Long time semi-classical evolution of wave packets in quantum chaos.
Examples of non quantum unique ergodicity with hyperbolic maps

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"Resonances and periodic orbits: spectrum and zeta functions in quantum and classical chaos"

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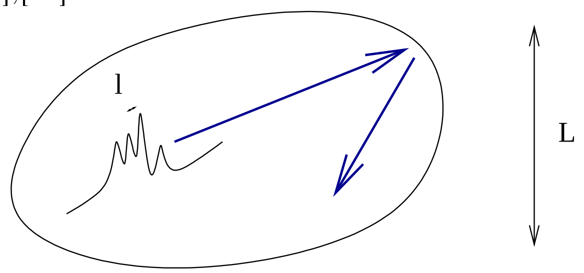
1 Introduction

1.1 Introduction on quantum chaos and scarred eigenstates

Semi-classical analysis: is a fruitful approach to understand wave equations in the regime where the wave length l *is small* in comparison with the size L of the domain (or size of the typical variation of the potential), where the wave evolves. The semi-classical parameter is $h = l/L \ll 1$. In that regime, one shows that the evolution of a wave can be described in terms of Hamiltonian classical dynamics in the same domain (or with the same potential) [27],[14].



For example in that regime, the Van-Vleck formula (1928) [55] expresses the evolving wave as a sum of the initial wave transported along several classical trajectories. Because the wave formalism enters in many area of physics (acoustic waves, seismic waves, electromagnetic waves, quantum waves ...), semi-classical analysis is an important mathematical tool to understand physical phenomena.

There is a standard way to pass from the classical mechanics formulation to the quantum (wave) mechanics formulation, and vis et versa. In each field there are some well defined objects or concepts. However, there is not always a simple and one to one correspondence between these two domains, especially when the classical dynamics is not integrable.

Quantum mechanics parameter $\hbar$	Semi-classical limit ($\hbar \to 0$)$\Longrightarrow$ $\Longleftarrow$ Quantization	**Classical mechanics**
Hilbert space		Phase space
Operator $\hat{H}$		Hamilton function H
Evolution $\lvert\psi(t)\rangle$		Ergodicity, mixing,...
Eigenstates of $\hat{H}$		Periodic orbits
		Invariant measures

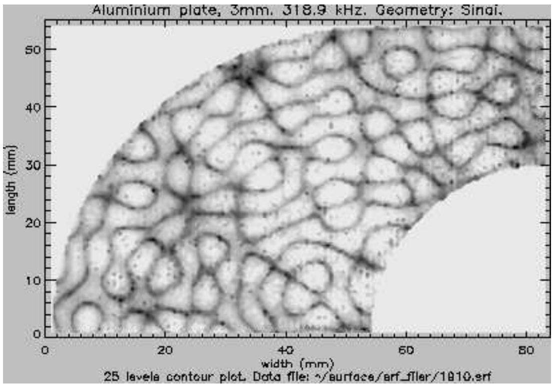
We will be concern with uniform hyperbolic dynamical systems, with the property that every classical trajectory has a hyperbolic instability (uniformly over phase space). This property implies a very strong chaotic behaviour of the dynamics [11][35],[48]. It is important to notice that although hyperbolic systems form an open set in the space of dynamical systems (i.e. properties are robust under perturbations), most of the "chaotic" systems which have some physical importance are not uniformly chaotic. They have mixed phase space, with some regular part, with K.A.M. tori, and some chaotic part with hyperbolic sets [27].

For a quantum hyperbolic dynamical system, there is a **(universal) characteristic time**, called **Ehrenfest time** t_E, which is the time when the wave length size l is expanded to the size of the cavity L, due to hyperbolic instabilities. The **Ehrenfest time** t_E, is given by $L \simeq l e^{\lambda t_E}$, where λ is the maximal Lyapounov coefficient, i.e.

$$t_E \simeq \frac{1}{\lambda} \log(1/h), \qquad h = l/L.$$

1.1.1 Examples of waves in cavities, in physics and mathematics

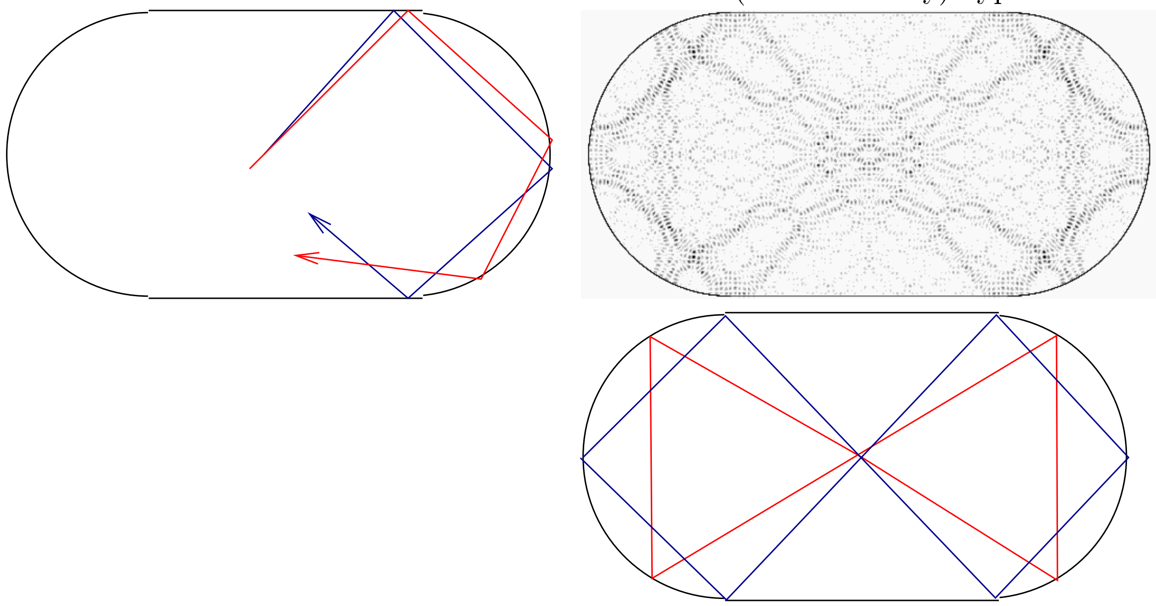
- Quantum waves (electrons) in atoms, molecules, nanoscopic electronic devices.
- Electromagnetic waves in cavities.
- stationary Acoustic wave function of Aluminium Sinai stadium (C. Ellegaard, M. Oxborrow, P. Bertelsen and K. Schaadt): size $L \sim 10cm$, wave length $\lambda \sim 3mm$.



- Seismic wave in earth or in geological valley.
- In mathematics: function $\psi(x)$ on a (negative curvature) compact Riemannian manifold. The wave equation $\partial^2\psi/\partial t^2 = \Delta\psi$ properties or properties of the eigenfunctions of the Laplacian $\Delta\psi = \lambda\psi$, for large $|\lambda|$ are closely related with geodesics properties on the manifold. Geodesics are the classical trajectories. Negative curvature is responsible for divergence of closed trajectories, and hyperbolicity (i.e. chaotic properties) of the flow.

1.1.2 Strange phenomena are observed: "Scarred" eigenfunctions:

The following figure shows a **stationary wave** $-\hbar^2\Delta\psi = E\psi$ in the stadium billiard with Dirichlet conditions. The classical flow of this billiard is (non uniformly) hyperbolic.



As this example shows, some (exceptional) **stationary "Scarred states",** are partially localized near (unstable) periodic orbits. Numerous studies of **scarred states** have been done by physicists (Heller 1984[32], micro-wave cavities, electron in quantum dots...).

1.1.3 Some questions which are considered in these lectures notes:

For an hyperbolic dynamical system,

1. Is it possible to understand the **distribution of stationary waves,** on configuration space or on phase space?
   Are they all equidistributed in the limit $h = l/L \to 0$? (This is the property of **Quantum Unique Ergodicity, Q.U.E)**

2. Is it possible to describe the **evolution of a wave packet** from a semi-classical point of view? (spreading and interferences after long time evolution, in terms of classical trajectories)

Recent reviews on quantum chaos: There are some recent reviews on the *mathematical aspects* of quantum chaos, and semi-classical measures. For example by S. De Bièvre *[15],[16]* or by S. Zelditch *[59][62]*.

1.2 Quantum map on the torus as convenient models for quantum chaos study

In this section, we summarize the main results reported in these lectures.

1.2.1 Linear map on the torus

In the first part, we will study a **linear hyperbolic map** $M \in SL(2,\mathbb{Z})$ on **Torus phase space** $\mathbb{T}^2$. For example

$$\begin{pmatrix} x_{i+1} \\ p_{i+1} \end{pmatrix} \equiv \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}_{M} \begin{pmatrix} x_i \\ p_i \end{pmatrix}, \qquad \text{modulo } [1] \tag{1}$$

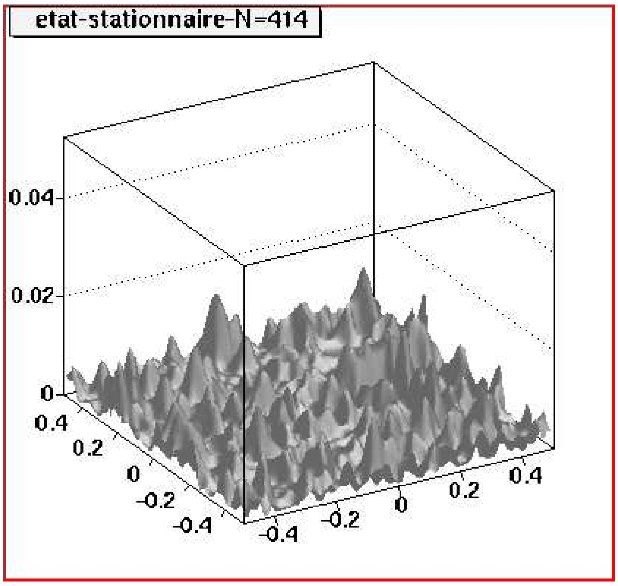
The hyperbolicity of the map together with the modulo 1 operation are responsible for hyperbolicity and chaotic properties of the map. This map is one of the most simple example of a symplectic chaotic map [11][35],[48]. It can be considered as a Poincaré section of a hyperbolic flow.

We will introduce a semi-classical parameter $h = 1/N$, $N \in \mathbb{N}$, quantize the map M, giving an operator $\hat{M}_h$, and consider **quantum stationary states** (i.e. eigenstates) in the semi-classical limit $h \to 0$:

$$\hat{M}_h |\phi_h\rangle = e^{i\varphi_h} |\phi_h\rangle.$$

The **Schnirelman theorem** ("the Quantum Ergodic theorem"), states that in the semi-classical limit $h = 1/N \to 0$, most of stationary states are **equidistributed**. Their limit measure on the torus phase space is the Lebesgue measure (for finite h, there may have fluctuations on a scale $o(1)$):

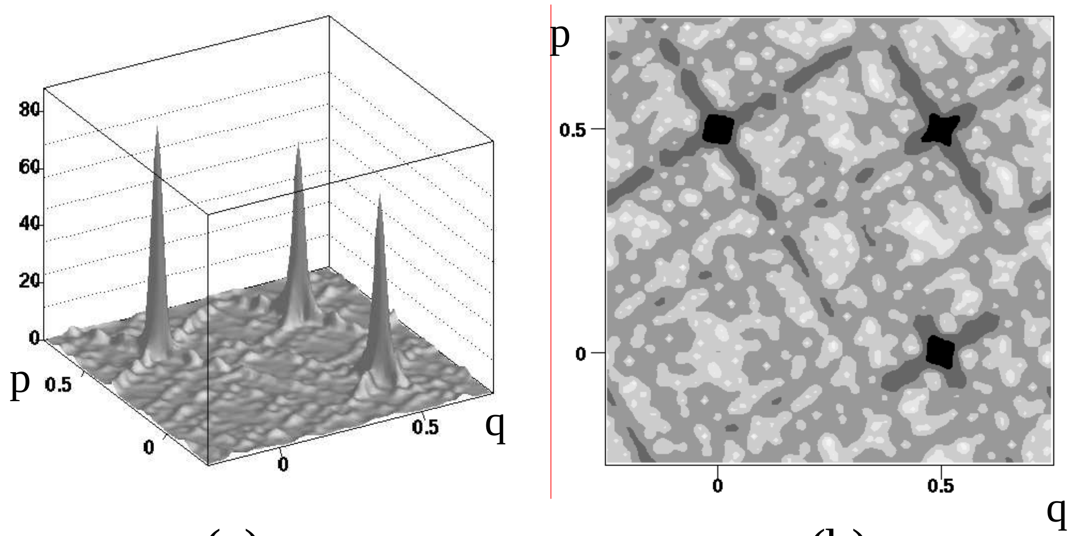
$$\boxed{\text{Probability measure}\left(|\phi_h\rangle\right) \equiv_{h\to 0} \mu_{Lebesgue}, \qquad \text{for almost all sequences.}}$$



However, for this system, we will see that some (exceptional) stationary states are **localized on periodic orbits** [23] (for a special subsequence $h_n \to 0$):

$$\boxed{\text{Probability measure}\,(|\phi_h\rangle) \equiv_{h=h_n\to 0} \frac{1}{2}\delta_{periodic\,orbit} + \frac{1}{2}\mu_{Lebesgue},}$$

giving a counter example to Quantum Unique Ergodicity. Notice that the measure is not totally localized ($\neq \delta_{periodic-orbit}$), but half of the measure is equidistributed on phase space.



(a) (b)

For quantum *linear* map on the torus there is a strange phenomenon of **"short quantum periods"**: $\hat{M}^P \propto \hat{Id}$, with a period P which is sometimes equal to twice the Ehrenfest time $P \simeq 2t_E$. This means that at time $t = 2t_E$, an initial localized wave packet has an **exact revival**. This revival phenomenon is very different from the mixing property of the classical map. The semi-classical explanation for this revival is that the wave packet spreads along the unstable manifold, and at time $t = t_E$ equidistributes over all the torus. Then, due to very particular interference effects, it reconstructs and comes back along the stable manifold to the initial localized state. We will indeed give a precise interpretation of this revival, in terms of **constructive interferences** along **classical homoclinic orbits** at time $t = 2t_E = 2\frac{1}{\lambda}\log(1/h)$. The existence of strong scarred states is a direct consequence of this revival: we superposing the time evolution of the wave packet from time $t = 0$ to time $t = 2t_E$, and obtain an eigenstate. In the interval $t \in [0, 2t_E]$, the wave packet is localized half of the time, and equidistributes the other half. This explains the classical measure $\mu = \frac{1}{2}\delta_{p.o.} + \frac{1}{2}\mu_{Leb.}$. Due to this unavoidable spreading, we will also prove that the maximal weight of a scarred state on a periodic orbit is 1/2 [8][22].

- The short quantum period implies high degeneracies in the spectrum. The scarred states are some particular vectors of the high dimensional eigenspaces. On the opposite, for the same example, P. Kurlberg and Z. Rudnick have shown that the *joint spectrum* of $\hat{M}$ and Hecke operators has **all eigenfunctions equidistributed [38, 39].** They also shown that for almost all sequence $\hbar_n \to 0$, **Q.U.E** holds.

1.3 Non linear hyperbolic map, and evolution of quantum wave packets

Linear map on the torus are very particular, but easy to handle. In the second part, we will consider "perturbed map" of the form

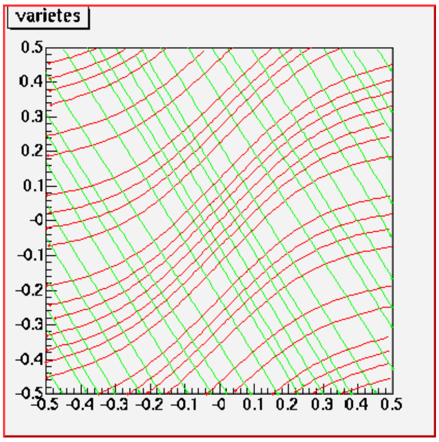
$$M = M_1 M_0 \quad \mathbb{T}^2 \to \mathbb{T}^2,$$

$$M_0 \in SL\,(2\mathbb{Z}) \text{ hyperbolic linear,} \qquad \text{Perturbation: } M_1 = \text{Hamiltonian Flow on}\mathbb{T}^2$$

Hyperbolic map have the very important property to be "stable" under perturbations (structural stability). It means that if the perturbation $M_1$is small enough (in the C^1topology), then the map M is conjugate to M_0:

$$M = QM_0Q^{-1}, \qquad Q : \mathbb{T}^2 \to \mathbb{T}^2 \quad : \text{Hölder continuous}$$

The map M is still hyperbolic (mixing,...), however stability Lyapunov coefficients of the periodic orbits, or classical actions have changed. The following figure shows the stable/unstable foliation for a non linear hyperbolic map:



It is not possible to extend in a simple way the construction of the scarred eigenstates of $\hat{M}_0$.

However extension of some results have been obtained recently:

- J.M. Bouclet and S. DeBièvre [9], have showed that a stationary scarred quantum states (if it exists!) can not localize to fast on a periodic orbit with $\hbar \to 0$: they obtain an upper bound on α, for the radius $r \sim \hbar^\alpha$, on which the Husimi function can fully concentrate.

- Stéphane Nonnenmacher [20] has shown a similar upper bound such that if some part of the function concentrates in the disk, then another part should equidistribute.
- Nalini Anantharaman has shown a similar result for eigenfunctions of the Laplacian on negative curvature manifold: the eigenfunctions of the Laplacian do not concentrate on sets of small topological entropy[3]. She has shown that an eigenfunction cannot be completely concentrated in a disk $o(1)$.

There remains the question of existence of scarred eigenstates for non linear map. Our construction of scarred states for the linear map relies on revival effect at time $2t_E$. We will see that revival can not appear before that time. So a generalization requires a description of the evolution of a wave packet beyond the Ehrenfest time. In the second part of the lectures, we will report some recent results on semi-classical description of wave packet evolution for long time ($t \simeq C \log(1/\hbar)/\lambda = C\,t_E$, with *any* $C > 0$), i.e. beyond the Ehrenfest time, for a hyperbolic non linear map [21].

1.3.1 Observation of the evolution of a wave packet, and characteristic time scales

One of the main challenges in quantum chaos is to deal with both the long time limit $t \to \infty$, and the semi-classical limit $\hbar \to 0$. Usual semi-classical results, such as the Ehrenfest theorem, or Egorov theorem concerns $\hbar \to 0$ first, and $t \to \infty$ after. The challenge is to try to reverse this order (in order to get informations on individual eigenfunctions and eigenvalues), or more modestly, make t depending on $\hbar$.

In order to have some intuition on typical characteristic time scales in quantum chaos and associated phenomena, we discuss now the evolution of a wave packet on a numerical example. Under a non linear hyperbolic map M on the torus, a wave packet (a coherent state $|x_0\rangle$) is launched at time $t = 0$, at a generic position x_0. Figure (1) shows the Husimi distribution (i.e. phase space representation) of the evolved state $|\psi(t)\rangle = \hat{M}^t|x_0\rangle$ at different time $t \in \mathbb{Z}$. We recall that the Husimi distribution of the initial state $|x_0\rangle$ has typical width $\Delta_0 \simeq \sqrt{\hbar}$, (due to uncertainty principle $\Delta x \Delta p \simeq \hbar$, and specific choice $\Delta x = \Delta p = \Delta_0 \simeq \sqrt{\hbar}$). During time evolution, the wave packet center is moving, and its distribution spreads, due to instabilities of the trajectories. Figure 2 summarizes the main effects we discuss below.

Finite time regime with "no dispersion": We first consider a fixed value of $t = Cste$, and $\hbar \to 0$ (of course t can be arbitrary large in principle). The evolved state $|\psi(t)\rangle$ is localized at the classical position $x(t) = M^t x_0$. In more precise words, the semi-classical measure of $|\psi(t)\rangle$ is a Dirac measure at $x(t) = M^t x_0$. The evolved state $|\psi(t)\rangle$ spreads but its width is $\Delta_t \simeq e^{\lambda t}\Delta_0 \simeq \sqrt{\hbar}$, still of order $\hbar^{1/2}$ [33][31][42]. Because t can be chosen arbitrary large a priori, the ergodic nature of the dynamics may have importance if $x(t)$ follows a dense trajectory for example. Some well known semi-classical results such as semiclassical Egorov theorem, or Schnirelman quantum ergodicity theorem use these finite range of time [12][58].

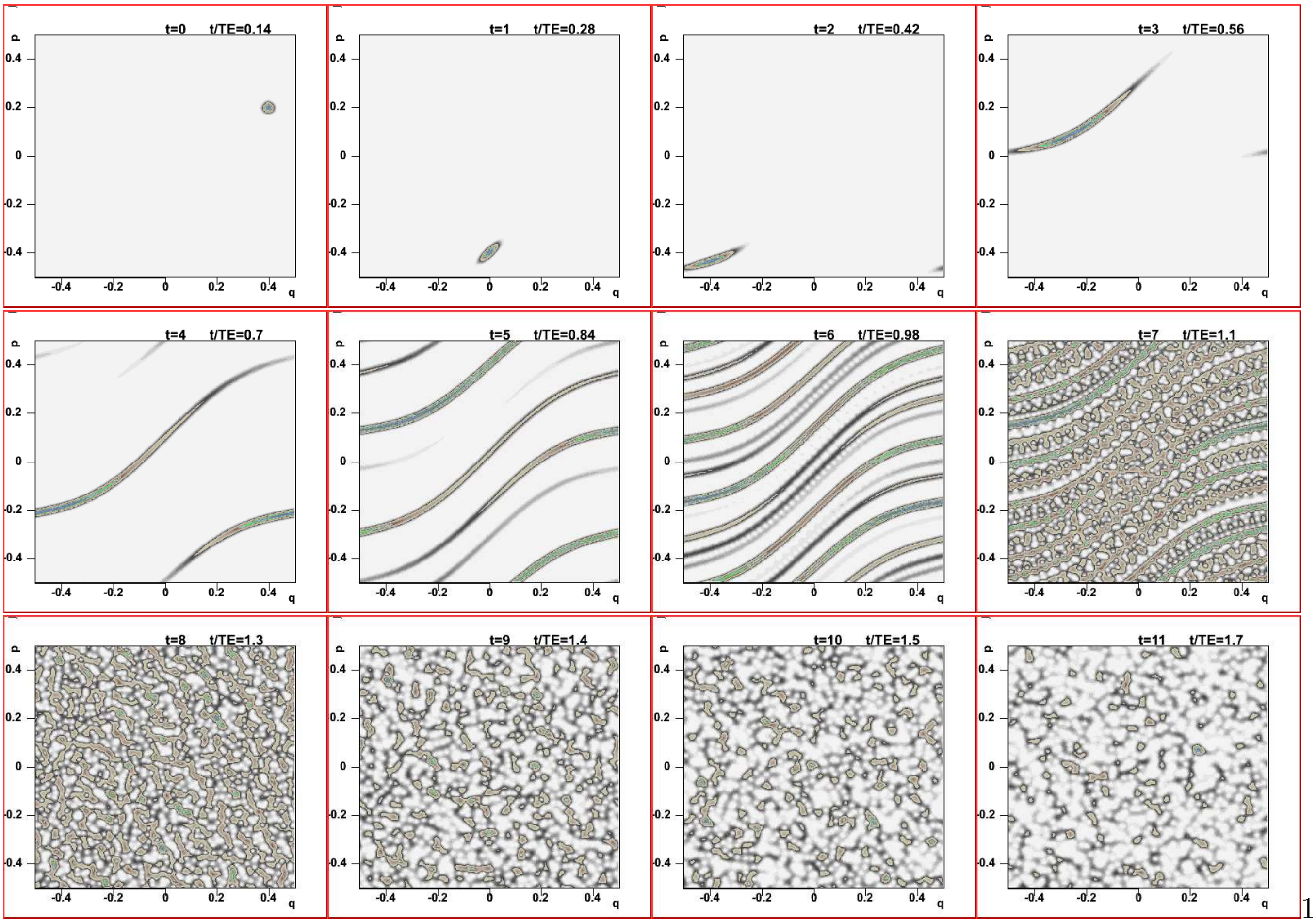


Figure 1: Husimi distribution of the evolution of a (generic) coherent state at initial position $x_0 = (0.4, 0.2)$, for different time $t = 0, 1, 2, \ldots$ and $h = 10^{-3}$.

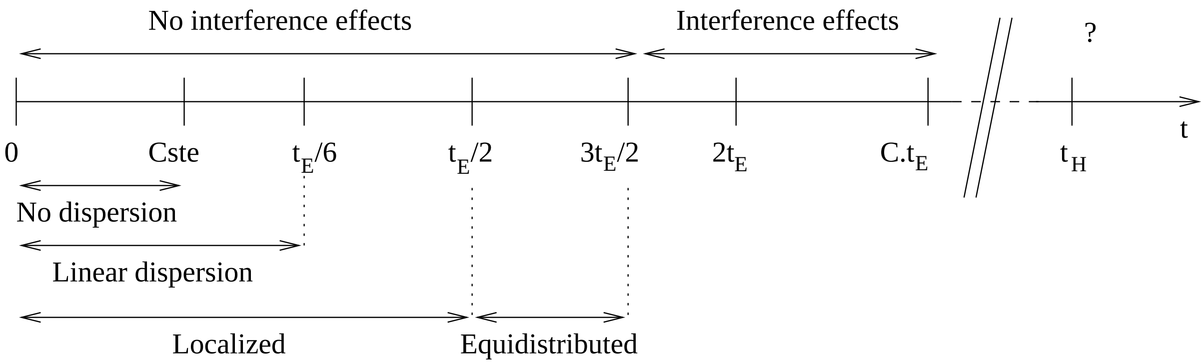


Figure 2: Characteristic times which appear in the semi-classical limit $h \to 0$, for the evolution of an initial coherent state. $t_E = \frac{1}{\lambda} \log(1/h)$is the Ehrenfest time, (very small "in practice") compared to the Heisenberg time $t_H = 1/h$.

Linear dispersion regime: Some recent and very general results [13][4][28][10] describe the evolved quantum state $|\psi(t)\rangle$, in the linear dispersion regime, which means that non linear effects on the dispersion of the coherent state are supposed to be negligible with respect to the linear effects. Because the first non linear effects correspond to cubic terms in the Hamiltonian, this imposes that $\Delta_t^3 \ll \hbar$, equivalently $e^{\lambda t}\hbar^{1/2} \ll \hbar^{1/3}$, or $t \ll \frac{1}{6}t_E$. In our numerical example $\frac{1}{6}t_E = 1.2$.

Localized regime: After that time, the coherent state spreads more and more. But its width is still of microscopic size if $\Delta_t \ll 1$, i.e. $t \ll \frac{1}{2}t_E$. In more precise words, the semi-classical measure of $|\psi(t)\rangle$ is still a Dirac measure at $x(t)$ in that range of time. In our example $\frac{1}{2}t_E = 3.6$.

At a time around $t \simeq \frac{1}{2}t_E$, the quantum state has size of order 1, and can be described as a "Lagrangian W.K.B. state" [20].

Equidistribution regime: For time t larger than $\frac{1}{2}t_E$, the wave packet spreads and wraps around the torus phase space, along unstable manifolds, like a classical probability measure.

Thanks to classical mixing, a smooth classical probability distribution is known to converge towards the uniform Liouville measure for large time. The Husimi distribution is expected to behave like a classical measure, and equidistributes, if the different branches do not "interfere" with each other on phase space. After the time $\frac{1}{2}t_E$, we evaluate that the distance between consecutive branches get smaller and smaller like $d \sim e^{-\lambda(t-t_E/2)}$until the critical value $\hbar$ is obtained at time $t = \frac{1}{2}t_E + t_E = \frac{3}{2}t_E$. This is the ultimate value, because if $d \gg \hbar$, one can still insert a (squeezed) localized wave packet between two consecutive branches, which means that the branches do not yet interfere.

Indeed, in [8], the authors show that for the *linear* map, the semi-classical measure $|\psi(t)\rangle$ converges towards the Liouville measure, in the range of time $\frac{1}{2}t_E \ll t \ll \frac{3}{2}t_E$. J.M. Bouclet and S. De Bièvre in [9] obtain a similar result for the non linear hyperbolic map., for $t \ll \frac{2}{3}t_E$. S. Nonnenmacher in [20] reach the time $t \ll \frac{3}{2}t_E$.

In [52], R. Schubert has described evolution of initial Lagrangian states under an hyperbolic flow. He obtained similar results, namely equidistribution up to time $t \ll t_E$. This is indeed similar, because an initial coherent state becomes a Lagrangian state at time $\frac{1}{2}t_E$ (see [20]). This range of time is also considered and controlled in [3].

Longer time and interference effects: For longer time very little is known. Some arguments and numerical observations in [54] suggest that semi-classical formula applies for longer time. In [21] it is shown that in the range of time $t \in [0, C.t_E]$, where C is any constant, the evolved state $|\psi(t)\rangle$ expressed by its Husimi distribution $Hus(x) = |\langle x|\psi(t)\rangle|^2$, or Bargmann distribution $\langle x|\psi(t)\rangle$, can be expressed in general as a (finite) sum over different classical trajectories starting from the vicinity of the initial state x_0, and ending in the vicinity of the point x at after time t (similar to the semi-classical Van-Vleck formula). These trajectories give unavoidable interferences effects for time $t \geq \frac{3}{2}t_E$.

Revival may occur at time $t \simeq 2t_E$, however it is expected that at least generically, these different contributions are somehow uncorrelated, and as a result, the state $|\psi(t)\rangle$ is "generically" equidistributed over phase space as can be observed on figure 1.

An important characteristic time which is not considered here, because far much larger than the actual semi-classical approach could reach, is the **Heisenberg time** $t_H = 1/h$ ($= 1000$ in our example). This time is related with the mean separation between eigenvalues of $\hat{M}$. Some important effect of quantum chaos are numerically observed at this range of time, and explained by a **Random Matrix Theory** approach [6]. Note that contrary to the mathematical works which are "stopped" by the Ehrenfest time, the Heisenberg time is extremely discussed in the physical litterature, essentially with the random matrix theory. This allows to describe statistical properties of individual eigenfunctions and eigenvalues.

1.4 Semi-classical formulas beyond the Ehrenfest time

This last section of the introduction is not presented with all the required explanations. (The course begins in section 2.)

We report here some recent results concerning the "long time evolution" of quantum states with a non linear hyperbolic map on the torus [21]. Long time means $t \simeq Ct_E$, with any $C > 0$, and $t_E = \frac{1}{\lambda}\log(1/\hbar)$, $\hbar \to 0$, i.e. possibly t larger than the Ehrenfest time t_E which is the frontier time where "interference" phenomena occur. We just present here the main idea without proofs. The author think that the same analysis and results could be easily generalize for any uniform hyperbolic dynamics, like hyperbolic flows. Also we think that it could be generalize for maximal hyperbolic sets in non uniformly hyperbolic dynamical systems (like systems with "mixed phase space").

We just present the semi-classical **"Gutzwiller trace formula"** which expresses $\mathrm{Tr}\left(\hat{M}^t\right)$ in terms of periodic orbits of period t, but the same approach gives also matrix elements between coherent states (or any localized states) $\langle x'|\hat{M}^t|x\rangle$. The (first) main objective of this work has been to try to answer to question (2.8), raised at the end of this lecture.

Our result is

Theorem 2. *[21] For any $K > 0$, for any $C > 0$, and $|t| < C\frac{1}{\lambda}\log(1/\hbar)$,*

$$\mathrm{Tr}\left(\hat{M}^t\right) = T_{semi,t,J} + \mathcal{O}\left(\hbar^K\right)$$

with

$$T_{semi,t,J} = \sum_{x_n \in Fix(M^t)} \exp\left(-i\frac{\mathcal{A}_n}{\hbar}\right)\frac{1}{2\sinh\left(\frac{\lambda_n t}{2}\right)} e^{S_n} \tag{2}$$

where x_n if a fixed point of M^t on $\mathbb{T}^2$ (a periodic orbit of M), $\mathcal{A}_n$ its classical action, λ_n its Lyapounov exponent, and

$$S_n = \hbar S_{(1)} + \hbar^2 S_{(2)} + \ldots + \hbar^{[J/2]-1} S_{([J/2]-1)}, \qquad e^{S_n} = 1 + \hbar S_{(1)} + \ldots$$

is a finite semi-classical series (semi-classical corrections), where each term $S_{(k)}$, is expressed in terms of the semiclassical normal form of the periodic orbit x_n, up to order J, with J large enough:

$$J > 2(K + C)$$

More precisely, $T_{semi,t,J}$ is expressed uniquely from (semiclassical) **cocyle invariants**[1] on the phase space torus (except for the action $\mathcal{A}_n$ which needs to introduce a line bundle over $\mathbb{T}^2$). We explain in the next paragraph the basic ingredient of this result.

1.4.1 Remarks:

- Notice that $2\sinh\left(\frac{\lambda_n t}{2}\right) = \sqrt{\det\left(DM^t_{x_n} - I\right)}$.
- Notice that at time $t \simeq Ct_E$, the number of fixed point x_n is $\mathcal{N}_t \simeq e^{\lambda t} = (1/h)^C$, for $t \simeq Ct_E$, which has to be compared to the number of Planck cells on phase space ($\simeq 1/h$) which is much less, if $C > 1$, i.e. after the Ehrenfest time.
- Each semi-classical correction is bounded $\left|S_{(j)}\right| < C_j t$, uniformly over trajectories and time. The error bound above can be understood from the naive requirements (if moduli of errors would add) $\mathcal{N}_t e^{-\lambda_{min} t/2}\left(\hbar^{J/2} t C_{J/2}\right) < \mathcal{O}\left(\hbar^K\right)$, implied by $J > 2(K + C)$. Figure (3), shows from numerical results, that the actual error seems to be much smaller. (this is because the complex numbers "annihilate" each over).

[1]For a given (continuous) function $\varphi : \mathbb{T}^2 \to \mathbb{R}$, one can construct a cocyle which is the Birkhoff time average starting from point x:

$$\varphi_t(x) = \sum_{t'=1}^{t} \varphi\left(M^{t'} x\right)$$

The cohomological class of φ is characterized by the values of $\varphi_t(x)$, for every periodic point x, of period t[35]. For example if $\varphi(x)$ is the expanding rate from $x \to Mx$ along the unstable Manifold, the associated cohom. cocycle classes is the collection of Lyapounov coefficient (of all the periodic orbits). As an obvious consequence, useful for us, $t \min\varphi \leq \varphi_t(x) \leq t \max\varphi$.

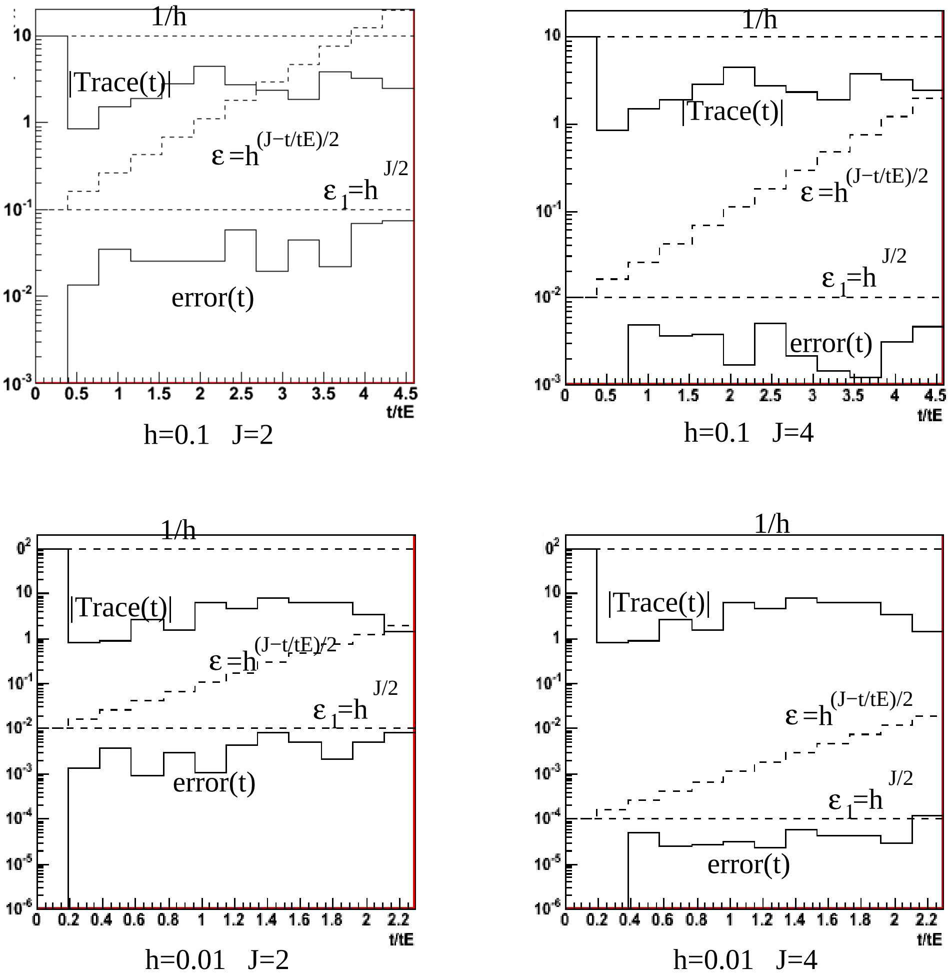


Figure 3: Numerical results and comparison to estimations of the error. $\left|Trace\left(\hat{M}^t\right)\right|$ and $Error_{t,J} = \left|Trace\left(\hat{M}^t\right) - Trace_{semi,t,J}\right|$ have been computed numerically for $t = 0 \rightarrow 11$. We plot the upper bound of the trace $1/h$, the upper bound of the error $\varepsilon_{t,J} = h^{\frac{1}{2}(J-t/t_E)}$, and the upper bound of the error for an individual term in the sum $\varepsilon_1 = h^{J/2}$. We have choose Planck constant $h = 0.1$ and $h = 0.01$, and computed normal forms of the periodic orbits up to order $J = 2$ and $J = 4$. The unexpected observed fact is that $Error_{t,J} < \varepsilon_1$, although $Error_{t,J} < \varepsilon_{t,J}$ has only been proved.

1.4.2 A non stationary semi-classical normal form of the map

The essential ingredient which allows to control the error for such long times in eq.(2), is that every semi-classical normal form is a **cocycle**, i.e. there is a global description of the hyperbolic dynamics by "semiclassical non stationary normal form", first established by D. Delattes [18], for the *classical* map. The main idea of **non stationary normal form** is that every point $x \in \mathbb{R}^2$ (of the universal cover), is sent to the point $M(x)$, by the action of the map M in the vicinity of x, and the map has the general form

$$M \equiv \mathcal{T}_{M(x)} N_x \mathcal{T}_x^{-1} \tag{3}$$

where $\mathcal{T}_x : \mathbb{R}^2 \to \mathbb{R}^2$ is a symplectic transformation (parametrized by x), and $N_x : \mathbb{R}^2 \to \mathbb{R}^2$ is a hyperbolic map expressed in **a normal form**. It means that N_x is the flow over time 1, of the **normal form hyperbolic Hamiltonian** (a total Weyl symbol in fact)

$$\begin{aligned} H_x(q', p') &= \lambda_{0,0,x} + \lambda_{0,1}(q'p') + \lambda_{0,2}(q'p')^2 + \dots \\ &\quad + \hbar\lambda_{1,0} + \hbar\lambda_{1,1}(q'p') + \hbar\lambda_{1,2}(q'p')^2 + \dots \end{aligned}$$

where each coefficient $\lambda_{j,l}(x)$ depends on $x \in \mathbb{T}^2$. See figure 4.

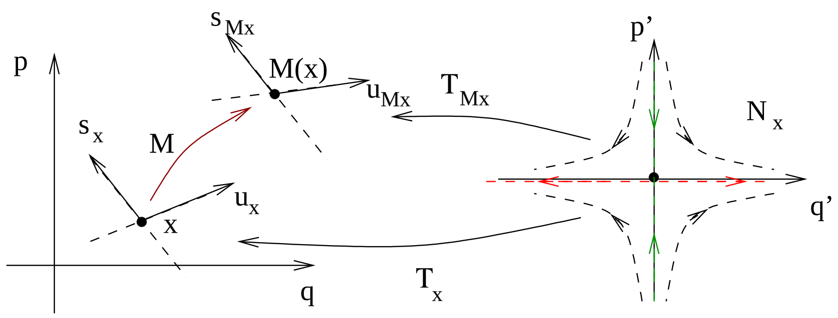


Figure 4: This picture traduces the conjugation relation of the normalization eq.(3). u_x, s_x is a unstable/stable frame at point $x \in \mathbb{T}^2$, sent respectively to axis q', p' by the conjugation.

$\mathcal{T}_x$ and N_x are continuous functions of $x \in \mathbb{T}^2$.

At the level of linear approximation, this result of non stationary normal form is nothing else but the definition of uniform hyperbolicity, with the result that the stable/unstable foliation is continuous. The construction of D. Delattes is a recursive scheme at all non linear orders, and can be extend to semi-classical corrections as well.

This normal form expression is well suited to express M^t for large time, by composition along any orbit $x \to M^t(x)$. For example

$$M^2 x \equiv \mathcal{T}_{M^2(x)} N_{M(x)} \mathcal{T}_{M(x)}^{-1} \mathcal{T}_{M(x)} N_x \mathcal{T}_x^{-1} = \mathcal{T}_{M^2(x)} \left(N_{M(x)} N_x \right) \mathcal{T}_x^{-1}, \qquad etc...$$

For a periodic orbit, $N_{x,t} \stackrel{\text{def}}{=} N_{M^t(x)} \dots N_{M(x)} N_x$ is again a normal form which can be expressed from the time averages of the function$\lambda_{j,k}(x)$ along the trajectory. The property of continuity of $\mathcal{T}_x$ and N_x is the most important for us, because it implies some uniform control of the resulting normal form for long (periodic) orbits.

Each indices j, l, the function $\lambda_{j,l}(x)$ generates a continuous **cocyle** on $\mathbb{T}^2$. The semi-classical trace formula $T_{semi,t,J}$ eq.(2), depends only on their cohomological class, i.e. the Birkhoff average along periodic orbits: $\tilde{\lambda}_{j,l}(x_{o.p.}) = \frac{1}{t}\sum_{t'=1}^{t} \lambda_{j,l}\left(M^{t'}x\right)$, up to order $2(j+l) \leq J$. For example the **Lyapounov exponent** is the cocyle $\lambda_n = \tilde{\lambda}_{0,1}(x_{o.p.})$. The **Anosov cocyle** is $\tilde{\lambda}_{0,2}(x_{o.p.})$, and is known to be an obstruction for the stable/unstable foliations to be smooth (C^2), cf [34], [30] p.289. The term $S_{(1)}$ in eq.(2) is given by

$$S_{(1)} = -it\left(\tilde{\lambda}_{2,0} + \tilde{\lambda}_{1,1} I_{(1)} + \tilde{\lambda}_{0,2} I_{(2)}\right)$$

where $I_{(j)} = \left(\frac{-i}{2}\right)^j + \mathcal{O}\left(e^{-\tilde{\lambda}_{0,1}t}\right)$ for large time. The other terms $S_{(j)}$ have similar expressions.

The semi-classical trace formula has been expressed using normal forms in [60, 61] , [26] and [53], but for finite time, because the cocyle property which relies on hyperbolicity of the flow, was not used.

1.4.3 An attempt to use thermodynamical theory for long time semi-classics

One aim of the school *"Resonances and periodic orbits: spectrum and zeta functions in quantum and classical chaos"* organised by Viviane Baladi is to conciliate recent ideas and results from one side "dynamical systems" (thermodynamical theory of Ruelle Pollicott and others,...) and from an other side "quantum chaos". We propose here in few words such an attempt suggested recently by the author in view of formula (2) (this work is under progress, but we present here some ideas in order to get some reactions).

We consider the prequantized line bundle over the torus $L \to \mathbb{T}^2$, with Chern index $N = 1/h \in \mathbb{N}$. The non linear map M can be lifted to act on L, and this induces a map $\tilde{M}$ in the space of $C^\infty(L)$ (C^∞sections of the line bundle L. This is an infinite dim. space). We can consider any potential function $\varphi : \mathbb{T}^2 \to \mathbb{C}$, and define similarly $\tilde{M}_\varphi$. Because of hyperbolicity, we expect the map $\tilde{M}_\varphi$ to have Ruelle-Pollicott resonances spectrum. Its regularized trace can be written (exactly):

$$\operatorname{Tr}\left(\tilde{M}_\varphi^t\right) = \sum_{x_n \in \operatorname{Fix}(M^t)} \exp\left(-i\frac{\mathcal{A}_n}{\hbar}\right) \frac{e^{\varphi_n}}{\left(2\sinh\left(\frac{\lambda_n t}{2}\right)\right)^2}$$

(notice the square from classical Jacobian, and the presence of the classical actions which are obtained as "holonomies" from the lifted action).

This has to be compared with eq. (2) which can be written:

$$\operatorname{Tr}\left(\hat{M}^t\right) \simeq \sum_{x_n \in \operatorname{Fix}(M^t)} \exp\left(-i\frac{\mathcal{A}_n}{\hbar}\right) \frac{1}{\left(2\sinh\left(\frac{\lambda_n t}{2}\right)\right)^2} e^{S_n}\left(e^{\lambda_n t/2} - e^{-\lambda_n t/2}\right)$$

Now for *large time* t, we said that S_n is a cocycle, so it is tempting to compare $\mathrm{Tr}\left(\hat{M}^t\right)$ with the classical traces

$$\mathrm{Tr}\left(\tilde{M}^t_{\varphi_1}\right)-\mathrm{Tr}\left(\tilde{M}^t_{\varphi_2}\right)\overbrace{\simeq}^{?}\mathrm{Tr}\left(\hat{M}^t\right) \tag{4}$$

with respective potential functions:

$$\varphi_1(x)=\frac{1}{2}\lambda_{0,1}(x)+S(x)$$

$$\varphi_2(x)=-\frac{1}{2}\lambda_{0,1}(x)+S(x)$$

Where $S(x)=\hbar S_{(1)}(x)+\hbar^2 S_{(2)}(x)+\dots$ is a semi-classical correction (due to non linearity).

Such a result would be of great interest, because from a knowledge of the resonance spectrum of $\tilde{M}_\varphi$, one could have a better control on the semi-classical Gutzwiller trace formula (2). We recall that physicists use the Gutzwiller trace formula for long time, without semi-classical correction terms, i.e. they use:

$$T_{Gutzwiller}(t)=\mathrm{Tr}\left(\tilde{M}^t_{\varphi_1}\right)-\mathrm{Tr}\left(\tilde{M}^t_{\varphi_2}\right)$$

with $\varphi_1(x)=\frac{1}{2}\lambda_{0,1}(x)$, $\varphi_2(x)=-\frac{1}{2}\lambda_{0,1}(x)$ (the second term is expect to be negligible for large time because of a smaller spectrum radius). They expect that this formula is a good approximation of $\mathrm{Tr}\left(\hat{M}^t\right)$ for large time (up to $t\simeq 1/\hbar$)[54]. Figure (3), supports this guess.

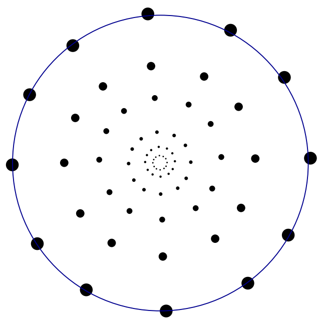
In the special case of a linear hyperbolic map $M\in SL(2,\mathbb{Z})$, then $S=0$, one has an exact relation:

$$\mathrm{Tr}\left(\tilde{M}^t_{\varphi_1}\right)-\mathrm{Tr}\left(\tilde{M}^t_{\varphi_2}\right)=\mathrm{Tr}\left(\hat{M}^t\right)$$

We have also

$$\mathrm{Tr}\left(\tilde{M}^t_{\varphi_1}\right)=\mathrm{Tr}\left(\hat{M}^t\right)\left(1-e^{\lambda t}\right)^{-1}=\mathrm{Tr}\left(\hat{M}^t\right)\left(1+e^{-\lambda t}+e^{-2\lambda t}+e^{-3\lambda t}\dots\right)$$

and we deduce the resonance spectrum of $\tilde{M}_{\varphi_1}$ (and $\tilde{M}_{\varphi_2}$) from the unitary spectrum $\left(e^{i\theta_k}\right)_{k=1\to N}$of $\hat{M}$. We obtain respectively $\left(e^{i\theta_k}e^{-l\lambda}\right)_{k=1\to N,l\geq 0}$ and $\left(e^{i\theta_k}e^{-(l+1)\lambda}\right)_{k=1\to N,l\geq 0}$. See figure.



2 Linear cat map

In this section we show how to quantize a hyperbolic map $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{Z})$ on $\mathbb{T}^2$. There is a standard way to quantize a Hamiltonian flow, so we have to express the map as an Hamiltonian flow. Both in quantum and classical mechanics, we will have first to express the dynamics on the plane $\mathbb{R}^2$ (universal cover of $\mathbb{T}^2$) and then pass to the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

2.1 Classical mechanics

2.1.1 Classical dynamics on the plane

Consider a quadratic Hamiltonian on plane phase space $(\mathbb{R}^2, \omega = dq \wedge dp)$:

$$H(q,p) = \frac{1}{2}\alpha q^2 + \frac{1}{2}\beta p^2 + \gamma qp, \tag{5}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$.

H generates a flow $x(t) = (q(t), p(t))$ on $\mathbb{R}^2$, given by $x(t) = M(t)x(0)$ ($t \in \mathbb{R}$), and explicitly (from Hamilton equations) by $dq(t)/dt = \partial_p H = \gamma q + \beta p$, $dp(t)/dt = -\partial_q H = -\alpha q - \gamma p$. For each $t \neq 0$, $M(t)$ is a matrix in $\mathrm{SL}(2, \mathbb{R})$, and for $t = 1$:

$$M \overset{\mathrm{def}}{=} M(1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \exp \begin{pmatrix} \gamma & \beta \\ -\alpha & -\gamma \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), \tag{6}$$

i.e. $\det(M) = AD - BC = 1$

Assuming $\gamma^2 > \alpha\beta$, M is a **hyperbolic map** with two real eigenvalues $e^{\pm\lambda}$ where $\lambda = \sqrt{\gamma^2 - \alpha\beta} > 0$ is the **Lyapounov exponent**. The two associated real eigenvectors corresponding to an unstable and a stable direction for the dynamics.

2.1.2 Classical dynamics on the torus

Suppose moreover that $A, B, C, D \in \mathbb{Z}$, i.e. $M \in SL(2, \mathbb{Z})$. Then for any $x \in \mathbb{R}^2, n \in \mathbb{Z}^2$,

$$M(x+n) = M(x) + M(n) \equiv M(x) \bmod 1$$

so M induces a map on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, see figure 5. This map is Anosov (uniformly hyperbolic), with strong chaotic properties, such as ergodicity and mixing, see [35] p. 154.

Remark: M is not the time 1 flow of an Hamiltonian *on the torus*, otherwise it should be homotopic to identity (by contraction with time variable).

The stable / unstable directions have an irrational slope. They fill densely the torus.

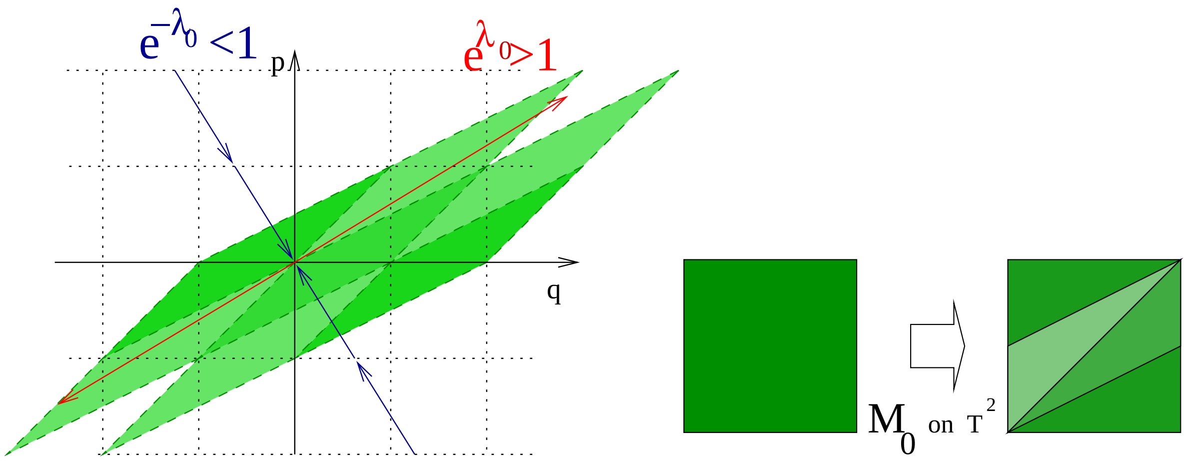
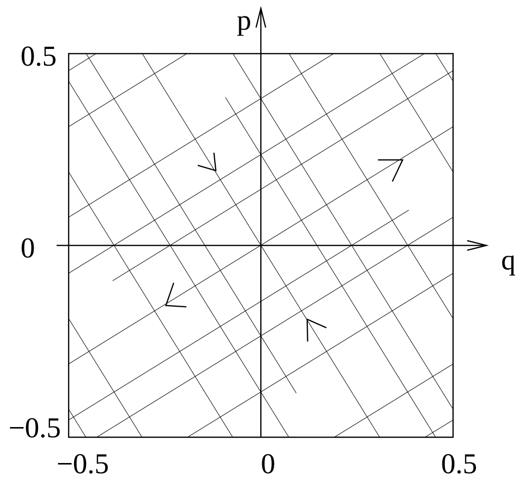


Figure 5: Dynamics of $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$ on $\mathbb{R}^2$ and on $\mathbb{T}^2$



2.2 Quantum dynamics

2.2.1 Quantum dynamics on the plane

The quantum **Hilbert space** associated to the plane phase space $\mathbb{R}^2$ is $\mathcal{H}_{plane} = L^2(\mathbb{R})$. A **quantum state** $\varphi(q) \in \mathcal{H}_{plane}$ is also written $|\varphi\rangle$ ($\varphi(q)$ is called the **quantum wave function**).

We introduce the Planck constant $\hbar > 0$, and set $h = 2\pi\hbar$. We will consider the **semi-classical limit** $\hbar \to 0$ later on.

Recall the usual position and momentum self-adjoint operators:

$$(\hat{q}\varphi)(q) \equiv q\varphi(q), \qquad (\hat{p}\varphi)(q) \equiv -i\hbar\frac{d\varphi}{dq}(q).$$

$$[\hat{q},\hat{p}] = i\hbar\hat{Id}$$

The Hamiltonian operator $\hat{H}$ is a self-adjoint operator, obtained by **Weyl quantization**[2] of H eq.(5):

$$\hat{H} = Op_{Weyl}\left(H\right) = \frac{\alpha}{2}\hat{q}^2 + \frac{\beta}{2}\hat{p}^2 + \frac{\gamma}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right),$$

The Schrödinger equation in $\mathcal{H}_{plane} = L^2\left(\mathbb{R}\right)$ governs time evolution of quantum states $|\varphi\left(t\right)\rangle \in \mathcal{H}_{plane}$:

$$i\hbar\frac{\partial|\varphi\left(t\right)\rangle}{\partial t} = \hat{H}|\varphi\rangle,$$

and generates a unitary evolution operator $\hat{M}$ between $t = 0 \to 1$, written:

$$|\varphi(1)\rangle = \hat{M}|\varphi\left(0\right)\rangle, \quad \hat{M} = \exp\left(-\frac{i}{\hbar}\hat{H}\right). \tag{7}$$

($\hat{M}$ is obtain by exponential of an operator quadratic in $(\hat{q},\hat{p})$. These operators form the **Metaplectic group**)

Unitary translation operators: For $v = (v_q, v_p) \in \mathbb{R}^2$, let

$$T_v : \mathbb{R}^2 \to \mathbb{R}^2 \tag{8}$$

be the translation on classical phase space by v, i.e. $T_v\left(x\right) = x + v$. T_v is understood as the flow generated by the linear Hamiltonian function $f\left(q,p\right) = \left(v_1 q - v_2 p\right)$.

The corresponding unitary quantum **translation operator** is defined by:

$$\hat{T}_v = \exp\left(-\frac{i}{\hbar}\left(v_1\hat{p} - v_2\hat{q}\right)\right). \tag{9}$$

These quantum translations satisfy the algebraic identity

$$\hat{T}_v\hat{T}_{v'} = e^{-iS/\hbar}\hat{T}_{v+v'}, \tag{10}$$

with $S = \frac{1}{2}\left(v_1 v_2' - v_2 v_1'\right) = \frac{1}{2}v \wedge v'$ (this comes from $[\hat{q},\hat{p}] = i\hbar\hat{Id}$). The translation operators generate an (irreducible) unitary representation of the **Heisenberg group**, with Lie algebra $\hat{q},\hat{p},\hat{Id}$, $[\hat{q},\hat{p}] = i\hbar\hat{Id}$.

---

[2] Remark: Op_{Weyl} : (quadratic function$f$ on $\mathbb{R}^2$)$\to\hat{f} = Op_{Weyl}\left(f\right)$ is characterized by requirements[15] (prop. 12.1)

1. $Op_{Weyl}\left(q\right) = \hat{q}$, $Op_{Weyl}\left(p\right) = \hat{p}$,
2. $\left[Op_{Weyl}\left(f\right), Op_{Weyl}\left(g\right)\right] = i\hbar Op_{Weyl}\left(\{f,g\}\right)$
3. $Op_{Weyl}\left(\overline{f}\right) = \left(Op_{Weyl}\left(f\right)\right)^{+}$

For any matrix $M \in SL(2,\mathbb{R})$, one trivially has $M\,T_v M^{-1} = T_{Mv}$. This intertwining persists at the quantum level:

$$\boxed{\hat{M}\,\hat{T}_v\,\hat{M}^{-1} = \hat{T}_{Mv}.} \tag{11}$$

This last relation will play a major role at many instances in these lectures. It is important to remark that this relation holds because M is a linear map. (It will be false in the non linear case).

2.2.2 Quantum mechanics on the torus

At the classical level, the torus phase space was obtained by introducing periodicity on $\mathbb{R}^2$ with respect to the $\mathbb{Z}^2$ lattice, generated by translations $T_{(1,0)}$ and $T_{(0,1)}$. The same construction can be done in quantum mechanics. The difference is that we have to check now that these two translation operators commute before we consider their common eigenspaces.

From (10), we have $\hat{T}_{(1,0)}\hat{T}_{(0,1)} = e^{-i/\hbar}\hat{T}_{(0,1)}\hat{T}_{(1,0)}$, so the two translation operators commute $\left[\hat{T}_{(1,0)}, \hat{T}_{(0,1)}\right] = 0$ if and only if

$$\boxed{N = \frac{1}{2\pi\hbar} \in \mathbb{N}^*}$$

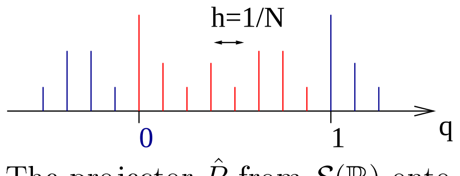
We will suppose this last condition from now on.

Then define the common eigenspace of $\hat{T}_{(1,0)}, \hat{T}_{(0,1)}$ with eigenvalue 1:

$$\mathcal{H}_N = \left\{ |\varphi\rangle \in \mathcal{S}'(\mathbb{R}) \ / \ \hat{T}_{(1,0)}|\varphi\rangle = |\varphi\rangle, \quad \hat{T}_{(0,1)}|\varphi\rangle = |\varphi\rangle \right\}$$

In order to have a concrete expression of $|\varphi\rangle \in \mathcal{H}_N$, remark that using $\hbar$-Fourier-Transform: $\tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dq\, \varphi(q)\, e^{-ipq/\hbar}$, we have that $\varphi \in \mathcal{H}_N$ iff $\tilde{\varphi}(p-1) = \tilde{\varphi}(p)$ and $\varphi(q-1) = \varphi(q)$. The quasi-periodicity of $\tilde{\varphi}$ implies that $\varphi(q) = \sum_{n\in\mathbb{Z}} a_n \delta(q - q_n)$, with $q_n = n/N$, $n \in \mathbb{Z}$, and $a_n \in \mathbb{C}$. The periodicity of $\varphi(q)$ implies that $a_{n-N} = a_n$. So a quantum state $|\varphi\rangle \in \mathcal{H}_N$ is specified by $(a_n)_{n=1\to N} \in \mathbb{C}^N$. We deduce that $\mathcal{H}_N$ is a finite dimensional phase space:

$$\dim \mathcal{H}_N = N = \frac{1}{h}.$$



The projector $\hat{P}$ from $\mathcal{S}(\mathbb{R})$ onto the space $\mathcal{H}_N$ is:

$$\hat{P} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \hat{T}_{(1,0)}^{n_1}\,\hat{T}_{(0,1)}^{n_2} = \sum_{n\in\mathbb{Z}^2} \hat{T}_n. \tag{12}$$

(assuming N even).

From (11) , we deduce:

$$\hat{M}\,\hat{P} = \hat{P}\,\hat{M} \tag{13}$$

and a commutative diagram:

$$\begin{array}{ccc} \mathcal{S}(\mathbb{R}) & \xrightarrow{\hat{M}} & \mathcal{S}(\mathbb{R}) \\ \downarrow \hat{P} & & \downarrow \hat{P} \\ \mathcal{H}_N & \xrightarrow{\hat{M}} & \mathcal{H}_N \end{array}$$

We have an endomorphism

$$\hat{M}:\ \mathcal{H}_N \rightarrow \mathcal{H}_N: \qquad (\equiv N \times N\,\text{matrix}) \tag{14}$$

We will sometimes write $\hat{M}_{torus}$ for this operator, to distinguish it from $\hat{M}_{plane}$ eq.(7), acting in $\mathcal{H}_{plane}$.

2.3 Phase space representation of Quantum dynamics

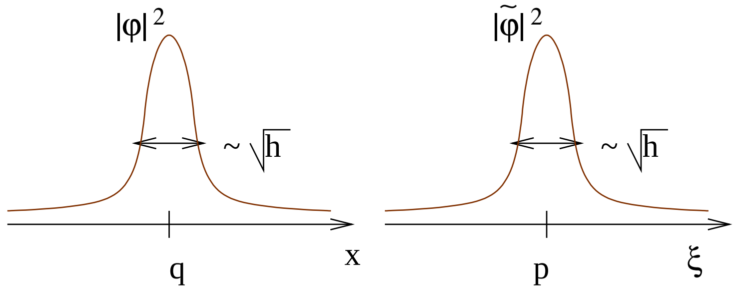
Classical dynamics takes place on the torus phase space $\mathbb{T}^2$. We have seen that a quantum state $|\varphi\rangle \in \mathcal{H}_N$ is a function (distribution) of $q \in \mathbb{R}$ alone. In order to investigate the semi-classical limit, it would be better to represent quantum states as distributions on phase space. For that purpose we introduce coherent states.

2.3.1 Coherent states [56],[47]

The normalized state $|0\rangle \in \mathcal{H}_{plane}$ is defined by $a|0\rangle = 0$, where $a = \frac{1}{\sqrt{2\hbar}}\left(\hat{q} + i\hat{p}\right)$. This states $|0\rangle$ is the ground state of the Harmonic oscillator $\frac{1}{2}\left(\hat{p}^2 + \hat{q}^2\right)$, and is semi-classically localized at the origin $x = (0,0) \in \mathbb{R}^2$. A "standard" coherent state of the plane is then defined as the translated

$$|x\rangle \stackrel{\text{def}}{=} \hat{T}_x|0\rangle, \quad x = (q,p) \in \mathbb{R}^2. \tag{15}$$

and is semi-classically localized at $x = (q,p) \in \mathbb{R}^2$. Indeed the wave function of $|x\rangle = |q,p\rangle$, is $\varphi_{q,p}\left(q'\right) = \frac{1}{(\pi\hbar)^{1/4}} \exp\left(i\frac{pq'}{\hbar}\right) \exp\left(-\frac{(q'-q)^2}{2\left(\sqrt{\hbar}\right)^2}\right)$, and is localized at q, with a width $\simeq \sqrt{\hbar}$ ($\rightarrow 0$ for $\hbar \rightarrow 0$). Its $\hbar$-Fourier transform is $\tilde{\varphi}_{q,p}\left(\xi\right) = \frac{1}{(\pi\hbar)^{1/4}} \exp\left(-i\frac{q\xi}{\hbar}\right) \exp\left(-\frac{(\xi-p)^2}{2\left(\sqrt{\hbar}\right)^2}\right)$, similarly localized at p.



There is a closure identity [56]:

$$\hat{I}d_{\mathcal{H}_{plane}} = \int_{\mathbb{R}^2} |x\rangle\langle x| \frac{dx}{h} \tag{16}$$

2.3.2 Husimi distribution

The **Husimi distribution** of a quantum state $|\psi\rangle \in \mathcal{H}_{plane}$ is the positive measure on phase space:

$$\boxed{Hus_{\psi}(x) \stackrel{\text{def}}{=} \frac{1}{h} |\langle x|\psi\rangle_n|^2 = \frac{1}{h} \left| \int \overline{\varphi}_{q,p}(q')\, \psi_n(q')\, dq' \right|^2}$$

Where $|\psi\rangle_n = |\psi\rangle / \sqrt{\langle\psi|\psi\rangle}$ is the *normalized* state. From (16), we check that this is a probability measure:

$$\int_{\mathbb{R}^2} Hus_{\psi}(x)\, d^2x = \|\psi_n\|^2 = 1$$

This definition is very intuitive: for a given x, an important value of $Hus_{\psi}(x)$ means that the quantum state has "high probability of presence" at point x of phase space. Semi-classical measure will give a more precise sense of that, below.

A **coherent state on the torus** is defined by periodicity:

$$|x\rangle_{torus} \stackrel{\text{def}}{=} \hat{P}|x\rangle \in \mathcal{H}_N$$

and we define similarly the Husimi distribution of a quantum state $|\varphi\rangle \in \mathcal{H}_N$.

There is a closure identity on the torus:

$$\hat{I}d_{\mathcal{H}_N} = \int_{\mathbb{T}^2} |x\rangle_{torus}\langle x| \frac{dx}{h}$$

So $\int_{\mathbb{T}^2} Hus_{\varphi}(x)\, d^2x = \|\varphi\|^2 = 1$.

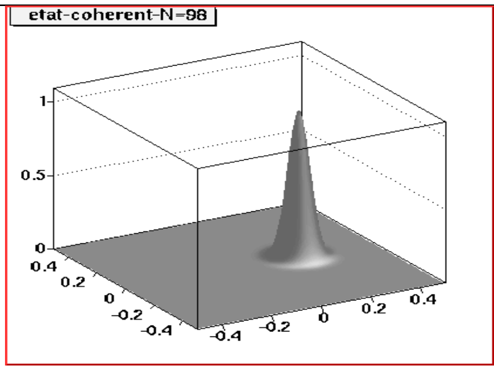
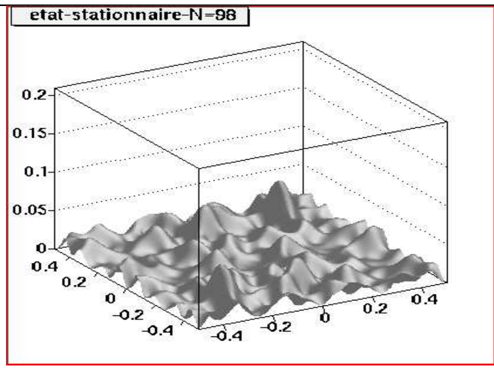
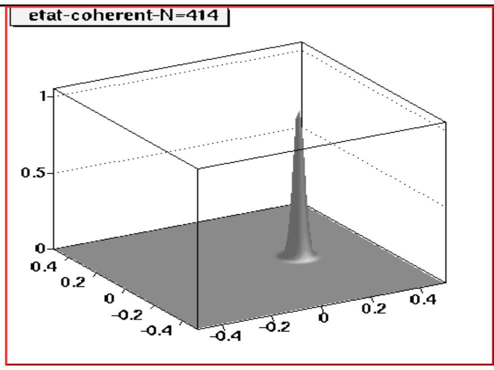
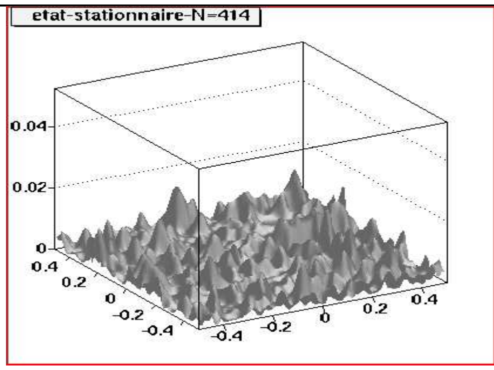
Remark: the definition of the Husimi distribution depends on the choice of coherent states family. In geometrical quantization terms, it depends on the choice of a complex polarization [57]. However, the "semi-classical measures" we will define below do not depend on this choice.

2.3.3 Example of Husimi distributions

Let $x_0 = (q_0, p_0) \in \mathbb{R}^2$. The Husimi distribution of the coherent state $|x_0\rangle = |q_0, p_0\rangle$ is $Hus_{x_0}(x) = \frac{1}{h} |\langle x|x_0\rangle|^2$. Standard calculation with the Heisenberg group gives ([56],[24])

$$Hus_{x_0}(x) = \frac{1}{h} \exp\left(-\frac{1}{2\hbar} \left((q-q_0)^2 + (p-p_0)^2 \right) \right)$$

The Husimi distribution is a Gaussian centered at point x_0 on phase space, with width $\Delta x \simeq \hbar^{1/2}$. This distribution get localized at point x_0 for $\hbar \to 0$, and converge towards the Dirac measure δ_{x_0}.

	Coherent state $\vert q_0, p_0\rangle$	Eigenstate $\hat{M}\vert\psi_h\rangle = e^{i\phi_h}\vert\psi_h\rangle$
$N = 1/h = 98$		
$N = 1/h = 414$		
Semi-classical Measure : measure for $h \to 0$	$\mu = \delta_{(q_0,p_0)}$	$\mu = \mu_{Lebesgues} (= dq.dp)$ (?)

2.3.4 Weyl Quantization on the torus

(Cf [15] page 45). If $f \in C^\infty(\mathbb{T}^2)$, $x = (q,p) \in \mathbb{R}^2$, $n = (n_1, n_2) \in \mathbb{Z}^2$, Fourier decomposition gives

$$f(x) = \sum_{n\in\mathbb{Z}^2} f_n \exp\left(-i2\pi\left(n_1 p - n_2 q\right)\right) = \sum_{n\in\mathbb{Z}^2} f_n \exp\left(-i2\pi\left(n \wedge x\right)\right)$$

then Weyl quantization of f is the operator:

$$\begin{aligned} \hat{f} &= Op_{Weyl}(f) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^2} f_n \exp\left(-i2\pi\left(n_1\hat{p} - n_2\hat{q}\right)\right) \qquad (17) \\ &= \sum_{n\in\mathbb{Z}^2} f_n \exp\left(-\frac{i}{\hbar}\left(\frac{n_1}{N}\hat{p} - \frac{n_2}{N}\hat{q}\right)\right) \\ &= \sum_{n\in\mathbb{Z}^2} f_n \hat{T}_{n/N} \end{aligned}$$

using eq.(9) and $2\pi\hbar = 1/N$.

2.4 Quantum periods phenomenon

The property of quantum periods described here is (at our level of understanding) the key point for the quantum revival phenomenon, and for the example of Non Quantum Unique Ergodicity described below. As we will see, this phenomenon is very specific to the action of quantum quadratic Hamiltonian on the torus Hilbert space.

2.4.1 Quantum periods

References: Etienne Ghys [25], or Hannay-Berry (1980)[29].

Theorem 3. *For any value $h = 1/N$, there exists $P_N \in \mathbb{N}^*$ and $\alpha_N \in \mathbb{R}$, such that $\hat{M}^{P_N} = \hat{I}d\ e^{i\alpha_N}$.*

Remarks

- it means that after time P_N a quantum state comes back to its initial value (up to a constant phase). This is very different with the classical dynamics, where due to mixing, any smooth distribution converges towards the uniform Liouville measure.
- Because $\hat{M}$ is a finite dimensional unitary matrix, we expect quasi-periodicity in general. Periodicity means that the eigenvalues of $\hat{M}$ are regularly distributed on the circle. In case where P_N is much smaller than N, it implies degeneracies in the spectrum.

Proof. A translation $\hat{T}_v$, $v \in \mathbb{R}^2$, act *inside* $\mathcal{H}_N$ if $\hat{T}_v$ commute with $\hat{T}_{(1,0)}, \hat{T}_{(0,1)}$ or with $\hat{T}_n$, $\forall n \in \mathbb{Z}^2$. From (10), $\hat{T}_v\hat{T}_n = \hat{T}_n\hat{T}_v \underbrace{\exp\left(-i2\pi N v \wedge n\right)}_{=1}$, if $Nv \in \mathbb{Z}^2$, i.e. $v = \frac{k}{N}$, $\quad k \in \mathbb{Z}^2$.

These translations are therefore:

$$\hat{T}_{k/N}, \qquad k \in \mathbb{Z}^2$$

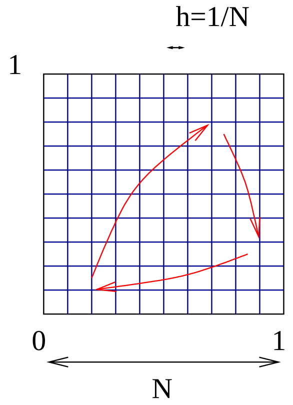
But for $k, m \in \mathbb{Z}^2$, $\hat{T}_{k/N+m}\hat{P} = \hat{T}_{k/N}\hat{T}_m \sum_{n\in\mathbb{Z}^2} \hat{T}_n = \hat{T}_{k/N} \sum_{n\in\mathbb{Z}^2} \hat{T}_{m+n} = \hat{T}_{k/N}\hat{P}$. So there is the periodicity relation $\left(\hat{T}_{k/N+m}\right)_{/\mathcal{H}_N} = \left(\hat{T}_{k/N}\right)_{/\mathcal{H}_N}$, $\forall m, k \in \mathbb{Z}^2$. The translation operators $\hat{T}_{k/N}$ form a group, and $\mathcal{H}_N$ is irreducible with respect to it. Using (11),

$$\begin{aligned}
\hat{M}^P \propto \hat{I} \ &\Leftrightarrow\ \left[\hat{M}^P, \hat{T}_{k/N}\right] = 0, \quad \forall k\, [1, N]^2, \\
&\Leftrightarrow\ \hat{M}^P \hat{T}_{k/N} \hat{M}^{-P} = \hat{T}_{k/N} \ \Leftrightarrow\ \hat{T}_{M^P k/N} = \hat{T}_{k/N} \\
&\Leftrightarrow\ M^P \frac{k}{N} \equiv \frac{k}{N}\ [1]
\end{aligned}$$

So

$$P(N) = min_P \left\{ P \,/\, M^P \equiv I\ [N] \right\}$$

The last relation has obviously a solution, because M has integer coefficients. Its action preserves the lattice on the torus with rational components k/N, and P_N is given by the least common multiple of the periods of this action.



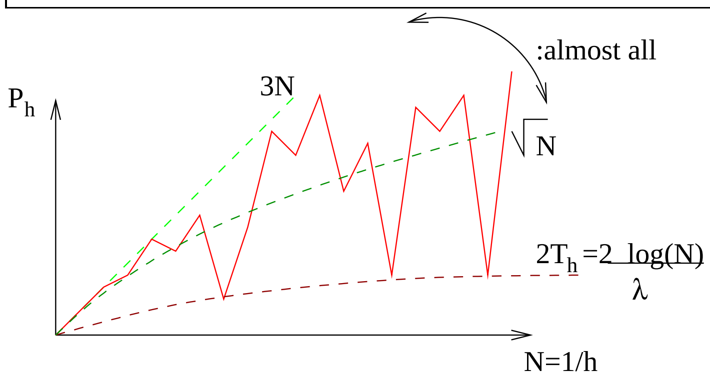
□

Theorem 4. *General bounds ([36],[25]):*

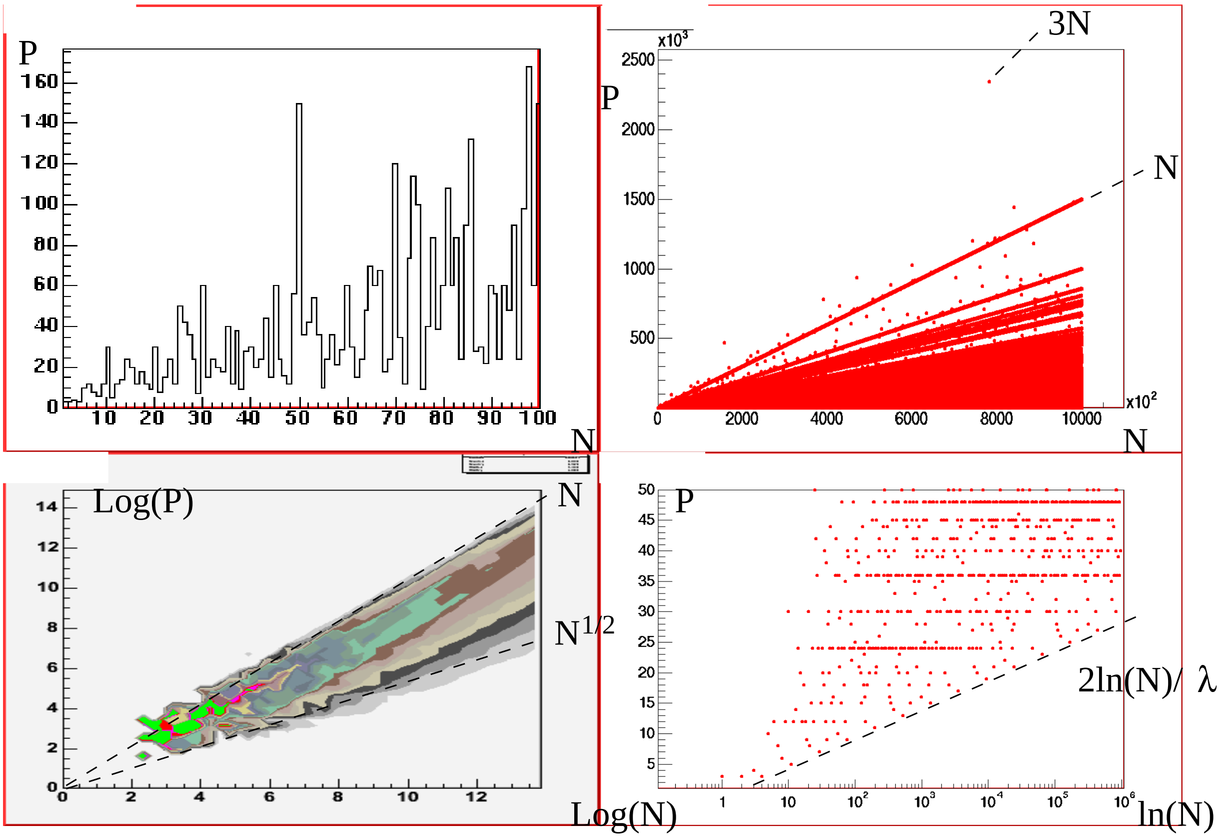
$$\exists C, \forall N, \quad \frac{2}{\lambda}\log N - C \leq P(N) \leq 3N$$

For almost all values of $N \in \mathbb{N}$ [39]:

$$P(N) \geq \sqrt{N}$$



Numerical calculations for $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$:



2.4.2 Short quantum periods

We calculate explicitly the "shortest quantum period", i.e. the lowest values of N for a given period $P \in \mathbb{N}$.

Proposition 5. *For $P = 2t_1$ even, $t_1 \in \mathbb{N}$, $N = a_{t_1}$ is given by the sequence:*

$$a_0 = 0, \quad a_1 = 1, \quad a_{t_1+2} = Ta_{t_1+1} - a_{t_1}$$

where $T = Trace\,(M) > 2$.
For $P = 2t_1 + 1 odd$, $t_1 \in \mathbb{N}$, $N = b_{t_1}$ is given by the sequence:

$$b_0 = 1, \quad b_1 = T + 1, \quad b_{t_1+2} = Tb_{t_1+1} - b_{t_1},$$

In both cases, for large N,

$$P = 2t_E + C + \mathcal{O}\left(\frac{1}{N}\right) \sim 2t_E$$

with the Ehrenfest time

$$t_E = \frac{1}{\lambda} \log N$$

and C is a constant which depends on the parity of P only.

Example : for $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, and P odd, one obtains the values

$$(N, P) = (1, 1), (4, 3), (11, 5), (29, 7), (76, 9), \ldots$$

Remark : we will obtain in section (2.6.6), that $P \sim 2t_E$ is the shortest quantum period.

Démonstration. One has

$$Det\,(M - \lambda I) = \lambda^2 - T\lambda + 1$$

with

$$T = Trace\,(M) > 2$$

So (Cayley relation)

$$M^2 = MT - I,$$

Let $t \in \mathbb{N}^*$, and

$$t_1 = t_2 = \frac{t}{2}, \qquad : \text{if } t \text{ is even}$$

$$t_1 = \frac{t-1}{2}, \quad t_2 = \frac{t+1}{2} \qquad : \text{if } t \text{ is odd}$$

Let

$$\mathcal{T}_t : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathcal{T}_t\,(x) = M^{t_2}x - M^{-t_1}x$$

then

$$det\,(\mathcal{T}_t) = det\,\big(M^t - I\big) = 2 - e^{\lambda t} - e^{-\lambda t} < 0.$$

From Cayley relation, one deduces

$$\mathcal{T}_{t+2} = T\mathcal{T}_t - \mathcal{T}_{t-2}$$

The first terms are

$$\mathcal{T}_{-1} = I - M, \quad \mathcal{T}_0 = 0, \quad \mathcal{T}_1 = M - I = -\mathcal{T}_{-1}, \qquad \mathcal{T}_2 = M - M^{-1}$$

so

$$\mathcal{T}_{2t_1} = a_{t_1}\left(M - M^{-1}\right), \qquad \mathcal{T}_{2t_1+1} = b_{t_1}\left(M - I\right) \tag{18}$$

with $a_{t_1}, b_{t_1} \in \mathbb{N}$,

$$a_0 = 0, \quad a_1 = 1, \quad a_{t_1+2} = Ta_{t_1+1} - a_{t_1},$$

$$b_0 = 1, \quad b_1 = T + 1, \quad b_{t_1+2} = Tb_{t_1+1} - b_{t_1},$$

and for large t :

$$a_{t_1} = \left(\frac{det\left(\mathcal{T}_{2t_1}\right)}{det\left(M^2 - I\right)}\right)^{1/2} = \frac{e^{\lambda t_1}}{\left(T^2 - 4\right)^{1/2}} + \mathcal{O}\left(e^{-\lambda t_1}\right)$$

$$b_{t_1} = \left(\frac{det\left(\mathcal{T}_{2t_1+1}\right)}{det\left(M - I\right)}\right)^{1/2} = \frac{e^{\lambda t_1} e^{\lambda/2}}{\left(T - 2\right)^{1/2}} + \mathcal{O}\left(e^{-\lambda t_1}\right)$$

Finally

$$M^{2t_1} - I = \left(M^{t_1} - M^{-t_1}\right) M^{t_1} = \mathcal{T}_{2t_1} M^{t_1} = a_{t_1}\left(M - M^{-1}\right) M^{t_1}$$

so :

$$M^{2t_1} \equiv I \ \left[a_{t_1}\right]$$

because $\left(M - M^{-1}\right) M^{t_1}$ is an integer matrix.

The sequence $N = a_{t_1}$,with $t_1 \in \mathbb{N}$, gives a short quantum period P which is even :

$$P = 2t_1 = 2t_E + \frac{1}{\lambda}\log\left(T^2 - 4\right) + \mathcal{O}\left(e^{-\lambda t_1}\right) \sim 2t_E$$

Similarly,

$$M^{2t_1+1} - I = \left(M^{t_1+1} - M^{-t_1}\right) M^{t_1} = \mathcal{T}_{2t_1+1} M^{t_1} = b_{t_1}\left(M - I\right) M^{t_1}$$

$$M^{2t_1+1} \equiv I \ \left[b_{t_1}\right]$$

The sequence $N = b_{t_1}$,with $t_1 \in \mathbb{N}$, gives an odd short quantum period

$$P = 2t_1 + 1 = 2t_E + \frac{1}{\lambda}\log\left(T - 2\right) - \lambda + 1 + \mathcal{O}\left(e^{-\lambda t_1}\right) \sim 2t_E$$

$\square$

2.5 Semi-classical measures and the quantum ergodic theorem

2.5.1 Semi-classical measures

Définition 6. Let $|\psi_h\rangle \in \mathcal{H}_{torus,h}$ be a sequence of quantum states, for $h \to 0$. Let μ_h be the Husimi measures, i.e. for $f \in C^\infty(\mathbb{T}^2)$, $\mu_h(f) = \int_{\mathbb{T}^2} dx\, Hus_\psi(x)\, f(x)$. Then the **semi-classical measure** μ is

$$\mu = weak - \lim_{h\to 0} \mu_h$$

if it exists.

Remarks:

- Borel probability measures on $\mathbb{T}^2$ is a compact set, so any sequence has at least one accumulation measure μ.
- The weak limit means that all the fluctuations of the Husimi distribution at the scale $\sqrt{h}$ are washed out. It remains only the variations at a finite scale with respect to $\hbar$.
- A sequence of states is said **localized at point** x, if $\mu = \delta_x$. A sequence of states is said **equidistributed** on the torus, if $\mu = \mu_{Lebesgue}$.
- The semi-classical measure of a coherent state $|x_0\rangle = |q_0, p_0\rangle$ is the Dirac measure δ_{x_0} at point x_0. So the coherent states $|x_0\rangle$ are **localized at point** x_0. See section 2.3.3.

Définition 7. If $|\psi_h\rangle$ is a sequence of eigenstates of $\hat{M}$ (i.e. $\hat{M}|\psi_h\rangle = e^{i\varphi_h}|\psi_h\rangle$), then μ is called **an invariant semi-classical measure.**

One important question is quantum chaos is to determine the set of invariant semi-classical measures denoted $\mathcal{M}_{inv.semi-class.}$.

2.5.2 Invariant semi-classical measure are classically invariant

Proposition 8. *Any invariant semi-classical measure is a classical invariant measure, i.e.:*

$$\mathcal{M}_{inv.semi-class} \subset \mathcal{M}_{inv.class}$$

Proof. Any test function $f \in C^\infty(\mathbb{T}^2)$ can be decomposed into Fourier modes:

$$f_n(x) = \exp(-i2\pi(n_1 p - n_2 q)) = \exp(-i2\pi(n \wedge x))$$

with $x = (q, p) \in \mathbb{R}^2$, and $n = (n_1, n_2) \in \mathbb{Z}^2$. Weyl quantization (17), gives $\hat{f}_n = \hat{T}_{n/N}$. Then one has an "exact Egorov property" for transportation of observables:

$$\begin{aligned} \hat{M}^{-1} \hat{f}_n \hat{M} &= \hat{M}^{-1} \hat{T}_{n/N} \hat{M} = \hat{T}_{M^{-1}n/N} = \hat{f}_{M^{-1}n} \\ &= Op_W \left(\exp \left(-i2\pi \left(M^{-1} n \wedge x \right) \right) \right) = Op_W \left(f \circ M \right) \end{aligned} \tag{19}$$

using $M^{-1} n \wedge x = n \wedge Mx$, because M preserves area.

Let μ be an invariant semi-classical measure defined as the limit of measures μ_h from a sequence of stationary states $|\psi_h\rangle, \hbar \to 0$. First, we use the fact that [45],[14]

$$\mu_h \left(f_n \right) = \langle \psi_h | \hat{f}_n | \psi_h \rangle + \mathcal{O} \left(\hbar \right) \tag{20}$$

The left hand side is the Husimi measure, whereas the right hand side is the Wigner measure of the normalized state $|\psi'\rangle$. Then

$$\mu_h \left(f_n \right) + \mathcal{O} \left(\hbar \right) = \langle \psi | \hat{f}_n | \psi \rangle = \langle \psi | \hat{M}^{-1} \hat{f}_n \hat{M} | \psi \rangle = \langle \psi | Op_W \left(f_n \circ M \right) | \psi \rangle = \mu_h \left(f_n \circ M \right) = \left(M \mu_h \right) \left(f_n \right)$$

One concludes that $\mu = M\mu$ is a classical invariant measure. $\square$

2.5.3 The Quantum Ergodic theorem

(Schnirelman 1974 [51], Zelditch 1987 [58], Colin de Verdière 1985 [12], Helffer Martinez Robert 1987 [2], Bouzouina DeBievre 1996 [1])

Theorem 9. *The "Quantum Ergodic theorem":*
*For an **ergodic** dynamics, **almost all** of the **invariant semi-classical measures** μ are **equidistributed:***

$$\mu = \mu_{Lebesgue}$$

More precisely, for each N, and any basis $|\psi_{j,N}\rangle \in \mathcal{H}_N$, $j = 1 \ldots N$, of eigenvectors of $\hat{M}_N$, there exists a subset $S(N) \subset \{1, \ldots, N\}$ of stationary states such that

1. $\lim_{N \to \infty} \frac{\sharp S(N)}{N} = 1$, i.e. the subset has "measure 1".
2. For any observable $f \in C^\infty \left(\mathbb{T}^2 \right)$, for any sequence $|\psi_{j_N,N}\rangle$of stationary states, with $j_N \in E(N), N \in \mathbb{N}$,
$$\lim_{N \to \infty} \langle \psi_{j,N} | \hat{f} | \psi_{j,N} \rangle = \int_{\mathbb{T}^2} f \, dx$$
i.e. the limit measure is $\mu = \mu_{Lebesgue}$.

Remarks:

- This theorem relies on ergodicity of the dynamics, and not on the mixing (which is a much stronger property).

- This theorem does not exclude some *exceptional* sequence of eigenvectors with semi-classical measure different from $\mu_{Lebesgue}$.
- This theorem can be rephrased in terms of the Variance of the distribution of diagonal matrix elements as follows.

2.5.4 Variance of the distribution of diagonal matrix elements

Following *[58], [1], [15] ,[62].*

Let $f \in C^{\infty}(\mathbb{T}^2)$, be an observable with spatial average $\langle f\rangle \stackrel{\text{def}}{=} \int_{\mathbb{T}^2} f dx$, and $\hat{f} = Op_{Weyl}(f)$. For any $N \in \mathbb{N}$, let $\left(|\psi_{j,N}\rangle\right)_{j=1\to N}$ be a basis of eigenstates of $\hat{M}$, and consider the distribution $\mathcal{D}_{f,N}$ of the N diagonal elements:

$$f_{j,N} = \langle\psi_{j,N}|\hat{f}|\psi_{j,N}\rangle \in \mathbb{R}, \quad j = 1 \to N$$

A first easy observation is that the mean value the distribution $\mathcal{D}_{f,N}$ is equal to $\langle f\rangle$ in the semi-classical limit $N \to \infty$ (from general properties on symbols):

$$\frac{1}{N}\sum_{j=1}^{N} f_{j,N} = \frac{1}{N}\mathrm{Tr}\left(\hat{f}\right) \to \langle f\rangle .$$

Theorem 10. *The variance of the distribution $\mathcal{D}_{f,N}$ is zero in the semi-classical limit $N \to \infty$:*

$$S_2(f) \stackrel{\text{def}}{=} Var\left(\mathcal{D}_{f,N}\right) = \frac{1}{N}\sum_{j=1}^{N} \left|f_{j,N} - \langle f\rangle\right|^2 \to 0$$

This last result implies $f_{j,N} \to \langle f\rangle$ for "almost every sequence", and gives the Quantum Ergodic theorem above *[15].*

Proof. See *[58], [1], [15] ,[62], [49].* Let us suppose that $\langle f\rangle = 0$, to simplify notations. Let $T \in \mathbb{N}$ be a fixed time.

$$\frac{1}{T}\sum_{t=1}^{T} \hat{M}^{-t}\hat{f}\hat{M}^{t} = \frac{1}{T}\sum_{t=1}^{T} \widehat{f \circ M^t} = \widehat{\langle f\rangle_T}$$

from Egorov relation (19), and with the time averaged observable

$$\langle f\rangle_T \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T} f \circ M^t$$

Because $\hat{M}|\psi_{j,N}\rangle = e^{i\varphi_{j,N}}|\psi_{j,N}\rangle$,

$$f_{j,N} = \langle\psi_{j,N}|\hat{f}|\psi_{j,N}\rangle = \frac{1}{T}\sum_{t=1}^{T}\langle\psi_{j,N}|\hat{M}^{-t}\hat{f}\hat{M}^{t}|\psi_{j,N}\rangle = \langle\psi_{j,N}|\widehat{\langle f\rangle_T}|\psi_{j,N}\rangle$$

so $S_2(f) = S_2(\langle f\rangle_T)$.

From Cauchy-Schwarz inequality, and semi-classical property on quantization of observables,

$$\left|\langle\psi_j|\widehat{\langle f\rangle_T}|\psi_j\rangle\right|^2 \leq \left\|\widehat{\langle f\rangle_T}|\psi_j\rangle\right\|^2 \||\psi_j\rangle\|^2 = \langle\psi_j|\widehat{\langle f\rangle_T}\widehat{\langle f\rangle_T}|\psi_j\rangle = \langle\psi_j|\widehat{(\langle f\rangle_T)^2}|\psi_j\rangle + \mathcal{O}(\hbar)$$

So

$$S_2(f) \leq \frac{1}{N}\mathrm{Tr}\left(\widehat{(\langle f\rangle_T)^2}\right) + \mathcal{O}(\hbar) = \int_{\mathbb{T}^2} (\langle f\rangle_T)^2 + \mathcal{O}(\hbar) = \|\langle f\rangle_T\|_{L^2} + \mathcal{O}(\hbar)$$

We use that $\|\langle f\rangle_T\|_{L^2} \to 0$, for $T \to \infty$, from the mean ergodic theorem, to deduce the theorem, with a suitable order of limits $\hbar \to 0$, and $T \to \infty$. $\square$

Remarks:

- In the proof, the time T (evolution of the dynamics) is fixed, the semi-classical limit is taken $\hbar \to 0$, and after $T \to \infty$. So the proof requires a control of semiclassical evolution at "finite time", with respect to $\hbar$.
- In [44], J. Markloff and S. Okeefe 2004 show Schnirelman theorem for a particular dynamics on "mixed phase" space. In [17] S. DeBièvre and M. Degli Esposti consider Sawtooth map and stationary map. In [20], M. Degli Espoti et al. show the Schnirelman theorem for the baker's map, together with a bound $S_2(f) \leq C/\log N$.
- An important question is related with the rate of ergodicity $S_2(f) \to 0$, and more precisely, to characterize the distribution $\mathcal{D}_f$, for $N \to \infty$. A general result for linear map is a log estimate [46]:

$$S_2(f) \leq \frac{\lambda}{2}(1+\varepsilon)\frac{(|f|_3)^2}{\log N}$$

  this bound $\mathcal{O}(1/\log(1/\hbar))$ has been first obtained by S. Zelditch for Anosov geodesic flows [63].
- For a "generic non linear hyperbolic map", physicists expect (from Random Matrix theory) that $S_2(f) \simeq \frac{1}{N}$, and more precisely that the rescaled distribution $\mathcal{D}_{f/\sqrt{\hbar}}$ is a Gaussian distribution with variance equal to the classical variance, see [62],[19].
- In [40], P. Kurlberg and Z. Rudnick show that the quantum variance is $S_2(f) \sim 1/N$ for Hecke eigenstates of cat maps. They also conjecture the form of the distribution.
- A quantum variance $\sim \sqrt{E} \sim \hbar$ was obtained also for Hecke eigenstates of arithmetic hyperbolic manifolds by W. Luo and P. Sarnak [43].

2.5.5 The problem of Quantum Unique Ergodicity or existence of "strong scars"

We are still concerned with hyperbolic dynamical systems. **Quantum Unique Ergodicity (Q.U.E.)** is the property that *every* invariant semi-classical measure is equal to the Liouville measure $\mu_{Liouville} = dqdp$, i.e. $\mathcal{M}_{inv.semi-class.} = \{\mu_{Liouville}\}$ only, with no exception in the Quantum ergodic theorem.

- This property has been conjectured for the Laplacian on compact negative curvature manifold by Z.Rudnick and P.Sarnak (1994)[50].
- For arithmetic surface of constant negative curvature, with the hypothesis that the spectrum of Δ is simple, E. Lindenstrauss has proved **Q.U.E.** in 2003 [41], cf Cours de N. Bergeron [5]. (Q.U.E. is proved for joint eigenstates of the Laplacian and Hecke operators).
- Similarly, P. Kurlberg and Z. Rudnick have proved Q.U.E. for linear hyperbolic map on the torus, for joint eigenstates of the Laplacian and Hecke operators [38].

On the opposite, one can wonder whether $\mathcal{M}_{inv.semi-class.} \neq \{\mu_{Liouville}\}$? For example, is it possible to find a sequence of eigenstates which localize on a periodic orbit, i.e. with $\mu = \delta_{periodic\,\mathrm{orbit}}$?

This last possibility seems to be in contradiction the uncertainty principle $\Delta q \Delta p \geq h$: an eigenstate should have a minimum size around a periodic orbit, and the unstable flow should spread it, along the unstable manifold. This argument is indeed correct, and rigorous results have been obtained using a control of the semiclassical spreading up to Ehrenfest time $t \leq t_E = \frac{1}{\lambda} \log(1/\hbar)$:

- Nalini Anantharaman has shown in 2004 [3] that eigenfunctions of Laplacian on compact negative curvature manifold **do not concentrate on sets of "small" topological entropy**.
- For hyperbolic linear map on the torus, F. Bonechi and S. DeBièvre[8] show that if
$$\mu = \beta \delta_{\text{periodic orbit}} + (1 - \beta)\, \nu$$
with $\nu(0) = 0$, then $\beta \leq 0.6$. (The bound has been improved to $\beta \leq 0.5$ in [22]).
- Similar results have been obtained for non linear hyperbolic map on the torus, by J.M. Bouclet, S. DeBièvre[9], S. Nonnenmacher[20].
- We show below the existence of exceptional sequences of "strong scarred eigenstates" for which
$$\mu = \frac{1}{2}\delta_{periodic\,orbit} + \frac{1}{2}\mu_{Lebesgue}$$
giving a counter example to **Q.U.E.** in a very particular example of linear hyperbolic map on the torus.

2.6 Evolution of coherent states

2.6.1 Description of the Husimi distribution on the plane

We first study the evolution of a coherent state on the plane $\mathbb{R}^2$. To simplify notations, in this section, the dynamics is suppose to be given by the hyperbolic normal form Hamiltonian $H = \lambda qp$, so:

$$\hat{H} = Op_W(H) = \lambda \frac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}), \qquad \hat{M}_t = \exp\left(-i\hat{H}t/\hbar\right)$$

With this Hamiltonian, the q axis is the unstable direction, whereas p is the stable direction. We suppose that at time $t = 0$, the quantum state is the coherent state at origin $|\psi(0)\rangle = |0\rangle$. The Husimi distribution at time t is:

$$Hus_{\psi(t)}(x) = \frac{1}{h} |\langle x|\psi(t)\rangle|^2 = \frac{1}{h} \left|\langle x|\hat{M}_t|0\rangle\right|^2$$

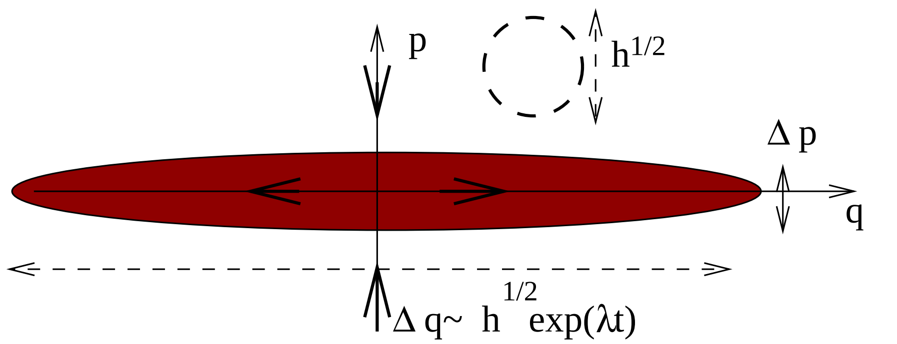
We can write $\langle x|\hat{M}_t|0\rangle = \langle 0|\hat{T}_{-x}\hat{M}_t|0\rangle$. This last quantity can be exactly calculated from Heisenberg group and Metaplectic group calculations techniques, see [56],[24]. We obtain:

$$Hus_{\psi(t)}(x) = \frac{1}{h}\frac{1}{\cosh(\lambda t)} \exp\left(-\frac{1}{2}\left(\left(\frac{q}{\Delta q_t}\right)^2 + \left(\frac{p}{\Delta p_t}\right)^2\right)\right) \tag{21}$$

This distribution is a normalized Gaussian distribution with widths:

$$\Delta q_t = \frac{\hbar^{1/2}}{\sqrt{1 - \tanh(\lambda t)}} \simeq \hbar^{1/2} \frac{e^{\lambda t}}{\sqrt{2}}, \qquad \text{for } t \gg 1$$

$$\Delta p_t = \frac{\hbar^{1/2}}{\sqrt{1 + \tanh(\lambda t)}} \simeq \hbar^{1/2} \frac{1}{\sqrt{2}}, \qquad \text{for } t \gg 1$$



So the main observation is that distribution spreads along the unstable direction like a classical distribution (length $\sim \hbar^{1/2}e^{\lambda t}$), whereas its width Δp does not vanishes, but saturates. This is due to the definition of Husimi distribution, which can be seen as a convolution with a circular Gaussian distribution of size $\simeq \hbar^{1/2}$.

Ehrenfest time: We observe that the state $|\psi(t)\rangle$ is **localized** in phase space if:

$$\Delta q_t \ll 1 \Leftrightarrow t \ll \frac{1}{2} t_E \tag{22}$$

with the **Ehrenfest time** defined by

$$t_E \stackrel{\text{def}}{=} \frac{1}{\lambda} \log(1/h) = \frac{1}{\lambda} \log(N)$$

In other words, for time $|t| \ll \frac{1}{2} t_E$, the sequence of states $|\psi(t)\rangle_\hbar$, $\hbar \to 0$, has semi-classical measure δ_0 (Dirac measure at the origin). Recall that we have already met the Ehrenfest time with the shortest quantum period $P \simeq 2t_E$.

2.6.2 Change of phase space representation

Choose $0 < \alpha < 1/2$, and define a new family of coherent states:

$$|0, \alpha\rangle \stackrel{\text{def}}{=} \hat{M}^{\alpha t_E} |0\rangle$$

$$|x, \alpha\rangle \stackrel{\text{def}}{=} \hat{T}_x |0, \alpha\rangle$$

From above, the (sequence of) state $|x, \alpha\rangle, \hbar \to 0$ is localized at x (α should not be greater than 1/2). These new coherent states have position and momentum uncertainty$\Delta q \simeq \hbar^{1/2-\alpha}, \Delta p \simeq \hbar^{1/2+\alpha}$.

The associated Husimi representation of a state $|\psi\rangle$ is:

$$Hus_{\psi,\alpha}(x) \stackrel{\text{def}}{=} \frac{1}{h} |\langle x, \alpha|\psi\rangle|^2$$

(In geometric quantization, this corresponds to choose a complex polarization which depends on $\hbar$).

Considering the state $|\psi(t)\rangle = \hat{M}^t|0\rangle$, we get

$$\langle x, \alpha|\psi(t)\rangle = \langle 0|\hat{M}^{-\alpha t_E} \hat{T}_x \hat{M}^t|0\rangle = \langle 0|\hat{T}_{M^{-\alpha t_E} x} \hat{M}^{t-\alpha t_E}|0\rangle$$

This gives

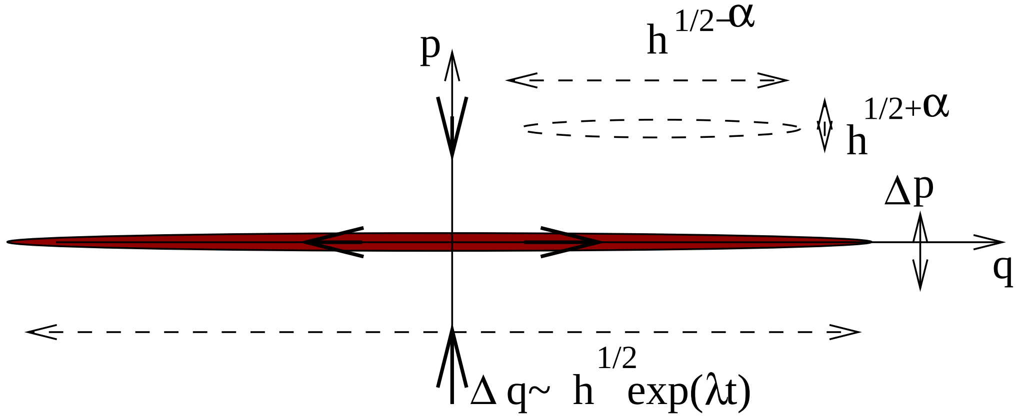
$$Hus_{\psi(t),\alpha}(x) = Hus_{\psi(t-\alpha t_E)}\left(M^{-\alpha t_E} x\right) = \frac{1}{h} \frac{1}{\cosh(\lambda(t - \alpha t_E))} \exp\left(-\frac{1}{2}\left(\left(\frac{q}{\Delta q_{\alpha,t}}\right)^2 + \left(\frac{p}{\Delta p_{\alpha,t}}\right)^2\right)\right)$$

with widths

$$\Delta q_{\alpha,t} \simeq \hbar^{1/2} \frac{e^{\lambda t}}{\sqrt{2}}, \qquad \text{for } t \gg \alpha t_E$$

$$\Delta p_{\alpha,t} \simeq \hbar^{1/2+\alpha} \frac{1}{\sqrt{2}}, \qquad \text{for } t \gg 1$$

So the Husimi distribution is has the same length Δq_t but a much finer width $\Delta p_{\alpha,t}$ which can reach $\hbar$ if α is close to $1/2$.



This change of phase space representation (i.e. choice of parameter α) does not affect the semi-classical measure.

2.6.3 Evolution of a coherent state on the torus

We come back to the initial hyperbolic dynamics $\hat{M}$ on the torus.

Let the initial state be the coherent state $|\psi(0)\rangle = |0\rangle_{torus}$ at time $t = 0$, be localized on the **fixed point** $x = 0$ on the torus.

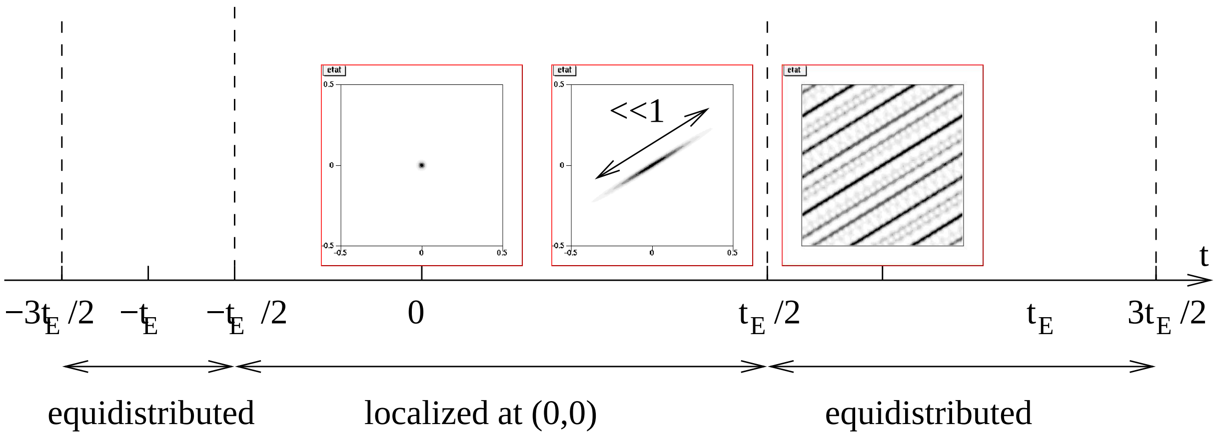
Recall that a quantum state on the torus is just obtained by periodization of its evolution on the plane:

$$|\psi(t)\rangle_{torus} = \hat{M}^t \hat{P} |0\rangle_{plane} = \hat{P}\left(\hat{M}^t |0\rangle_{plane}\right) = \hat{P}|\psi(t)\rangle_{plane}$$

where $\hat{P} = \sum_{n\in\mathbb{Z}^2} \hat{T}_n$, eq.(12), "makes a copy" in each cell of $\mathbb{Z}^2$.

From the above description of $|\psi(t)\rangle_{plane}$ we expect the Husimi distribution of $|\psi(t)\rangle_{torus}$ to behave like (rigorous results are given below):

- The distribution *spreads along the unstable direction* and has length $\Delta_t \simeq \hbar^{1/2} e^{\lambda t}$. It is still localized at 0 for $|t| \ll \frac{1}{2} t_E$. So we expect the semi-classical measure to be Dirac measure at 0.
- For $\frac{1}{2} t_E \ll t$, the distribution on the plane has length $\Delta \gg 1$, so the distribution on the torus wraps along the unstable manifold, which fills densely the torus. The branches get closer and closer as time increases. The minimal distance behaves like $\delta \simeq \exp\left(-\lambda\left(t - \frac{1}{2} t_E\right)\right)$. Because each branch has thickness $\hbar^{1/2+\alpha}$ (with $0 < \alpha < 1/2$), different branches do not interfere if $\delta \gg \hbar^{1/2+\alpha} \Leftrightarrow t \ll (1+\alpha)\, t_E$. Choosing α close to $1/2$, we expect that the semi-classical measure is the Liouville measure, if $\frac{1}{2} t_E \ll t \ll \left(\frac{3}{2} - \varepsilon\right) t_E$, with any $\varepsilon > 0$.



Theorem 11. *(F. Bonechi-S. DeBièvre [7], and [23])*
(1) For any $\varepsilon > 0$*, and* $|t_h| < \frac{1}{2}t_E(1-\varepsilon)$ *the sequence* $|\psi(t_h)\rangle$ *is* ***localized*** *at* 0*:* $\mu = \delta_{(0,0)}$.
(2) For $\frac{1}{2}t_E(1+\varepsilon) < |t_h| < \frac{3}{2}t_E(1-\varepsilon)$ *the sequence* $|\psi(t_h)\rangle$ *is* ***equidistributed*** *:* $\mu = \mu_{Lebesgue}$.

(1) is quite obvious from the description of the Husimi distribution we gave above eq.(22). The following theorem is a general result, that shows that (2) is a consequence of (1):

Theorem 12. *(F. Bonechi-S. DeBièvre [7], in the ArXiv preprint version)*
For any localized *sequence of states* $|\psi_\hbar\rangle$ *(i.e.* $\mu = \delta_0$*), the sequence* $|\psi'_\hbar\rangle = \hat{M}^{t_E}|\psi_h\rangle$ *is* equidistributed *(i.e.* $\mu' = \mu_{Lebesgue}$*).*

Proof. . To show that the semi-classical measure μ' is Lebesgue, one has to show that for any Fourier mode on $\mathbb{T}^2$, $f_n(q,p) = \exp(-i2\pi(n_1 p - n_2 q))$, with $n \in \mathbb{Z}^2$:

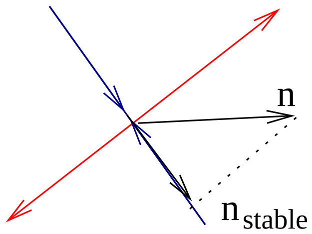
$$\mu'_h(f_n) \rightarrow_{h\rightarrow 0} \int_{\mathbb{T}^2} \mathrm{d}x\, f_n = \delta_{n,0}$$

Suppose $n \neq 0$. One computes (using eq.(20) and (11))

$$\begin{aligned} \mu'_h(f_n) + \mathcal{O}(\hbar) &= \langle\psi'_h|\exp(-i2\pi(n_1\hat{p} - n_2\hat{q}))|\psi'_h\rangle = \langle\psi_h|\hat{M}^{-t_E}\hat{T}_{hn}\hat{M}^{t_E}|\psi_h\rangle \\ &= \langle\psi_h|\hat{T}_{hM^{-t_E}n}|\psi_h\rangle \end{aligned}$$

Let us consider the decomposition of the vector $n = (n_u, n_s)$ in the frame of unstable/stable directions. Then

$$hM^{-t_E}n \equiv h\left(e^{-\lambda t_E}n_u, e^{\lambda t_E}n_s\right) = \left(h^2 n_u, n_s\right) \simeq (0, n_s)$$



Now $|\psi_h\rangle$ is localized at point 0, whereas $\hat{T}_{hM^{-t_E}n}|\psi_h\rangle$ is localized at point $(0, n_s) \neq (0,0)$, so $\langle\psi_h|\hat{T}_{hM^{-t_E}n}|\psi_h\rangle \to 0$ for $\hbar \to 0$. $\square$

Remarks: The preceding proof relies on specific property of the linear map (11).

2.6.4

2.6.5 Interpretation of the short quantum period as constructive interference effects along homoclinic orbits

Let us again consider

$$|\psi(t)\rangle = \hat{M}^t|0\rangle_{torus}$$

with $|0\rangle_{torus} = \hat{P}|0\rangle_{plane}$ being the coherent state at fixed point $x = 0$. In the case of short quantum periods, we have seen that at time $t = P \simeq 2t_E$, $\hat{M}^P \propto \hat{Id}$, so $|\psi(P)\rangle \propto |0\rangle_{torus}$ is again a coherent state localized at $x = 0$.

We propose to explain this surprising fact from a semiclassical point of view. For that purpose, we consider the **auto-correlation function** $C(t) \stackrel{\text{def}}{=} \langle\psi(0)|\psi(t)\rangle_{torus}$. Of course $C(0) = 1$, and the **revival** at time $t = P$ means that $|C(P)| = 1$. To simplify the notations, suppose t even. We have:

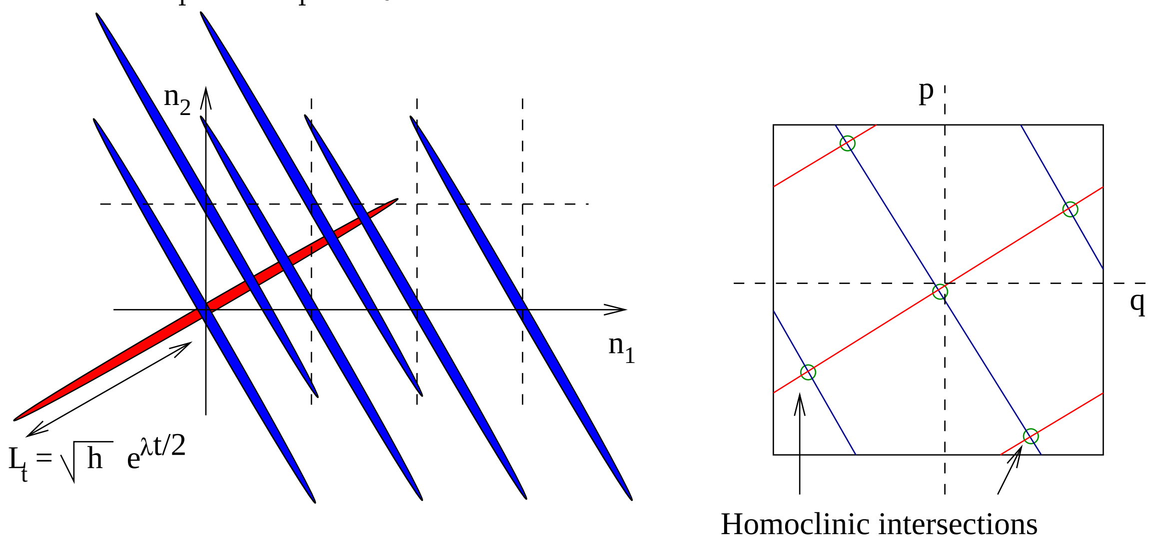
$$\begin{aligned} C(t) &\stackrel{\text{def}}{=} \langle\psi(0)|\psi(t)\rangle_{torus} \\ &= \langle\psi(-t/2)|\psi(t/2)\rangle_{torus} \\ &= \sum_{n\in\mathbb{Z}^2}\langle\hat{T}_n\psi(-t/2)|\psi(t/2)\rangle_{plane} \\ &= \sum_{n\in\mathbb{Z}^2} A_n e^{iS_n/\hbar} \end{aligned}$$

with $A_n, S_n \in \mathbb{R}$. We can calculate $\langle\hat{T}_n\psi(-t/2)|\psi(t/2)\rangle_{plane} = A_n e^{iS_n/\hbar}$ exactly, using metaplectic algebra, see [56],[24]. Let $(q_n, p_n) \in \mathbb{R}^2$ be the coordinates of $n \in \mathbb{Z}^2$ expressed in the frame of stable/unstable directions. To simplify notations we treat a case of symmetric matrix M, for which the stable/unstable directions are orthogonal. For example $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It gives:

$$A_n = \frac{1}{\sqrt{\cosh(\lambda t)}} \exp\left(-\frac{1}{2}\left(\frac{q_n}{L_t}\right)^2 - \frac{1}{2}\left(\frac{p_n}{L_t}\right)^2\right)$$
$$L_t = \hbar^{1/2} e^{\lambda t/2},$$
$$S_n = -\frac{1}{2}\tanh(\lambda t)\, q_n p_n \simeq -\frac{1}{2} q_n p_n, \qquad \text{if } t \gg \frac{1}{2} t_E$$

The function $A_n(q_n, p_n)$ has a circular Gaussian shape, with radius L_t. So the sum $\sum_n$ is convergent, and terms who contribute significantly are those such that $|q_n, p_n| \lesssim L_t$.

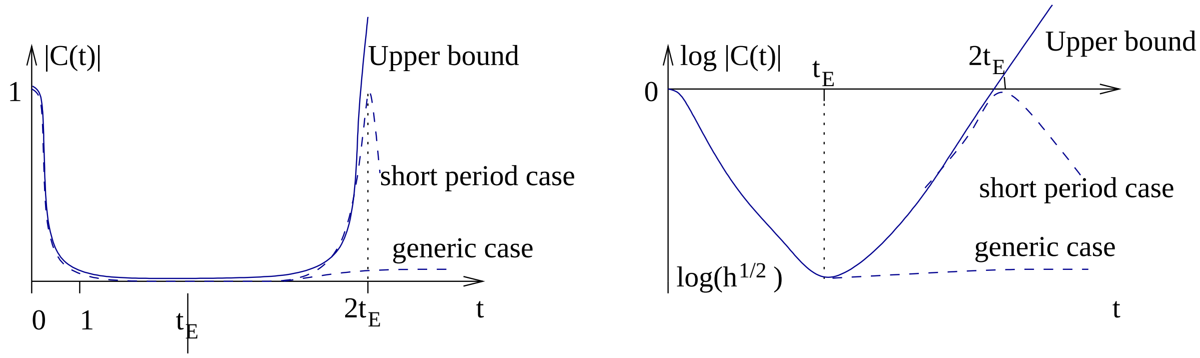
Let us discuss the geometrical meaning of the scalar product $\langle \hat{T}_n \psi(-t/2) | \psi(t/2) \rangle_{plane}$. We have seen that on the plane, the Husimi distribution of $|\psi(t/2)\rangle$ is centered at 0, and spreads along the unstable manifold, with length $L_t \simeq \hbar^{1/2} e^{\lambda t/2}$. Similarly, the state $\hat{T}_n |\psi(-t/2)\rangle$ is centered at point $n \in \mathbb{Z}^2$, and spreads along the stable direction, with length $L_t \simeq \hbar^{1/2} e^{\lambda t/2}$. So the scalar product $\langle \hat{T}_n \psi(-t/2) | \psi(t/2) \rangle_{plane}$ is associated with a **homoclinic intersection,** i.e. an intersection between the unstable and stable manifold issued from the periodic point 0.



- For time $|t| \ll t_E$, the length is $L_t = \sqrt{h} e^{\lambda |t|/2} \ll 1$, so $|\psi(t/2)\rangle$is still localized at $x = 0$, and $\hat{T}_n |\psi(-t/2)\rangle$ is still localized at $x = n$. Only the term $n = 0$ contributes to the sum, giving$|C(t)| = 1/\sqrt{\cosh(\lambda t)} \simeq \sqrt{2} e^{-\lambda t/2}$.

- For time $|t| \gg t_E$, then $L_t \gg 1$. From the figure, it is easy to guess that the states $|\psi(t/2)\rangle$ and $\hat{T}_n |\psi(-t/2)\rangle$ will intersect and have a significant scalar product, if $|n| \lesssim L_t$. There are $\mathcal{N}_t \simeq L_t^2 = \left(\sqrt{h} e^{\lambda t/2}\right)^2 = h e^{\lambda t}$ such terms who contribute to the sum. Each term is bounded by $e^{-\lambda t/2}$. So we can bound $|C(t)|$ from above:

$$|C(t)| < B_t \stackrel{\text{def}}{=} \text{Cste}\, \mathcal{N}_t e^{-\lambda t/2} \simeq h e^{\lambda t/2} \qquad (23)$$



Remarks:

- Note the competition between $\mathcal{N}_t = he^{\lambda t}$ which grows exponentially due to the increasing number of homoclinic orbits, and $e^{-\lambda t/2}$ which decreases slower due to spreading. At time $t = 2t_E$, the bound reaches $B_{2t_E} \simeq 1$. This shows that a revival $|C(t)| = 1$ can not appear before time $t = 2t_E$. After that time, some cancellations are necessary to make the scalar product be less than 1.

- Ehrenfest time t_E is an important qualitative time for the autocorrelation function: after that time $C(t)$ is the sum of many terms, and interference effects begin to appear. Because the same analysis can be made for negative times, it is more correct to say that **interference effects are absent in the autocorrelation function on a time interval of length** $2t_E$.

- Heuristically, for $t \gg t_E$, Let us suppose a **generic case** where the phases $S_n/\hbar$ are "not correlated", and behave as random variables. So the complex numbers $A_n e^{iS_n/\hbar}$ add randomly on the complex plane. $C(t)$ is then a random variable with a Gaussian distribution centered at 0, and $\langle |C(t)| \rangle \simeq \sqrt{\mathcal{N}_t} e^{-\lambda t/2} \simeq \sqrt{h}$, **independent of** t. This gives that the Husimi distribution at point $x = 0$, $Hus_{\psi_t}(x) = \frac{1}{h} |\langle x|\psi(t)\rangle|^2 = \frac{1}{h} |C(t)|^2$ is on average$\langle Hus_{\psi_t}(x = 0)\rangle \simeq 1$. A similar heuristic with other points x, gives similarly that the Husimi distribution is $\langle Hus_{\psi_t}(x)\rangle \simeq 1, \forall x$, **equal to the Liouville measure** (this is what gives mixing for evolution of smooth classical densities). One of **the main challenge in quantum chaos** is to make this last heuristic more rigorous.

- On the opposite, in the **short quantum period case**, $\hat{M}^P \propto \hat{I}d$, with $P \simeq 2t_E$, the phases $\exp(iS_n/\hbar)$ **add constructively** up to time $t = 2t_E$ (in some special cases they are all close to 1). So all the complex numbers A_n add constructively.

- For future purpose, we have showed that for any $0 < \alpha < 2$, then $|C(\alpha t_E)| \to 0$, for $\hbar \to 0$, i.e. $|\psi(0)\rangle_{torus}$ and $|\psi(\alpha t_E)\rangle_{torus}$ become orthogonal to each other.

- In ref. [23], we use Diophantine properties of number to deduce the upper bound of $|C(t)|$. The present explanations, using the splitting $t = t/2 + t/2$, shows that it is not necessary.

2.6.6 Interpretation of the short quantum period as constructive interference effects along periodic orbits

We show now an alternative way to interpret the short quantum period, in terms of **periodic orbits.** This is not in contradiction with the previous paragraph because, it is well known that in hyperbolic systems, there is a "strong rigidity" in the symbolic dynamics and classification of periodic orbits or homoclinic orbits.

A **periodic point of period** $t \in \mathbb{Z}$, is a fixed point x of M^t, i.e a solution of $M^t x = x\,[1] \Leftrightarrow \tilde{\mathcal{T}}_t(x) = n, \quad n \in \mathbb{Z}$, with the map $\tilde{\mathcal{T}}_t(x) = M^t x - x$. The map $\tilde{\mathcal{T}}_t$ is expansive, and the number of periodic point on the torus (i.e. $x \in [0,1]^2$) is $\mathcal{N}_t = \left|\det \tilde{\mathcal{T}}_t\right| = e^{\lambda t} - e^{-\lambda t} - 2$. The **classical action of the periodic trajectory** passing through the periodic point $x_{n,t} = \tilde{\mathcal{T}}_t^{-1}(n)$ is $A_n = \frac{1}{2} M^t(x_{n,t}) \wedge x_{n,t} \in \mathbb{Q}$.

For the linear map on the torus, there is an **exact trace formula** [37], which gives:

$$\mathrm{Tr}\left(\hat{M}^t\right) = \sum_{x_{n,t} \in [0,1]^2} \frac{e^{iA_n/\hbar}}{\sqrt{\left|\det\left(M^t - I\right)\right|}}$$

Upper bound: Similarly with eq.(23), we can express an upper bound of $\mathrm{Tr}\left(\hat{M}^t\right)$from the sum of moduli. For $t > 0$, $\sqrt{\left|\det\left(M^t - I\right)\right|} = e^{\lambda t/2} - e^{-\lambda t/2} = \sqrt{\mathcal{N}_t}$, and

$$\left|\mathrm{Tr}\left(\hat{M}^t\right)\right| \leq \tilde{B}_t \stackrel{\mathrm{def}}{=} \sqrt{\mathcal{N}_t} \simeq e^{\lambda t/2}$$

Of course $\left|\mathrm{Tr}\left(\hat{M}^t\right)\right| \leq \dim\left(\mathcal{H}_N\right) = N$, and The bound $\tilde{B}_t$ reach this maximal value $N = e^{\lambda t_E}$ for $t = 2t_E$, giving the same conclusion as (23). This bound can be effectively reach at time $t \simeq 2t_E$ if the complex phases $\exp\left(i2\pi A_n/h\right)$ add constructively together. This is the case for short quantum periods.

2.7 Construction of strong scars, non Quantum Unique Ergodicity example

We consider the subsequence of $N = 1/h \to \infty$, giving short quantum periods $P \simeq 2t_E$, $\hat{M}^P = e^{i\alpha}\hat{Id}$. Let $\varphi \in [0, 2\pi]$ such that $P\varphi \equiv \alpha\ [2\pi]$. (There are P possible values for φ, one for each eigen-space of $\hat{M}$).

Let $|\psi(t)\rangle = \hat{M}^t |0\rangle_{torus}$ and define (we suppose P even to simplify notations):

$$|\phi\rangle \stackrel{\mathrm{def}}{=} \sum_{t=P/2+1}^{P/2} e^{-i\varphi t} |\psi(t)\rangle \tag{24}$$

Proposition 13. *$|\phi\rangle$ is an eigenstate of $\hat{M}$:*

$$\hat{M}|\phi\rangle = e^{i\varphi}|\phi\rangle$$

Proof. Indeed,

$$\begin{aligned}
\hat{M}|\phi\rangle &= \sum_{t=-P/2+1}^{P/2} e^{-i\varphi t}|\psi(t+1)\rangle = e^{i\varphi}\left(e^{-i\varphi P/2-i\varphi}|\psi(P/2+1)\rangle + \sum_{t=-P/2+2}^{P/2} e^{-i\varphi t}|\psi(t)\rangle\right)\\
&= e^{i\varphi}\left(e^{-i\varphi(P/2-P+1)}|\psi(-P/2+1)\rangle + \sum_{t=-P/2+2}^{P/2} e^{-i\varphi t}|\psi(t)\rangle\right) = e^{i\varphi}|\phi\rangle
\end{aligned}$$

Because $|\psi(P/2+1)\rangle = \hat{M}^P|\psi(-P/2+1)\rangle = e^{iP\varphi}|\psi(-P/2+1)\rangle$.

As explained in (11), half of the terms $|\psi(t)\rangle$ in this sum are localized, whereas half of the terms are equidistributed. This gives ☐

Theorem 14. *[23]$|\phi\rangle$ is a sequence of eigenstate of $\hat{M}$ with* ***non uniform semi-classical measure:***

$$\mu = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\mu_{Lebesgue}$$

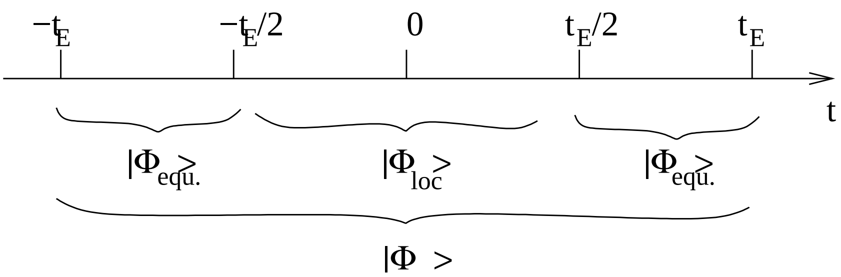
The partial sum (half part)

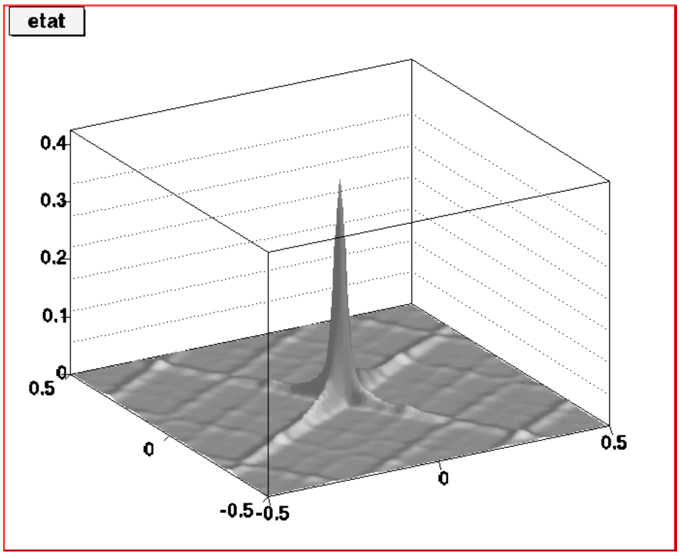
$$|\phi_{loc}\rangle \stackrel{\text{def}}{=} \sum_{t=-t_E/2}^{t_E/2} e^{-i\varphi t}|\psi(t)\rangle$$

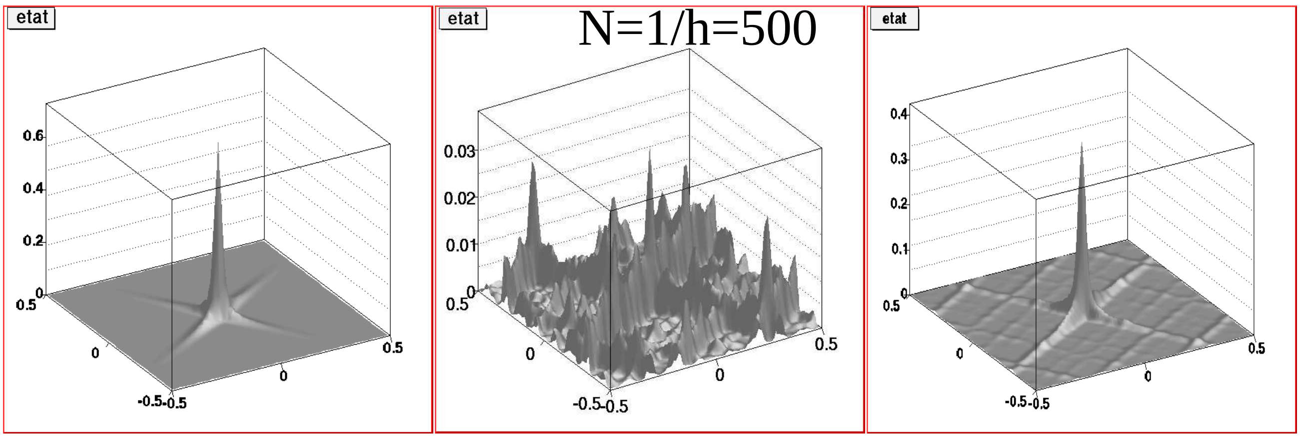
is ***localized*** *($\mu = \delta_0$), whereas*

$$|\phi_{equid.}\rangle = |\phi\rangle - |\phi_{loc}\rangle$$

is ***equidistributed*** *($\mu = \mu_{Lebesgue}$).*







$|\phi_{loc.}>$ $\qquad$ $|\phi_{equi.}>$ $\qquad$ $|\phi \ \ >$

Proof. (idea of the proof.)

We decompose the sum (24) in 4 similar terms (see figure):

$$|\phi_1\rangle = \sum_{t=-t_E}^{-t_E/2} e^{-i\varphi t}|\psi(t)\rangle, \qquad |\phi_2\rangle = \sum_{t=-t_E/2+1}^{0} e^{-i\varphi t}|\psi(t)\rangle,$$

$$|\phi_3\rangle = \sum_{t=1}^{t_E/2} e^{-i\varphi t}|\psi(t)\rangle, \qquad |\phi_4\rangle = \sum_{t=t_E/2+1}^{t_E} e^{-i\varphi t}|\psi(t)\rangle,$$

$$|\phi_{loc}\rangle = |\phi_2\rangle + |\phi_3\rangle, \qquad |\phi_{erg}\rangle = |\phi_1\rangle + |\phi_4\rangle.$$

In the study of $C(t)$, we have showed that for any $0 < \alpha < 2$, then $|C(\alpha t_E)| \to 0$, for $\hbar \to 0$, i.e. $|\psi(0)\rangle_{torus}$ and $|\psi(\alpha t_E)\rangle_{torus}$ become orthogonal to each other. Equivalently, for any $\alpha, \beta \in]-1, 1[$, $\alpha \neq \beta$, then $\langle \psi(\alpha t_E) | \psi(\beta t_E) \rangle \to 0$, for $\hbar \to 0$. It means that each term of the sum (24) is (quasi-) orthogonal to the others. This gives: $\exists C > 0$, $\langle \phi_{loc} | \phi_{loc} \rangle \simeq C.t_E$, $\langle \phi_{erg} | \phi_{erg} \rangle \simeq C.t_E$, $\langle \phi_i | \phi_j \rangle \simeq C(t_E/2)\,\delta_{ij}$, for $\hbar \to 0$.

Remark that the number of terms in each sum is $\simeq t_E$, and goes to ∞, for $\hbar \to 0$. The result we obtain are not affected if we add or substract a finite number of terms.

First $|\phi_2\rangle, |\phi_3\rangle, |\phi_{loc}\rangle$ are localized, because each term $|\psi(t)\rangle$ in their sum is localized, from to (11).

One can write

$$|\phi_1\rangle = \hat{M}^{-t_E} |\phi_3\rangle, \qquad |\phi_4\rangle = \hat{M}^{t_E} |\phi_2\rangle.$$

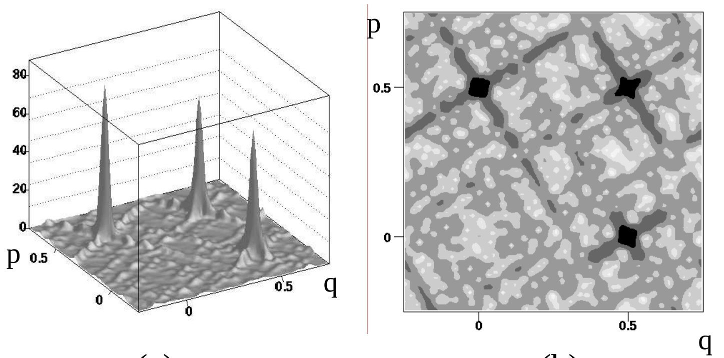
From (12), we deduce that $|\phi_1\rangle, |\phi_4\rangle$, are equidistributed.

To show that $|\phi_{equid.}\rangle = |\phi_1\rangle + |\phi_4\rangle$ is also equidistributed, one has to show that the cross term ${}_n\langle \phi_1 | \hat{f} | \phi_4 \rangle_n$ vanishes on any observable. As in the proof of (12), we take a Fourier function $\hat{f}_n = \hat{T}_{n/N}$, and write ${}_n\langle \phi_1 | \hat{f}_n | \phi_4 \rangle_n = {}_n\langle \phi_2 | \hat{M}^{t_E/2} \hat{f}_n \hat{M}^{-t_E/2} | \phi_5 \rangle_n = {}_n\langle \phi_2 | \hat{T}_{M^{t_E/2} n/N} | \phi_5 \rangle_n$, using (11), and where $|\phi_5\rangle = \hat{M}^{t_E} |\phi_3\rangle$ is equidistributed from (12). We get the scalar product between $|\phi_5\rangle_n$ which is equidistributed and $\hat{T}_{-M^{t_E/2} n/N} |\phi_2\rangle_n$ which is localized; so the scalar product vanishes for $\hbar \to 0$.

To show finally that $|\phi\rangle = |\phi_{loc}\rangle + |\phi_{equid.}\rangle$ has semi-classical measure $\frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\mu_{Lebesgue}$, it remains to show that the cross term ${}_n\langle \phi_{equid.} | \hat{f}_n | \phi_{loc} \rangle_n = {}_n\langle \phi_{equid.} | \hat{T}_{n/N} | \phi_{loc} \rangle_n$ vanishes. This is again a scalar product between $\hat{T}_{n/N} | \phi_{loc} \rangle_n$ which is localized and $|\phi_{equid.}\rangle$ which equidistributes. $\square$

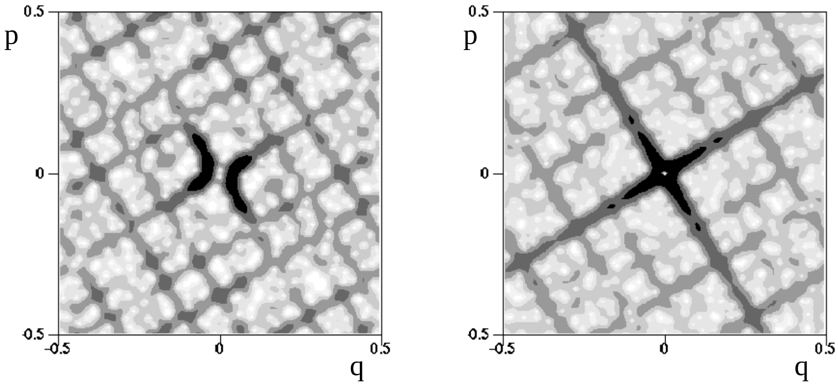
Remarks:

- Similar scarred eigenstates can be constructed on any periodic orbit. Example : on a period 3 orbit :



(a) (b)

  – For any given periodic orbit, a set of mutually orthogonal **"excited scarred eigenstates"**, can be constructed [23] (they have the same semi-classical measure) :



- If you don't suppose that N corresponds to a short quantum period, you can still define $|\phi\rangle$, but it is no more an eigenstate: it is a quasi-mode with error $\varepsilon_h = C\sqrt{1/t_E}$ [23]:
$$\left\| \left(\hat{M} - e^{i\phi} \right) |\phi\rangle_{normalized} \right\| \leq \varepsilon_h$$

- The following theorem shows that an invariant semi-classical measure can not be a pure Dirac measure, and that the factor 1/2 is optimal.

Theorem 15. *[8][22] If $|\psi\rangle_h$ is a sequence of eigenstates with semi-classical measure μ such that*
$$\mu = \beta \delta_{(0,0)} + (1-\beta)\,\nu$$
with $\nu(0) = 0$, then $\beta \leq 1/2$.

Remark: the bound was $\beta \leq 0.6$ in [8] and has been improved to 1/2 in [22].

Proof. (idea of the proof) Decompose
$$|\psi\rangle_h = \underbrace{|\psi_0\rangle_h}_{loc:\beta} + \underbrace{|\psi_\nu\rangle_h}_{equidis:1-\beta}$$

with $|\psi_0\rangle$ localized, i.e. $\mu_{\psi_0} = \beta\delta_0$ and orthogonal to $|\psi_\nu\rangle$: $\langle\psi_0|\psi_\nu\rangle \to_{h\to 0} 0$.
Then
$$|\psi\rangle_h \propto \hat{M}^{t_E}|\psi_h\rangle = \underbrace{\hat{M}^{t_E}|\psi_0\rangle}_{equidis.:\beta} + \hat{M}^{t_E}|\psi_\nu\rangle$$

From th.(12), $\hat{M}^{t_E}|\psi_0\rangle$ is equidistributed and has total "probability β". If we compare with the initial expression of $|\psi\rangle_h$, we deduce that $\hat{M}^{t_E}|\psi_0\rangle$ must "be a part" of $|\psi_\nu\rangle_h$, i.e. $\beta \leq \left\| |\psi_\nu\rangle_h \right\|^2 \simeq 1-\beta$. Then $\beta \leq 1/2$. $\square$

2.8 Conclusion. Suggestions to generalize these results

In this section, we have considered only *linear hyperbolic map* on the Torus. There is an important property in dynamical system theory that hyperbolic map are "structurally stable", i.e. there hyperbolicity properties (and therefore mixing) are kept under any small enough enough perturbation. So a linear hyperbolic map, can be seen as a very particular dynamical system in a (infinite dimensional) open set of *non linear hyperbolic maps* on the torus.

We have seen that for *a linear hyperbolic map* on the torus, there exists some particular sequences of eigenstates which have a non uniform semi-classical measure: they are strongly "scarred" on a periodic orbit (1/2 the Dirac measure), and this gives a **counter example to Quantum Unique Ergodicity (Q.U.E)** for hyperbolic systems. A natural question is:

Question: "Is this example of scarred eigenstates unique or not, in the set of non linear hyperbolic map on the torus? are there other examples of non Q.U.E. for non linear maps? or Q.U.E. is true everywhere else?"

To my knowledge, there is no answer to that question at that moment. The example we gave relies on the existence of a "short quantum period". This phenomenon of short quantum period is very particular to the linear map, and is due to arithmetical properties of $M \in SL(2, \mathbb{Z})$. From that point of view, it seems difficult to have a direct generalisation to the non linear case.

But the construction of these "scarred eigenstates" relies on a less stronger condition: we essentially need that starting from a coherent state on periodic orbit $|\psi(0)\rangle = |0\rangle$, then $|\psi(t)\rangle$ returns to this original state at time $2t_E$. In section (2.6.6) we have interpreted this revival in terms of constructive interference effects among homoclinic orbits. We have shown from a general upper bound of the auto-correlation function that **this revival can not appear before time $2t_E$**.

1. This remark shows that a first task would be to **"describe the semi-classical evolution of quantum states for time larger than the Ehrenfest time t_E"** in the non linear case. Long time such as $t \simeq C.t_E$, for any $C > 0$, would be enough for that purpose.

2. Theorem (15) does not require to control the semi-classical evolution for such long times. As its proof shows, time evolution up to time t_E only is needed. A generalization of this theorem in the non linear case would not permit to show Q.U.E nor the existence of scarred states, but would constrain the invariant semi-classical measure (not to be a Dirac measure).

As explained in section 1.3, there are some recent results concerning the second direction, for non linear map, or non linear hyperbolic flows [9][3][20]. In [21], one can find some recent results concerning the first direction .

In [3], Nalini Anantharaman control time evolution along individual orbits (with insertions of quasi-projectors) up to times $C \log(1/\hbar)$, with any $C > 0$.

Références

[1] A. Bouzouina and S. DeBièvre. Equipartition of the eigenfunctions of quantized ergodic maps on the torus. *Commun. Math. Phys.* , 178(1) :83–105, 1996.

[2] B. Hellfer an A. Martinez and D. Robert. Ergodicité et limite semi-classique. *Comm. Math. Phys.*, 109 :313–326, 1987.

[3] Nalini Anantharaman. The eigenfuncions of the laplacian do not concentrate on sets of small topological entropy. *preprint*, 2004.

[4] Dario Bambusi, Sandro Graffi, and Thierry Paul. Long time semiclassical approximation of quantum flows : A proof of the Ehrenfest time. *Asymptotic Anal.*, 21(2) :149–160, 1999.

[5] N. Bergeron. Spectre des surfaces hyperboliques. *lecture notes for IHP trimester "Time at work", http ://www.math.jussieu.fr/ baladi/ihp.html*, 2005.

[6] O. Bohigas. *Random Matrix Theories and Chaotic Dynamics*, volume 45. 1991.

[7] Francesco Bonechi and Stephan DeBièvre. Exponential mixing and ln(h) timescales in quantized hyperbolic maps on the torus. *Comm. Math. Phys.*, 211 :659–686, 2000.

[8] Francesco Bonechi and Stephan DeBièvre. Controlling strong scarring for quantized ergodic toral automorphisms. *Duke Math. J.*, 117(3) :571–587, 2003.

[9] J.M. Bouclet and S. De Bièvre. Long time propagation and control on scarring for perturbated quantized hyperbolic toral automorphisms. *preprint*, 117, 2004.

[10] A. Bouzouina and D. Robert. Uniform semiclassical estimates for the propagation of quantum observables. *Duke Math. J.*, 111(2) :223–252, 2002.

[11] Michael Brin and Garrett Stuck. *Introduction to Dynamical Systems.* Cambridge University Press, 2002.

[12] Yves Colin de Verdière. Ergodicité et fonctions propres du laplacien. (Ergodicity and eigenfunctions of the Laplacian). *Commun. Math. Phys.*, 102 :497–502, 1985.

[13] M. Combescure and D. Robert. Semiclassical spreading of quantum wave packets and applications near unstable fixed points of the classical flow. *Asymptotic Anal.*, 14(4) :377–404, 1997.

[14] Yves Colin de Verdière. *Méthodes semi-classiques et théorie spectrale. Book in progress. http ://www-fourier.ujf-grenoble.fr/ ycolver/.* 2002.

[15] S. DeBièvre. Quantum chaos : a brief first visit. *Contemporary mathematics*, 2000.

[16] S. DeBièvre. Recent results on quantum map eigenstates. *Proceedings of Workshop QMath 9*, 2004.

[17] S. DeBièvre and M. Degli Esposti. Egorov theorems and equidistribution of eigenfunctions for the quantized sawtooth and baker maps. *Ann. Inst. H. Poincaré*, 69 :1–30, 1998.

[18] David DeLatte. Nonstationnary normal forms and cocycle invariants. *Random and Computational dynamics*, 1 :229–259, 1992.

[19] B. Eckhardt, S. Fishman, J. Keating, O. Agam, J. Main, and K. Müller. Approach to ergodicity in quantum wave functions. *Phys. Rev. E*, 52, 1995.

[20] Mirko Degli Esposti, Stéphane Nonnenmacher, and Brian Winn. Quantum Variance and Ergodicity for the baker's map. *preprint math-ph/0412058*, 2004.

[21] Frédéric Faure. Semiclassical formula beyond the ehrenfest time in quantum chaos. *preprint*, 2005.

[22] Frédéric Faure and Stéphane Nonnenmacher. On the maximal scarring for quantum cat map eigenstates. *Communications in Mathematical Physics*, 245, 2004.

[23] Frédéric Faure, Stéphane Nonnenmacher, and Stephan De Bièvre. Scarred eigenstates for quantum cat maps of minimal periods. *Communications in Mathematical Physics*, 239, 2003.

[24] G. Folland. *Harmonic Analysis in phase space*. Princeton University Press, 1988.

[25] E. Ghys. Variations autour du théorème de récurrence de poincaré. *Le journal de maths des élèves*, 1 :3–11, 1994.

[26] Victor Guillemin. Wave-trace invariants. *Duke Math. J.*, 83(2) :287–352, 1996.

[27] M. Gutzwiller. *Chaos in classical and quantum mechanics*. Springer-Verlag, 1991.

[28] G.A. Hagedorn and A. Joye. Exponentially acurrate semiclassical dynamics : Propagation, localization, ehrenfest times, scattering, and more general states. *Ann. Henri Poincaré*, 1 :837–883, 2000.

[29] J.H. Hannay and M. Berry. Quantization of linear maps-fresnel diffraction by a periodic grating. *Physica D*, 1 :267–290, 1980.

[30] B. Hasselblatt. Hyperbolic dynamics. *Handbook of Dynamical Systems, North Holland*, 1A :239–320, 2002.

[31] E.J. Heller. Time dependant approach to semiclassical dynamics. *J. Chem. Phys.*, 62, 1975.

[32] E.J. Heller. Bound state eigenfunctions of classically chaotic hamiltonian systems : scars of periodic orbits. *Phys. Rev. Lett.*, 53, 1984.

[33] K. Hepp. The classical limit of quantum mechanical correlation funtions. *Comm. Math. Phys.*, 35, 1974.

[34] S. Hurder and A. Katok. Differentiability, rigidity and Godbillon-Vey classes for Anosov flows. *Publ. Math., Inst. Hautes Étud. Sci.*, 72, 1990.

[35] Anatole Katok and Boris Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[36] J.P. Keating. Asymptotic properties of the periodic orbits of the cat maps. *Nonlinearity*, 4 :277–307, 1991.

[37] J.P. Keating. The cat maps : quantum mechanics and classical motion. *Nonlinearity*, 4 :309–341, 1991.

[38] P. Kurlberg and Z. Rudnick. Hecke theory and equidistribution for the quantization of linear maps of the torus. *Duke Math. J.*, 103 :47–77, 2000.

[39] P. Kurlberg and Z. Rudnick. On quantum ergodicity for linear maps of the torus. *Comm. Math. Phys.*, 222 :201–227, 2001.

[40] P. Kurlberg and Z. Rudnick. On the distribution of matrix elements for the quantum cat map. *Annals of Maths*, 161 :489–507, 2005.

[41] E. Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *preprint*, pages 1–54, 2003.

[42] R.G. Littlejohn. "the semiclassical evolution of wave-packets". *Phys. Rep.*, 138, 1986.

[43] W. Luo and P. Sarnak. Quantum variance for hecke eigenforms. *Ann. Sci. Ecole Norm. Sup.*, 37 :769–799, 2004.

[44] J. Marklof and S. O'Keefe. Weyl's law and quantum ergodicity for maps with divided phase space. *Nonlinearity*, 18 :277–304, 2005.

[45] A. Martinez. *An Introduction to Semiclassical and Microlocal Analysis.* 2001.

[46] Stéphane Nonnenmacher. On the rate of quantum ergodicity for cat maps. *unpublished notes*, 2004.

[47] Perelomov, A. . *Generalized coherent states and their applications.* Springer-Verlag., 1986.

[48] M. Pollicott and M Yuri. *Dynamical Systems and Ergodic theory.* Cambridge University Press, 1998.

[49] Z. Rudnick. On quantum unique ergodicity for linear maps of the torus. pages 1–9.

[50] Z. Rudnick and P. Sarnak. The behaviour eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161 :195–231, 1994.

[51] A. Schnirelman. Ergodic properties of eigenfunctions. *Usp. Math. Nauk.*, 29 :181–182, 1974.

[52] Roman Schubert. Semi-classical behaviour of expectation values in time evolved lagrangian states for large times. *Commun. Math. Phys.*, 256 :239–254, 2005.

[53] Johannes Sjöstrand and Maciej Zworski. Quantum monodromy and semiclassical trace formulae. *J. Math. Pures Appl., IX. Sér. 81. [ISSN 0021-7824] http ://dx.doi.org/10.1016/S0021-7824(01)01230-2*, 1 :1–33, 2002.

[54] S. Tomsovic and E.J. Heller. Long-time semi-classical dynamics of chaos : the stadium billard. *Physical Review E*, 47, 1993.

[55] J.H. Van Vleck. *Proc. Natl. Acad. Sci. U.S.A.*, 14, 1928.

[56] W. M. Zhang, D. H. Feng, and R. Gilmore. Coherent states : theory and some applications. *Rev. Mod. Phys.*, 62, 1990.

[57] N.M.J. Woodhouse. *Geometric quantization.* Clarendon Press, Oxford, 1992.

[58] S. Zelditch. Uniform distribution of the eigenfunctions on compact hyperbolic surfaces. *Duke Math. J*, 55 :919–941, 1987.

[59] S. Zelditch. Quantum dynamics from the semi-classical viewpoint. *Lectures at I.H.P. http ://mathnt.mat.jhu.edu/zelditch*, 1996.

[60] S. Zelditch. Wave invariants at elliptic closed geodesics. *Geom. Funct. Anal.*, 7(1) :145–213, 1997.

[61] S. Zelditch. Wave invariants for non-degenerate closed geodesics. *Geom. Funct. Anal.*, 8(1) :179–217, 1998.

[62] S. Zelditch. Quantum ergodicity and mixing of eigenfunctions. *Elsevier Encyclopedia of Math. Phys.*, 2005.

[63] Steven Zelditch. On the rate of quantum ergodicity. I : Upper bounds. *Commun. Math. Phys.*, 160(1) :81–92, 1994.