{R})$, is
$$\varphi_{(x,\xi)}(y) := \frac{1}{(\pi\hbar)^{1/4}} e^{i\frac{\xi y}{\hbar}} e^{-\frac{(x-y)^2}{2\hbar}} e^{-\frac{i}{2\hbar}\xi x}$$



• rem:

Its $\hbar$-Fourier Transform is

2
$$\left(\mathcal{F}_\hbar \varphi_{(x,\xi)}\right)(\eta) = \frac{1}{(\pi\hbar)^{1/4}} e^{-\frac{ix\eta}{\hbar}} e^{-\frac{(\eta-\xi)^2}{2\hbar}} e^{i''}$$



• proof: Gaussian integral.

• So $\varphi_{(x,\xi)}$ becomes "localized" in

3 phase space $T^*\mathbb{R}$ at point (x,ξ).

as $\hbar \to 0$.

• **def**: the <u>Bargmann transform</u> is the continuous operator

4
$$\mathcal{B}_\hbar : \begin{cases} \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}^2) \\ u \longmapsto (\mathcal{B}_\hbar u)(x,\xi) := \int \overline{\varphi_{(x,\xi)}(y)} \cdot u(y)\, dy \end{cases}$$

rem: its formal adjoint is (defined from $\langle u, \mathcal{B}_\hbar^* v\rangle = \langle \mathcal{B}_\hbar u, v\rangle$)

measure

5
$$\mathcal{B}_\hbar^* : \begin{cases} \mathcal{S}(\mathbb{R}^2) \longrightarrow \mathcal{S}(\mathbb{R}) \\ v \longmapsto (\mathcal{B}_\hbar^* v)(y) = \int \varphi_{(x,\xi)}(y)\, v(x,\xi) \frac{dx\, d\xi}{(2\pi\hbar)} \end{cases}$$

• proof: p.17

6 • **prop**: $\mathcal{B}_\hbar^* \circ \mathcal{B}_\hbar = Id_{/\mathcal{S}(\mathbb{R})}$: "resolution of identity formula"

proof: p.17.

Corollary: so $\mathcal{B}_\hbar$ extends uniquely as:

1 $$\mathcal{B}_\hbar : L^2(\mathbb{R}, dx) \longrightarrow L^2\left(\mathbb{R}^2, \frac{dx\,d\xi}{2\pi\hbar}\right) \text{ isometry}$$

proof: $\|\mathcal{B}_\hbar u\|^2 = \langle \mathcal{B}_\hbar u, \mathcal{B}_\hbar u\rangle = \langle u, \underbrace{\mathcal{B}^*_\hbar \mathcal{B}_\hbar}_{Id} u\rangle = \langle u,u\rangle = \|u\|^2$ ∎

Prop:

2 $$\mathcal{B}_\hbar : \mathcal{H}_\hbar \longrightarrow \tilde{\mathcal{H}}_\hbar \quad : \text{isometry}$$

condition of quasi-periodicity

where

3 $$\tilde{\mathcal{H}}_\hbar := \Big\{ v \in C^\infty(\mathbb{R}^2), \quad v(x+n_1, \xi+n_2) = v(x,\xi) \times e^{+i2\pi\frac{N}{2}(n_1\xi - n_2 x)}\, e^{+i2\pi\frac{N}{2} n_1 n_2} \quad \forall(x,\xi)\ \forall(n_1,n_2) \Big\}$$

completed with scalar product

4 $$\langle v | w \rangle := \int_{\mathbb{T}^2} \overline{v(x,\xi)}\, w(x,\xi)\, \frac{dx\,d\xi}{2\pi\hbar}$$

proof: p. 17

Rem:

Prop:

5 $$\mathrm{Im}(\mathcal{B}_\hbar) = \Big\{ e^{-\frac{|z|^2}{2\hbar}}\, h(\bar{z}),$$

with $z := \frac{1}{\sqrt{2}}(x + i\xi)$

6 $\partial_z h = 0$: "antiholomorphic"

7 is called "Bargman-Segal space"

proof: see p. 18.

Def: The orthogonal projector:

$$\mathcal{P} := \mathcal{B}_\hbar \circ \mathcal{B}^*_\hbar : L^2(\mathbb{R}^2) \longrightarrow \mathrm{Im}(\mathcal{B}_\hbar) \subset L^2(\mathbb{R}^2)$$

is called the Bargman or Toeplitz projector.

9 proof: $\mathcal{P}$ is an orthogonal projector because

$$\mathcal{P}^2 = \mathcal{B}\underbrace{\mathcal{B}^*\mathcal{B}}_{Id}\mathcal{B}^* = \mathcal{P} \quad \text{and} \quad \mathcal{P}^* = (\mathcal{B}\mathcal{B}^*)^* = \mathcal{B}\mathcal{B}^* = \mathcal{P}$$ ∎

- proof of (15-5):

1 $$\langle u, B_\hbar^* v\rangle = \langle B_\hbar u, v\rangle \quad , \quad \forall u \in \mathcal{S}(\mathbb{R}),\ v \in \mathcal{S}(\mathbb{R}^2),$$

2 $$\Longleftrightarrow \int \overline{u(y)}\, (B_\hbar^* v)(y)\, dy = \iint \overline{\overline{\varphi_{(x,\xi)}(y)} \cdot u(y)\, dy}\ v(x,\xi)\, \frac{dx\, d\xi}{2\pi\hbar}$$

3 $$= \int \overline{u(y)} \cdot \left(\int \varphi_{(x,\xi)}(y)\, v(x,\xi)\, \frac{dx\, d\xi}{2\pi\hbar} \right) dy$$

giving (15-5). ∎

- proof of (15-6):

The Schwartz kernel of $B_\hbar^* \circ B_\hbar : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is from (15-4), (15-5):

4 $$K_{B_\hbar^* B_\hbar}(y', y) = \int \varphi_{(x,\xi)}(y')\ \overline{\varphi_{(x,\xi)}(y)}\ \frac{dx\, d\xi}{(2\pi\hbar)}$$

5 $$\underset{(15-1)}{=} \delta(y'-y) \quad : \text{Schwartz kernel of } \mathrm{Id}/\mathcal{S}(\mathbb{R}).$$

and Gaussian integral

∎

- proof of (16-3)

6 let $u \in \mathcal{H}_\hbar$, and $v := (B_\hbar u) \in \tilde{\mathcal{H}}_\hbar$, hence $u \underset{(15-6)}{=} B^* B u = B^* v$

For any $n = (n_1, n_2) \in \mathbb{Z}^2$, from def (13-6),

7 $$\hat{T}_{(1,0)}^{n_1}\, \hat{T}_{(0,1)}^{n_2}\, u = u \ \xrightarrow[(23-6)]{}\ \hat{T}_n u \cdot \exp\left(-i2\pi \frac{N}{2} n_1 n_2\right) = u$$

8 $$\rightarrow v = \left(B_\hbar\, \hat{T}_n\, B_\hbar^* v\right) \cdot \exp\left(-i2\pi \frac{N}{2} n_1 n_2\right)$$

but in Faure-Masato arxiv 1206.0282v1 p. 44, it is showed (3.20)

9 that $$\left(B_\hbar\, \hat{T}_n\, B_\hbar^* v\right)(x,\xi) = \exp\left(i2\pi \frac{N}{2} (n_2 x - n_1 \xi)\right) v(x - n_1, \xi - n_2)$$

we deduce (16-3) from (17-8) and (17-9). ∎

• **proof of (16-5)**

1 $\mathcal{B}_\hbar : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}^2)$ extends to $\mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}^2)$.

Let $u \in \mathcal{S}'(\mathbb{R})$, then

2 $$v(x,\xi) = \left(\mathcal{B}_\hbar u\right)(x,\xi) \underset{\substack{(15-4)\\(5-1)}}{=} \frac{e^{+\frac{i}{2\hbar}\xi x}}{(\pi\hbar)^{1/4}} \int e^{-i\frac{\xi y}{\hbar}} e^{-\frac{(x-y)^2}{2\hbar}} u(y)\, dy$$

put $z := \frac{1}{\sqrt{2}}(x+i\xi) \Longleftrightarrow \begin{cases} x = \frac{1}{\sqrt{2}}(z+\bar{z}) \\ \xi = \frac{i}{\sqrt{2}}(\bar{z}-z) \end{cases}$

replace in (18-2) and get:

$$v(x,\xi) = e^{-\frac{|z|^2}{2\hbar}} \cdot h(\bar{z})$$

with
$$h(\bar{z}) = \frac{1}{(\pi\hbar)^{1/4}} \int e^{-\frac{1}{2\hbar}y^2} e^{\frac{\sqrt{2}}{\hbar}y\bar{z}} u(y)\, dy$$

anti holomorphic in z. ∎

- *example* : considerer a wave packet $\varphi_{(x_0,\xi_0)} \in \mathcal{S}(\mathbb{R})$.

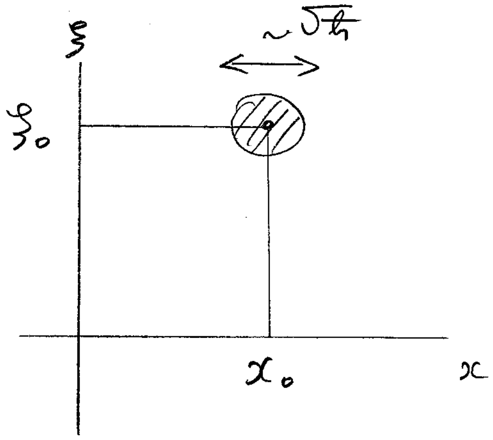
1 Then $\frac{1}{2\pi\hbar}\left|\left(\mathcal{B}\varphi_{(x_0,\xi_0)}\right)(x,\xi)\right|^2 = C.\exp\left(-\frac{1}{2\hbar}\left((x-x_0)^2+(\xi-\xi_0)^2\right)\right)$

is a Gaussian.

and converges (in the sense of distribution)

2 to the Dirac measure $\delta_{(x_0,\xi_0)}$

as $\hbar \to 0$.



$\times$ - *proof*:

3 - *Déf* : giving a sequence $u_\hbar \in \mathcal{H}_\hbar$, $\hbar = \frac{1}{2\pi N}$, $N \geq 1$

we associate the probability measures on $\mathbb{T}^2$:

4 $$\mathrm{Hus}_{u_\hbar}(x,\xi) := \frac{1}{2\pi\hbar}\left|\mathcal{B}u_\hbar(x,\xi)\right|^2$$: "*Husimi distribution*"

if it exists, the limit:

5 $$\mu_u := \lim_{\hbar\to 0}^{*} \mathrm{Hus}_{u_\hbar}$$

6 is a positive measure on $\mathbb{T}^2$, called "*semiclassical measure*".

- proof that Hus is a probability measure :

II.3 Construction of the quantum map $\hat{b}_\hbar : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$

(using Weyl quantization)

1 **Lemma**: The symplectic map $b_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$
is time-1 Hamiltonian flow generated by the quadratic Hamilt. function:

2 $$H(x, \xi) = \frac{1}{2} \alpha x^2 + \frac{1}{2} \beta \xi^2 + \gamma x \xi$$

with (α, β, γ) such that

3 $$b_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \exp \begin{pmatrix} \gamma & \beta \\ -\alpha & -\gamma \end{pmatrix}$$

proof: Hamilton equ. reads:

4 $$V_H : \begin{cases} \dot{x} = \dfrac{\partial H}{\partial \xi} = \gamma x + \beta \xi \\ \dot{\xi} = -\dfrac{\partial H}{\partial x} = -\alpha x - \gamma \xi \end{cases}$$

Integrate them to get (20-3).

Some semiclassical analysis:

Déf: Weyl quantization:

5 $$\mathrm{Op} : \begin{cases} \mathcal{S}(\mathbb{R}^2) \longrightarrow \mathcal{L}(L^2(\mathbb{R})) \\ f \longmapsto \hat{f} = \mathrm{Op}(f) := \iint (\mathcal{F} f)(\eta_x, \eta_\xi)\, e^{i(\eta_x \hat{x} + \eta_\xi \hat{\xi})}\, d\eta_x\, d\eta_\xi \end{cases}$$

Fourier Transform

6 with $\begin{cases} \hat{x} : u(x) \longmapsto x \cdot u(x) \\ \hat{\xi} : u(x) \longmapsto \left(-i\hbar \dfrac{du}{dx}\right)(x) \end{cases}$

<u>Prop</u> (after extension to distributions) one has:

1 $\cdot\ Op(1) = Id, \qquad Op(x) = \hat{x}, \qquad Op(\xi) = \hat{\xi}$

2 $\cdot\ Op(\bar{f}) = \left(Op(f)\right)^*$

<u>Prop</u>: <u>"Metaplectic Group"</u>

$\cdot$ If $f(x,\xi) = \frac{1}{2}\alpha x^2 + \frac{1}{2}\beta\xi^2 + \gamma x\xi + u x + v\xi + w$

: degree ≤ 2

3 then $Op(f) = \frac{1}{2}\alpha\hat{x}^2 + \frac{1}{2}\beta\hat{\xi}^2 + \frac{\gamma}{2}(\hat{x}\hat{\xi} + \hat{\xi}\hat{x})$
$+ u\hat{x} + v\hat{\xi} + w\cdot Id$

$$\left[\left(\frac{-i}{\hbar}\right)Op(f), \left(\frac{-i}{\hbar}\right)Op(g)\right] = \left(\frac{-i}{\hbar}\right)Op\Big(\underbrace{\{f,g\}}_{(\partial_x f\,\partial_\xi g - \partial_\xi f\,\partial_x g)}\Big)$$

<u>ex</u>: $[\hat{x}, \hat{\xi}] = i\hbar\, Id$

$\circ$ hence:

$$\left(\frac{-i}{\hbar}\right)Op : \begin{cases} sp_2(\mathbb{R}) \cong \{f(x,\xi) \text{ homog. degree } 2, \{\cdot,\cdot\}\} \longrightarrow mp_2(\mathbb{R}) \\ f \longmapsto \left(\frac{-i}{\hbar}\right)Op(f) \end{cases}$$

is an homomorphism of Lie Algebra.

here : $sp_2(\mathbb{R}) \cong mp_2(\mathbb{R})$: symplect. Lie algebra

$\cdot$ $f(x,\xi)$ defines a Hamiltonian vector field:

$$V_f = V^x\frac{\partial}{\partial x} + V^\xi\frac{\partial}{\partial \xi}, \qquad \begin{cases} V^x = \frac{\partial H}{\partial \xi} = \gamma x + \beta\xi \\ V^\xi = -\frac{\partial H}{\partial x} = -\alpha x - \gamma\cdot \end{cases}$$

which defines the time 1 flow:

$$\exp(-V_f) \in Sp(2,\mathbb{R}) \cong SL(2,\mathbb{R})$$: <u>linear symplect. group</u>

$\cdot$ Also,

$$\exp\left(-\frac{i}{\hbar}Op(f)\right) \in Mp(2,\mathbb{R})$$: <u>Metaplectic group</u>.

Rem:

- Notice that

1 $$\begin{cases} Mp(2,\mathbb{R}) \longrightarrow Sp(2,\mathbb{R}) & \text{is } 1:2 \text{ cover} \\ \exp\left(-\frac{i}{\hbar} Op(f)\right) \longmapsto \exp(-V_f) \end{cases}$$

2 explicitely, for $f(x,\xi) = \frac{1}{2}\left(\xi^2 + x^2\right)$: the "Harmonic Oscillator"

3 let $M_\alpha := \exp(\alpha V_f)$, $\widehat{M}_\alpha := \exp\left(-\frac{i}{\hbar}\alpha\, Op(f)\right)$, $\alpha \in \mathbb{R}$.

4 Then $M_{\alpha=2\pi} = Id_{\mathbb{R}^2}$, $\widehat{M}_{\alpha=2\pi} = -Id_{/L^2(\mathbb{R})}$.

__Proof__ of (22-1), (22-4);

We know the spectrum of the Harmonic oscillator:

5 $$Op(f) = \hbar \sum_{n\geq 0} \left(n + \frac{1}{2}\right) |\varphi_n\rangle\langle\varphi_n|$$

↳ eigenfunctions (Hermite)

6 $$\longrightarrow \widehat{M}_{\alpha=2\pi} = \exp\left(-\frac{i}{\hbar} 2\pi\, Op(f)\right) = \exp\left(-i\, 2\pi \sum_{n\geq 0}\left(n+\frac{1}{2}\right)|\varphi_n\rangle\langle\varphi_n|\right)$$

7 $$= \exp\Big(-i\pi \underbrace{\sum_n |\varphi_n\rangle\langle\varphi_n|}_{Id}\Big) = -Id.$$

$\blacksquare$

- __proof of (21-1)__:

8 $$\left(\mathcal{F}(x)\right)(\eta) = -i\, \delta'(\eta_x)\,\delta(\eta_\xi) \qquad \text{(check on test functions)}$$

9 therefore $$Op(x) = \iint (-i)\,\delta'(\eta_x)\,\delta(\eta_\xi)\, e^{i(\eta_x \hat{x} + \eta_\xi \hat{\xi})}\, d\eta = \hat{x}$$

etc ...

$\blacksquare$

Prop

10 $$\cdot\ Tr\left(Op(f)\right) = \frac{1}{2\pi\hbar}\int_{T^*\mathbb{R}} f(x,\xi)\, dx\, d\xi \quad \text{: "Weyl law"}$$

11 $$\cdot\ Op(f)\circ Op(g) = Op(f\cdot g) + \mathcal{O}(\hbar) \quad \text{: "Composition formula"}$$

↑ series in $\hbar$

Rem: in general we have an Lie algebra homomorphism

1 only at first order in $\hbar$:

prop: for $f, g \in \mathcal{S}(\mathbb{R}^2)$

2 $$\left[\left(\frac{-i}{\hbar}\right) Op(f), \left(\frac{-i}{\hbar}\right) Op(g)\right] = \left(\frac{-i}{\hbar}\right) Op\left(\{f, g\}\right)\left(1 + \underbrace{O(\hbar)}_{\text{asymptotic series in } \hbar}\right)$$

(this the difficulty and basis of semiclassical analysis).

- Weyl - Heisenberg algebra and group:

For $v = (v_1, v_2) \in \mathbb{R}^2$, let $f_v(x, \xi) = v_1 \cdot \xi - v_2 \cdot x$

3 $\to$ Ham. vector field $V_{f_v} = \begin{cases} \partial_\xi f = v_1 \\ -\partial_x f = v_2 \end{cases} = v$

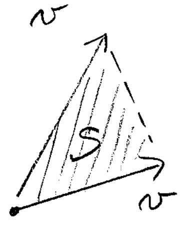
4 $T_v := \exp(-V_{f_v})$;

5 $\hat{T}_v := \exp\left(-\frac{i}{\hbar} Op(f_v)\right) = \exp\left(-\frac{i}{\hbar}\left(v_1 \hat{\xi} - v_2 \hat{x}\right)\right)$

satisfies: Weyl - Heisenberg group relations:

6 $$\boxed{\hat{T}_v \cdot \hat{T}_{v'} = e^{-i\frac{S}{\hbar}} \hat{T}_{v+v'}}$$

7 with $S = \frac{1}{2}(v_1 v_2' - v_2 v_1') = \frac{1}{2} v \wedge v'$ $\longrightarrow$



8 (proof: from $[\hat{x}, \hat{\xi}] = i\hbar \, Id$).

- Conjoint relations between Weyl-Heisenberg and Metaplectic groups:

- rem: if $M \in Sp(2,\mathbb{R})$, and $v \in \mathbb{R}^2$,

1 then $M \circ T_v = T_{Mv} \circ M$

2 proof: $M T_v x = M(x+v) = Mx + Mv = T_{Mv} Mx$ ▨

Similarly:

3 prop: if $M = \exp(-V_b) \in Sp(2,\mathbb{R})$,

4 $\hat{M} = \exp\left(-\frac{i}{\hbar} Op(b)\right) \in Mp(2,\mathbb{R})$,

$v \in \mathbb{R}^2$,

5 then $\hat{M} \circ \hat{T}_v = \hat{T}_{Mv} \cdot \hat{M}$

6 proof: use Schur lemma, see Segal's lectures.

- Prop: from (23-5) (20-5), on can write:

7 $Op(b) = \iint (\mathcal{F}b) \cdot \hat{T}_{(-\hbar\eta_\xi, \hbar\eta_x)}$,

then (24-5) gives "Exact Egorov relation":

$$\hat{M}^{-1} Op(b) \hat{M} = Op(b \circ M)$$

Rem: "Weyl quantization on the Torus", from déf (20-5) (extended to distributions) is:

$$Op: \begin{cases} C^\infty(\mathbb{T}^2) \longrightarrow \mathcal{L}(\mathcal{H}_\hbar) \\ b = \sum_{n \in \mathbb{Z}^2} \underbrace{b_n}_{\text{Fourier components}} e^{i2\pi(n_1 x + n_2 \xi)} \longmapsto Op(b) = \sum_n b_n \cdot e^{i2\pi(n_1 \hat{x} + n_2 \hat{\xi})} \end{cases}$$

and remark as (24-7) that:

$$Op(b) = \sum_{n \in \mathbb{Z}^2} b_n \cdot \hat{T}_{\left(-\frac{n_2}{N}, \frac{n_1}{N}\right)}$$

1 Let $\hat{H} := Op(H)$, with H : (20-2)

2 $$\hat{H} \underset{(21-3)}{=} \frac{1}{2}\alpha \hat{x}^2 + \frac{1}{2}\beta \hat{\xi}^2 + \frac{1}{2}\gamma\left(\hat{x}\hat{\xi} + \hat{\xi}\hat{x}\right)$$

3 Let $\hat{f}_\hbar := \exp\left(-\frac{i}{\hbar}\hat{H}\right) \quad : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$

(similar to (20-3)).

4 rem: $\hat{f}_\hbar$ is unitary in $L^2(\mathbb{R})$.

5 **Prop**: if N is even, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{S}(\mathbb{R}) & \xrightarrow{\hat{f}_\hbar} & \mathcal{S}(\mathbb{R}) \\ \downarrow \hat{P} & & \downarrow \hat{P} \\ \mathcal{H}_\hbar & \xrightarrow{\hat{f}_\hbar} & \mathcal{H}_\hbar \end{array}$$

6 : called "quantum map" on the Torus.

$\hat{f}_\hbar$ is a unitary operator in $\mathcal{H}_\hbar$.

proof: from (14-8):

$$\hat{P} = \sum_{n \in \mathbb{Z}^2} \hat{T}^{n_1}_{(1,0)} \hat{T}^{n_2}_{(0,1)} \underset{(23-6)}{=} \sum_n e^{-i 2\pi N \frac{1}{2}} \hat{T}_n$$

$$= \sum_{n \in \mathbb{Z}^2} \hat{T}_n \quad : \text{if } N \text{ is even.}$$

Then
$$\hat{f}_\hbar \hat{P} = \sum_n \hat{f}_\hbar \hat{T}_n \underset{(24-5)}{=} \left(\sum_n \hat{T}_{f_0 n}\right) \hat{f}_\hbar$$

$$= \left(\sum_{n' \in \mathbb{Z}^2} \hat{T}_{n'}\right) \hat{f}_\hbar = \hat{P}\hat{f}_\hbar$$

with $n' := f_0 n \in \mathbb{Z}^2 \Longleftrightarrow n = f_0^{-1} n'$.

III Quantum Ergodicity

III.1 The Quantum ergodic theorem

- For every $\hbar = \frac{1}{2\pi N}$, $N \geq 1$, even we consider the eigenvectors, eigenvalues of the unitary operator (25-6).

1 $$\boxed{\hat{f}_\hbar \psi_{\hbar,j} = e^{i\varphi_{\hbar,j}} \psi_{\hbar,j}}, \quad \psi_{\hbar,j} \in \mathcal{H}_\hbar, \ \|\psi_{\hbar,j}\| = 1$$

$j \in \{1, .., N\}$

$N = \dim \mathcal{H}_\hbar = \frac{1}{2\pi\hbar}$

2 - **Def**: an invariant semi-classical measure is a semiclassical measure (19-5) obtained by a sequence of eigenvectors of $\hat{f}_\hbar$: $\left(\psi_{\hbar,j_\hbar}\right)_{\hbar = \frac{1}{2\pi N}}$, $N \to \infty$.

3 - Prop: an invariant semi classical measure μ is invariant by the map f_0: $f_0^* \mu = \mu$.

proof: One has for every $\hbar$ and eigenvector ψ (27-1):

$\forall g \in C^\infty(\mathbb{T}^2)$,

4 $$\langle \psi | Op(g) \psi \rangle \overset{(27\text{-}1)}{=} \langle \hat{f}_\hbar \psi | Op(g) \hat{f}_\hbar \psi \rangle$$

5 $$= \langle \psi | \hat{f}_\hbar^{-1} Op(g) \hat{f}_\hbar \psi \rangle$$

6 $$\underset{\substack{\text{Exact Egorov} \\ (24\text{-}7)}}{=} \langle \psi | Op(g \circ f_0) \psi \rangle$$

On the other Hand one has:

$$\mathrm{Hus}_\psi(g) \underset{(19\text{-}4)}{=:} \langle \psi \mid \mathrm{Op}^{AW}(g)\psi \rangle = \langle \psi \mid \mathrm{Op}(g)\psi \rangle + O(\hbar)\cdot\|\psi\|^2$$

hence with (27-6):

$$\mathrm{Hus}_\psi(g) = \mathrm{Hus}_\psi(g\circ f_0) + O(\hbar)\cdot\|\psi\|^2$$

In the limit $\hbar \to 0$ this gives from déf (19-5), (27-2),

$$\mu(g) = \mu(g\circ f_0), \quad \forall g$$

$$\Leftrightarrow \quad \mu = f_0^* \mu .$$

rem: for the Anosov map $f_0 : \mathbb{T}^2 \to \mathbb{T}^2$ there exists many different invariant measures μ i.e $f_0^*\mu = \mu$,

e.g $\mu = \delta_{\text{periodic orbit}}$ or $\mu = \text{Lebesgues} = (dx\,d\xi)$

or fractal measures etc ...

The natural question after (27-3) is

"which of these invariant measures $f_0^*\mu = \mu$ are invariant semiclassical measures?".

- The quantum ergodic theorem says that if the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is ergodic then almost all invariant semiclassical measure μ is Lebesgues $(dx\,d\xi)$ (equidistributed on $\mathbb{T}^2$).

réf: Schnirelman 74, Zelditch 87, Colin de Verdière 85
Helffer Martinez Robert 87, Bouzouina De bièvre 96.

· Th "quantum ergodic theorem" on $\mathbb{T}^2$ for f_0.

1 · for every $N \geq 1$ (even), there exists a subset $S(N) \subset \{1, 2, \dots, N\}$

s.t:

1) $\forall g \in C^\infty(\mathbb{T}^2)$, any sequence $\psi_{\hbar, j_\hbar}$ with $j_\hbar \in S(N)$,

2 $$\lim_{\hbar \to 0} \langle \psi_{\hbar, j_\hbar} \,|\, Op(g)\, \psi_{\hbar, j_\hbar} \rangle = \int_{\mathbb{T}^2} g \, dx\,d\xi$$

ie: $\mu_\psi = \text{Lebesgues}_{\mathbb{T}^2}$.

3 2) $\lim_{N \to \infty} \frac{1}{N} (\# S(N)) = 1$: ie such sequence is "generic"

4 <u>proof</u>: Let $g_{j,\hbar} := \langle \psi_{\hbar, j} \,|\, Op(g)\, \psi_{\hbar, j} \rangle$, $j = 1 \longrightarrow N$.

5 $$\langle g \rangle := \int_{\mathbb{T}^2} g \, dx\,d\xi \text{ : space average.}$$

6 one has: $$\frac{1}{N} \sum_{j=1}^{N} g_{j,\hbar} = \frac{1}{N} \mathrm{Tr}\left(Op(g)\right) \xrightarrow[N \to \infty]{} \langle g \rangle$$

from <u>Weyl law</u>. (22-10).

Let us consider the Variance of the distribution of values (29-4) also called <u>quantum variance</u>:

<u>Lemma</u>:

7 $$S_2(g) := \frac{1}{N} \sum_{j=1}^{N} \left| g_{j,\hbar} - \langle g \rangle \right|^2 \xrightarrow[\hbar \to 0]{} 0$$

This Lemma is equivalent to (29-2) (29-3).

<u>Proof of the Lemma (29-7)</u>:

1 suppose $\langle g\rangle = 0$ for simplicity.

Let $T \geq 1$.

2 $$\frac{1}{T}\sum_{n=1}^{T} \hat{f}_\hbar^{-n}\, Op(g)\, \hat{f}_\hbar^{n} \underset{\text{Egorov (24-7)}}{=} \frac{1}{T}\sum_{n=1}^{T} Op\left(g\circ f_0^{+n}\right)$$

3 $$= Op\left(\langle g\rangle_T\right)$$

4 $$\text{with} \quad \langle g\rangle_T := \frac{1}{T}\sum_{n=0}^{T-1} g\circ f_0^{+n} \quad : \text{"time average of } g\text{"}$$

We use this time average to express that $\psi_{\hbar,j}$ is eigenvector:

5 $$g_{j,\hbar} := \langle \psi_{\hbar,j} | Op(g)\, \psi_{\hbar,j}\rangle \underset{(27-1)}{=} \langle \psi_{\hbar,j} | Op(\langle g\rangle_T)\, \psi_{\hbar,j}\rangle$$

Also

6 $$\left|\langle \psi_{\hbar,j} | Op(\langle g\rangle_T)\, \psi_{\hbar,j}\rangle\right|^2 \underset{\text{Cauchy Schwartz}}{\leq} \underbrace{\|\psi_{\hbar,j}\|^2}_{=1} \cdot \|Op(\langle g\rangle_T)\,\psi_{\hbar,j}\|^2$$

7 $$= \langle \psi_j | Op(\langle g\rangle_T)^2\, \psi_{\hbar,j}\rangle \underset{(22-11)}{=} \langle \psi_j | Op\left((\langle g\rangle_T)^2\right) \psi_j\rangle + O(\hbar)$$

So

8 $$S_2(g) \underset{(29-7)}{=} \frac{1}{N}\sum_{j=1}^{N} |g_{j,\hbar}|^2 \underset{\substack{(30-5)\\(30-7)}}{=} \frac{1}{N}\, Tr\left(Op\left(\langle g\rangle_T^2\right)\right) + O(\hbar)$$

9 $$\underset{(22-10)}{=} \int_{\mathbb{T}^2} \langle g\rangle_T^2 + O_T(\hbar)$$

- Finally let us show that ergodicity of the map f_0

1 implies that $\displaystyle\int_{\mathbb{T}^2} \langle g \rangle_T^2 \xrightarrow[T\to\infty]{} 0$.

With this

2 in (30-9) (as $S_2(g)$ is independent of T) we deduce (29-7).

proof of (31-1);

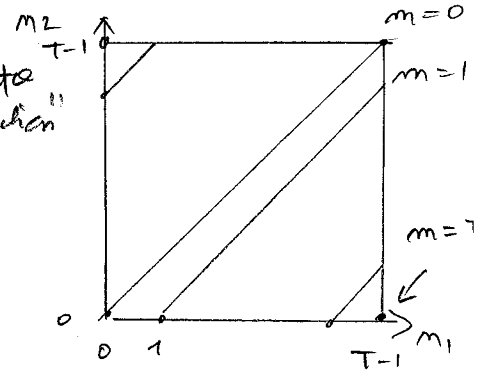
. We have indeed

3 $$\int_{\mathbb{T}^2} \langle g \rangle_T^2 \underset{(30-4)}{=} \frac{1}{T^2} \sum_{n_1, n_2 = 0}^{T-1} \int_{\mathbb{T}^2} \left(g \circ f_0^{n_1}\right) \cdot \left(g \circ f_0^{n_2}\right) \, dx \, d\xi$$

but since f_0 preserve the measure $(dx\, d\xi) =: dz$, we change $z' = f_0^{n_2}(z$

4 $$\int_{\mathbb{T}^2} \left(g \circ f_0^{n_1}\right)\left(g \circ f_0^{n_2}\right) dx\, d\xi = \int_{\mathbb{T}^2} \left(g \circ f^{n_1 - n_2}\right) \cdot g$$

5 put $m := n_1 - n_2$, $C(m) := \int_{\mathbb{T}^2} \left(g \circ f^m\right) \cdot g$: "correlation function"



6 $$\int_{\mathbb{T}^2} \langle g \rangle_T^2 = \frac{1}{T^2} \left(T \cdot C(0) + 2 \sum_{m=1}^{T-1} (T-m)\, C(m) \right)$$

7 $$= 2 \left(\frac{1}{T} \sum_{m=0}^{T-1} C(m) \right) - \frac{1}{T} C(0) - \frac{2}{T^2} \sum_{m=1}^{T-1} m\, C(m)$$

Ergodicity, (3-1) gives for the first term of (31-7) that:

8 $$\frac{1}{T} \sum_{m=0}^{T-1} C(m) \xrightarrow[T\to\infty]{} \langle g \rangle^2 \underset{(30-1)}{=} 0 .$$

The other terms also tend to zero because of the bound:

$$|C(m)| \leq |g|_\infty^2 .$$

So we have obtained (31-1). ∎

Remarks :

- It is clear in the proof that quantum ergodicity is equivalent that the Quantum Variance (29-7).

1 $$S_2(g) \xrightarrow[\hbar \to 0]{} 0$$

- R. Schubert (05), Zelditch, show that for maps which are mixing (2-6), as f_0 here, (32-1) can be improved to : $\boxed{S_2(g) = \mathcal{O}_g\left(\frac{1}{\log(1/\hbar)}\right)}$ which is optimal for the cat-map f_0. (Schubert[05], using "scared states")

- Kurlberg - Rudnick (2005), for "Hecke Basis", shows: $\boxed{S_2(g) \sim \hbar \cdot C_g}$, but $C_g \neq V_{cl}(g)$ below.

- From numerical observations, it is conjectured that for "generic" Anosov map, the rescaled distribution of $\left(g_{j,\hbar}\right)_{j=1 \to N}$ (29-4), as $\hbar \to 0$, converge towards a Gaussian with variance $\boxed{S_2(g) \sim \hbar \, V_{cl}(g)}$ where $V_{cl.}(g)$ is the "classical variance" of values of $\left(\sqrt{T}\langle g\rangle_T\right) = \frac{1}{\sqrt{T}} \sum_{n=0}^{T-1} g \circ f_0^n$, $T \to \infty$, with respect to the measure $(dx\,d\xi)$.

- The irrational translation map (3-4) is uniquely ergodic (3-9). Therefore from (27-3), the only invariant semiclassical measure is $(dx\,d\xi)$.
  rem : in that model, eigenvectors (27-1) are explicit: Fourier modes.

- Recent results related to quantum ergodic theorem

1 for hyperbolic maps

Scars :

- (F. Faure, S. Nonnenmacher, S. DBièvre, 03)
  For the linear cat map f_0, (6-10), existence of invariant semiclassical measure of the form :

2 $$\mu = \frac{1}{2} \delta_{\text{periodic orbit}} + \frac{1}{2} \text{Lebesg.}$$

and for f_0, if $\mu = \mu_{pp} + \mu_{Leb} + \mu_{sc}$

3 then $\mu_{pp}(\mathbb{T}^2) \leq \mu_{Leb}(\mathbb{T}^2)$ hence $\mu_{pp}(\mathbb{T}^2) \leq 1/2$

so (33-2) is the "extreme possibility".

- Kelmer (2010), for linear hyperbolic map on $\mathbb{T}^{2d}$, (6-1) constructs invariant semiclassical measure on submanifolds $\mathcal{L}$ of $\mathbb{T}^{2d}$. $\left(\text{ex: } M = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, A \in Sl_d(\ldots \right.$
  $\left. \mathcal{L} = \{(0, \xi)\} \right)$

Entropy of eigenfunctions

- N. Anantharaman, S. Non., Koch, for general Anosov map or flow.

$$h_{KS}(\mu) \geq \sum_{i=1}^{2d} \max\left(\log|\underbrace{\lambda_i}_{\text{eigenvalues of } M}|, 0\right) - \frac{d}{2} \underbrace{\lambda_{max}}_{\max_i \log|\lambda_i|}$$

(it is ≤ 0 if λ_{max} is large).

They conjecture :
$$h_{KS}(\mu) \geq \sum_{i=1}^{2d} \max\left(\frac{1}{2}\log|\lambda_i|, 0\right)$$

- G. Rivière for linear Anosov map :
$$h_{KS}(\mu) \geq \sum_{i=1}^{2d} \max\left(\log|\lambda_i| - \frac{1}{2}\lambda_{max}, 0\right)$$

$$h_{KS}(\mu) \geq \frac{1}{2}\lambda \quad \text{on } \mathbb{T}^2.$$

- Counter example of Quantum unique ergodicity for Anosov map:
  see lectures notes "Peyresq 2007", on F. Faure web page.

IV Geometric quantization of symplectic map $f : (M,\omega) \circlearrowleft$

ref : - Woodhouse's book : for flows
- Borthwick " , Kähler manifold
- Zelditch 05 : for maps
- Faure - Tsujii 2011

1 Let (M,ω) : compact smooth symplectic manifold,

2 $f : M \to M$: smooth symplectic map.

IV.1 Prequantization

Assumptions :

3 1) $[\omega] \in H^2(M,\mathbb{Z})$ (integral cohomology class).

4 2) $H_1(M,\mathbb{Z})$ has no torsion

5 3) 1 is not eigen-value of $f_* : H_1(M,\mathbb{R}) \to H_1(M,\mathbb{R})$ (induced by f)

Then :

Thm : . with assumption 1), there exists a principal bundle:

$$U(1) \to P \xrightarrow{\pi} M \quad \text{with connection } A \in C^\infty(P, \Lambda^1 \otimes i\mathbb{R})$$

with curvature $\Theta = dA = -i(2\pi)(\pi^*\omega)$

. with assumption 2), A can be chosen s.t. there exists

$\tilde{f} : P \to P$ s.t

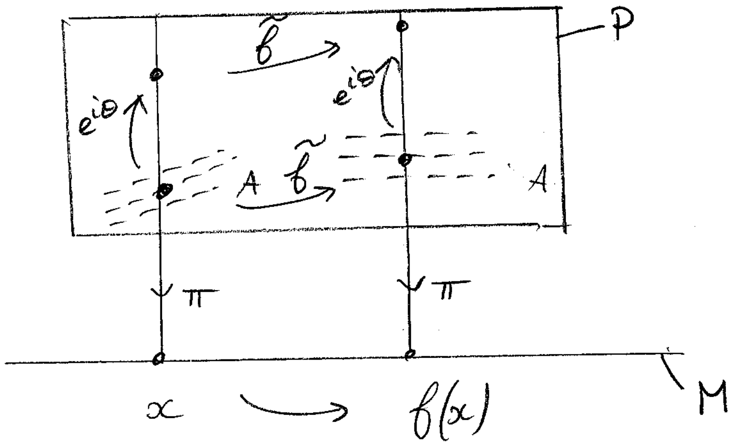
1) $\pi \circ \tilde{f} = f \circ \pi$: $\tilde{f}$ lift f.

2) $\tilde{f}(e^{i\theta} p) = e^{i\theta} \tilde{f}(p)$, $\forall p \in P$, $\forall e^{i\theta} \in U(1)$: equivariant

3) $\tilde{f}^* A = A$: preserves the connection

P is called the prequantum bundle

$\tilde{f} : P \to P$ is the prequantum map



def: Let $V \in C^\infty(M)$ called "potential function".
The prequantum transfer operator is

$$\hat{F} : \begin{cases} C^\infty(P) \longrightarrow C^\infty(P) \\ u \longmapsto \hat{F}(u) = e^{V\circ\pi}\left(u \circ \tilde{f}^{-1}\right) \end{cases}$$

Let $\hat{F}_N : C_N^\infty(P) \longrightarrow C_N^\infty(P)$ be its restriction to Fourier mode $N \in \mathbb{Z}$:

$$C_N^\infty(P) := \left\{ u \in C^\infty(P),\ \forall p \in P,\ \forall e^{i\theta} \in U(1),\ u(e^{i\theta}p) = e^{iN\theta} u(p) \right\}$$

rem: . $C_N^\infty(P) \cong C^\infty(M, L^{\otimes N})$: space of smooth sections.
$\hookrightarrow$ associate complex line bundle over M.

. define for $N > 0$: $\hbar := \frac{1}{2\pi N}$

def: the <u>covariant derivative</u> is

$$D : \begin{cases} C^\infty(P) \longrightarrow C^\infty(P;\Lambda^1) \\ u \longmapsto (Du)(V) = (HV)(u) \end{cases}$$

HV: Horizontal component of $V \in T_p P$

and restricts to
$$D : C^\infty_N(P) \longrightarrow C^\infty_N(P;\Lambda^1)$$

IV.2 Quantization

- Let g be some Riem. metric on M compatible with ω:

$$\omega(Ju, Jv) = \omega(u,v) \qquad \forall u, v \in T_xM, \ \forall x \in M$$
$$g(u,v) = \omega(u, Jv)$$

for some almost complex structure J.

$\to$ this induce an equivariant metric on P.

and defines as usual:

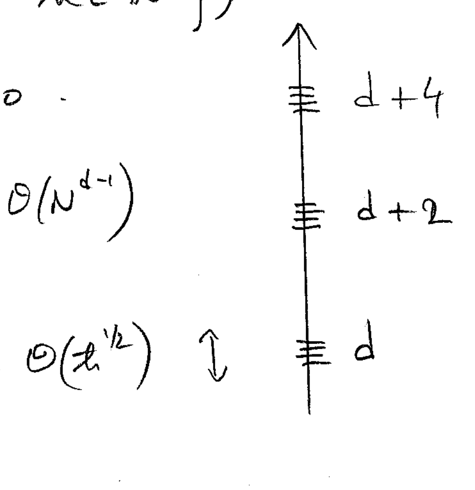
$$D^* : C^\infty(P,\Lambda^1) \longrightarrow C^\infty(P) \quad : \text{"adjoint operator"}$$

$$\Delta := D^* D : C^\infty(P) \longrightarrow C^\infty(P) \quad : \text{"rough Laplacian" self adj in } L^2(P).$$

Thm: for any $\alpha > 0$, the spectrum:
$\sigma\left(\hbar \Delta_{/L^2_N(P)}\right) \cap [0,\alpha]$ is contained in a $O(\hbar^{1/2})$ neighborhood of the integers $\{d + 2k, \ k \in \mathbb{N}\}$, $2d = \dim M$.

Let $\mathcal{P}_0$: spectral projector on the "cluster" $k=0$.

Then:
$$\operatorname{ran}(\mathcal{P}_0) = \int_M \left[e^{N\omega}\, \mathrm{Todd}(TM)\right]_{2d} = N^d \, \mathrm{Vol}_\omega(M) + O(N^{d-1})$$



• definitions:

$\mathcal{H}_\hbar := \mathrm{Im}(\mathcal{P}_0)$ is the quantum space

$\hat{f}_\hbar := \mathcal{P}_0 \hat{F}_N \mathcal{P}_0 : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$ is the quantum map.

rem: • with some good choice of V, $\hat{f}_\hbar$ is unitary.

• In the case $M = \mathbb{T}^2$, f = linear cat map, the precious construction gives $\mathcal{H}_\hbar \equiv$ Bargmann space (holomorphic sections of L^N) and $\hat{f}_\hbar$ coincides with the Weyl quantization.

In particular, $\dim \mathcal{H}_\hbar = N$.

Rem: The use of an additional metric g is not "natural". In paper with M. Tsuji we show that the prequantum operator itself has band spectrum, and that we can obtain a "natural quantum space $\mathcal{H}_\hbar$" and a "natural quantum map $\hat{f}_\hbar : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$" if $f: M \to M$ is Anosov.