Le pendule inversé oscillant

(1)
$$\begin{cases} x = l \sin(\theta) \\ z = l \cos(\theta) + z_A(t) \end{cases}$$

$$\longrightarrow \begin{cases} \dot{x} = l \cos(\theta)\ \dot{\theta} \\ \dot{z} = -l \sin\theta\ \dot{\theta} + \dot{z}_A \end{cases}$$

$$U = mgz = mg\left(l\cos(\theta) + z_A(t)\right)$$

$$E_c = \frac{1}{2} m \left(\dot{x}^2 + \dot{z}^2\right) = \frac{1}{2} m \Big(l^2 \cos^2\theta\ \dot{\theta}^2 + l^2 \sin^2\theta\ \dot{\theta}^2 + \dot{z}_A^2 - 2 \dot{z}_A l \sin\theta\ \dot{\theta} \Big)$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m \dot{z}_A^2 - m \dot{z}_A l \sin\theta\ \dot{\theta}$$

$$L(\theta, V_\theta, t) = E_c - U = \frac{1}{2} m l^2 V_\theta^2 - m \dot{z}_A l \sin\theta\ V_\theta + \frac{1}{2} m \dot{z}_A^2 - mgl\cos\theta - mg\, z_A(t)$$

(2)
$$p_\theta = \frac{\partial L}{\partial V_\theta} = m l^2 V_\theta - m \dot{z}_A l \sin\theta$$

$$\longrightarrow \quad V_\theta = \frac{1}{ml^2}\left(p_\theta + m \dot{z}_A l \sin\theta \right)$$

$$H(\theta, p_\theta, t) = p_\theta \cdot V_\theta - L = \frac{1}{ml^2} p_\theta^2 + \frac{\dot{z}_A}{l} \sin\theta\ p_\theta - \frac{1}{2ml^2}\left(p_\theta + m \dot{z}_A l \sin\theta\right)^2$$
$$+ m \dot{z}_A l \sin\theta \frac{1}{ml^2}\left(p_\theta + m \dot{z}_A l \sin\theta\right) - \frac{1}{2} m \dot{z}_A^2$$
$$+ mgl\cos\theta + mg\, z_A(t)$$

$$= \frac{1}{2ml^2} p_\theta^2 + \cancel{\frac{\dot{z}_A}{l} \sin\theta\, p_\theta} - \cancel{\frac{\dot{z}_A}{l} \sin\theta\, p_\theta} - \frac{1}{2} m \dot{z}_A^2 \sin^2\theta$$

$$+ \frac{\dot{z}_A}{l} \sin\theta\, p_\theta + m \dot{z}_A^2 \sin^2\theta + mgl\cos\theta$$

$$- \frac{1}{2} m \dot{z}_A^2 + mg z_A$$

$$= \frac{p^2}{2ml^2} + \frac{\dot{z}_A}{l} \sin\theta\, p + \frac{1}{2} m (\dot{z}_A)^2 \sin^2\theta$$

$$+ mgl\cos\theta \qquad + f(t)$$

$\to$ fonction indépendante de θ et p

(3)

$$\begin{cases} \dfrac{d\theta}{dt} = \dfrac{\partial H}{\partial p} = \dfrac{p}{ml^2} + \dfrac{\dot{z}}{l} \sin\theta \\[2ex] \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial \theta} = -\dfrac{\dot{z}}{l} p\cos\theta - m(\dot{z})^2 \sin(\theta)\cos(\theta) + mgl\sin \end{cases}$$

(4) au premier ordre : $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$

$$\begin{cases} \dot{\theta} = \dfrac{p}{ml^2} + \dfrac{\dot{z}}{l}\theta \\[2ex] \dot{p} = -\dfrac{\dot{z}}{l} p - m(\dot{z})^2 \theta + mgl\theta \end{cases}$$

$\Leftrightarrow$ $\dot{X} = A.X$ avec $A(t) = \begin{pmatrix} \dfrac{\dot{z}}{l} & \dfrac{1}{ml^2} \\ mgl - m(\dot{z})^2 & -\dfrac{\dot{z}}{l} \end{pmatrix}$

$X(t) = \begin{pmatrix} \theta(t) \\ p(t) \end{pmatrix}$

Le théorème de Liouville s'écrit :

$$\mathrm{div}(v) = 0 \quad \text{où} \quad v = \begin{cases} v_\theta = \dot{\theta} \\ v_p = \dot{p} \end{cases} \quad \text{est le}$$

champ de vecteur dans l'espace de phase.

$$\mathrm{div}(v) = \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_p}{\partial p} = \frac{\partial \dot{\theta}}{\partial \theta} + \frac{\partial \dot{p}}{\partial p}$$

ici $\begin{cases} \dot{\theta} = A_{11}\,\theta + A_{12}\,p \\ \dot{p} = A_{21}\,\theta + A_{22}\,p \end{cases}$ système linéaire

donc $0 = \mathrm{div}(v) = A_{11} + A_{22} = \mathrm{Trace}(A)$

Pour un système linéaire $\dot{X} = AX$, le théorème de Liouville s'exprime donc par

$$\mathrm{Tr}(A) = 0 \quad , \quad \text{vérifié ici.}$$

(5) $\tilde{\Sigma}$, $\tilde{p}$ et $\tilde{t}$ sont sans dimension. Remarquer que θ en radian est sans dimension physique

on obtient :

$$\frac{d\theta}{dt} = \frac{g^{1/2}\,d\theta}{\ell^{1/2}\,d\tilde{t}} = \frac{\tilde{p}\; m\, g^{1/2}\,\ell^{3/2}}{m\,\ell^2} + \frac{g^{1/2}\,\ell\, d\tilde{\Sigma}}{\ell^{1/2}\,\ell\, d\tilde{t}}\,\theta$$

$$\Longleftrightarrow \quad \frac{d\theta}{d\tilde{t}} = \tilde{p} + \left(\frac{d\tilde{\Sigma}}{d\tilde{t}}\right)\cdot\theta$$

et
$$\frac{dp}{dt} = \frac{d\tilde{p}}{d\tilde{t}}\cdot\frac{m\,g^{1/2}\,\ell^{3/2}\,g^{1/2}}{\ell^{1/2}} = -\frac{1}{\ell}\frac{\ell\, d\tilde{\Sigma}}{d\tilde{t}}\frac{g^{1/2}}{\ell^{1/2}}\,\tilde{p}\; m\, g^{1/2}\,\ell^{3/2}$$

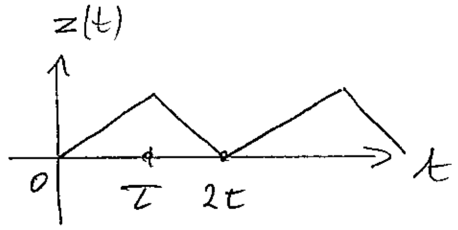
$$- \, m\left(\frac{d\tilde{\Sigma}}{d\tilde{t}}\right)^2 \frac{\ell^2\, g\,\theta}{\ell} + m\,g\,\ell\,\theta$$

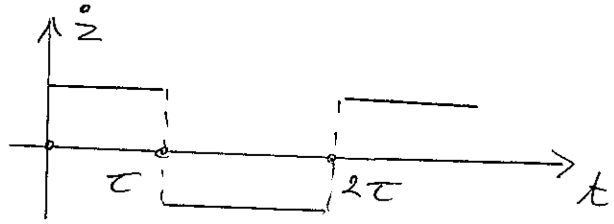
$$\Longleftrightarrow \quad \frac{d\tilde{p}}{d\tilde{t}} = -\left(\frac{d\tilde{\Sigma}}{d\tilde{t}}\right)\tilde{p} + \left(1 - \left(\frac{d\tilde{\Sigma}}{d\tilde{t}}\right)^2\right)\theta$$

soit $\quad \dot{X} = A(t)\cdot X \quad$ avec $\quad X = \begin{pmatrix}\theta \\ \rho\end{pmatrix}$ (on oublie les tildes)

et avec $\quad A(t) = \begin{pmatrix} \dot{z} & 1 \\ 1-\dot{z}^2 & -\dot{z} \end{pmatrix}$

(6)





pour $\quad t \in I_1 = [0\,;\tau]$ on

a $\quad A = \begin{pmatrix} V & 1 \\ 1-V^2 & -V \end{pmatrix} = P\,D\,P^{-1}$

avec $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad P = \begin{pmatrix} 2V+2 & 2V-2 \\ 2(1-V^2) & 2(1-V^2) \end{pmatrix}$

en divisant la 1ère colonne par $2(V+1)$ et la 2ème par $2(V-1)$
on peut choisir : $\quad P = \begin{pmatrix} 1 & 1 \\ 1-V & -(1+V) \end{pmatrix}$

alors $\quad P^{-1} = \frac{1}{(-2)}\begin{pmatrix} -1-V & -1 \\ -1+V & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+V & 1 \\ 1-V & -1 \end{pmatrix}$

$e^{\log 2\cdot D} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad M_1 = P\,e^{\log 2\cdot D}\,P^{-1} = \frac{1}{4}\begin{pmatrix} 3V+5 & 3 \\ -3V^2+3 & -3V+5 \end{pmatrix}$

de même, pour $t \in I_2 = [\tau; 2\tau]$, on change $V \to (-V)$, donnant

$$A = \begin{pmatrix} -V & 1 \\ 1-V^2 & V \end{pmatrix},$$

$$M_2 = \frac{1}{4}\begin{pmatrix} -3V+5 & 3 \\ -3V^2+3 & 3V+5 \end{pmatrix}$$

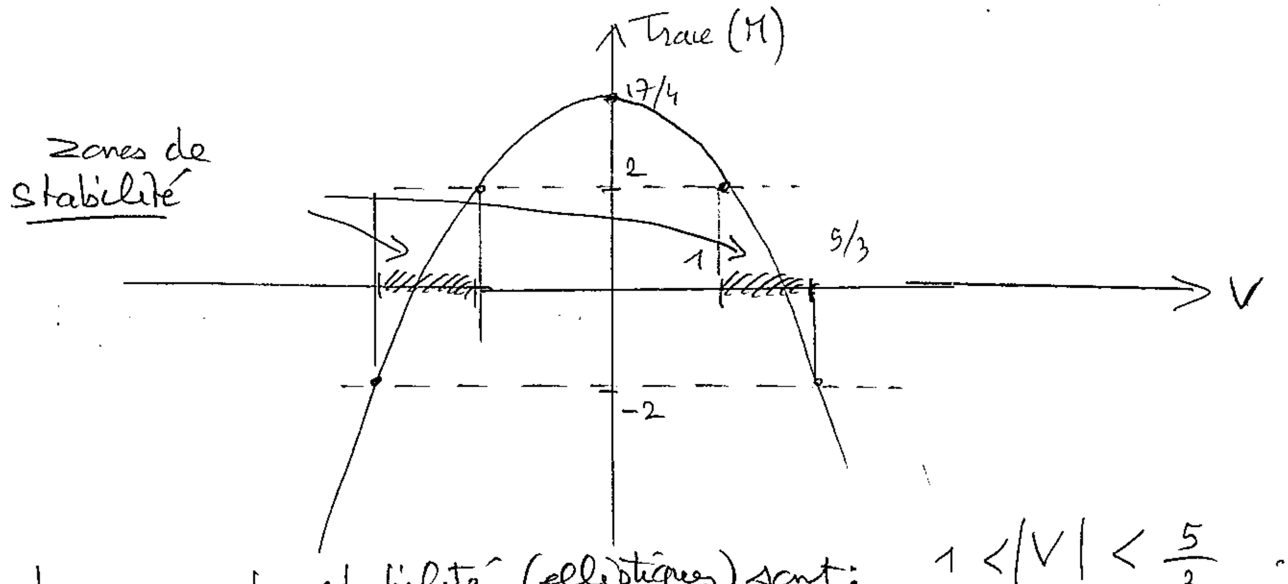
Alors

$$M = M_2\, M_1 = \frac{1}{8}\begin{pmatrix} -9V^2+17 & -9V+15 \\ 9V^3-15V^2+9V+15 & -9V^2+17 \end{pmatrix}$$

7 On a $\mathrm{Trace}(M) = \frac{1}{4}\left(-9V^2+17\right)$

donc $\mathrm{Trace}(M) > 2 \iff -9V^2+17 > 8$
$\iff V^2 < 9 \iff -1 < V < 1$

et $\mathrm{Trace}(M) < -2 \iff -9V^2+17 < -8$
$\iff V^2 > \frac{25}{9} \iff |V| > \frac{5}{3} \approx 1.66$



les zones de stabilité (elliptiques) sont: $1 < |V| < \frac{5}{3}$.