$\boxed{\text{Exercice 4}}$

$$F(x) = \underbrace{-kx}_{\substack{\text{force conservation} \\ \text{(ressort)}}} - \gamma \frac{dx}{dt}$$

force de frottement (non conservative)

. équations du mouvement :

$$\begin{cases} \dot{x} = \frac{p}{m} \\ \dot{p} = -kx - \gamma \dot{x} = -kx - \frac{\gamma}{m} p \end{cases} = \underbrace{\begin{pmatrix} 0 & \frac{1}{m} \\ -k & -\frac{\gamma}{m} \end{pmatrix}}_{A} \begin{pmatrix} x \\ p \end{pmatrix}$$

$\Longleftrightarrow \quad \dot{E} = A.E \quad$ avec $E = \begin{pmatrix} x \\ p \end{pmatrix}$

. On diagonalise: $A = P.D.P^{-1}$ avec $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$.

on pose $X = P^{-1}.E = (X_1, X_2)$: changement de variables $(x, p) \longrightarrow (X_1, X_2)$

alors $\dot{E} = A E = P D P^{-1} E$

$\Longleftrightarrow \quad \dot{X} = D X \quad \Longleftrightarrow \quad \begin{cases} \dot{X}_1 = d_1 X_1 \\ \dot{X}_2 = d_2 X_2 \end{cases}$

$$(*) \quad \Longleftrightarrow \begin{cases} X_1(t) = X_1(0)\, e^{d_1 t} \\ X_2(t) = X_2(0)\, e^{d_2 t} \end{cases}$$

avec $d_1 = -\frac{\gamma}{2m} + \frac{1}{2}\sqrt{\Delta}$, $\quad d_2 = -\frac{\gamma}{2m} - \frac{1}{2}\sqrt{\Delta}$

$$\Delta = \frac{1}{m}\left(\frac{\gamma^2}{m} - 4k\right)$$

$$P = \begin{pmatrix} \sqrt{\Delta} + \frac{\gamma}{m} & -\sqrt{\Delta} + \frac{\gamma}{m} \\ -2k & -2k \end{pmatrix}$$

$$\qquad \underbrace{\qquad\quad}_{V_1} \qquad \underbrace{\qquad\quad}_{V_2}$$

• <u>1er cas</u> : <u>si $\Delta \geq 0$</u> alors d_1, d_2, P réels.

(il faut $\gamma^2 \geq 4km$)

on a $0 \leq \sqrt{\Delta} \leq \frac{\gamma}{m}$ donc $-\frac{\gamma}{m} \leq d_2 \leq d_1 \leq 0$

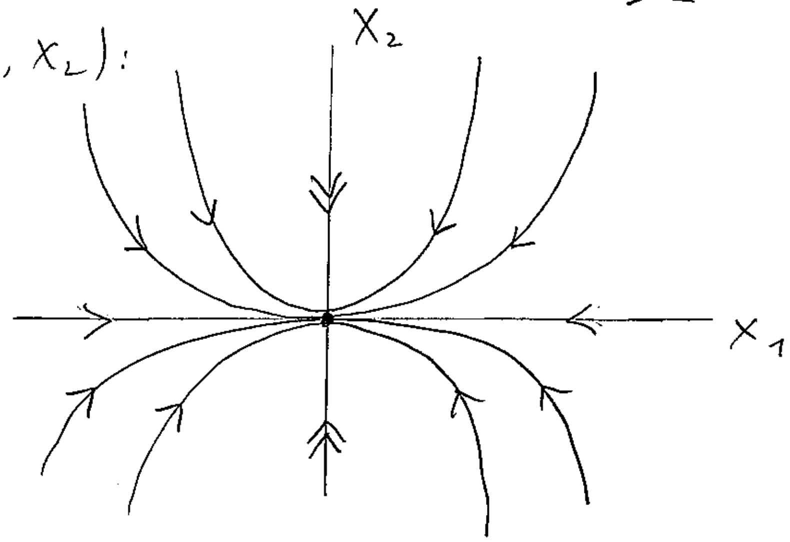
d'après (*), $X_1(t), X_2(t) \longrightarrow 0$ exponentiellemt vite

On peut éliminer t dans (*) :

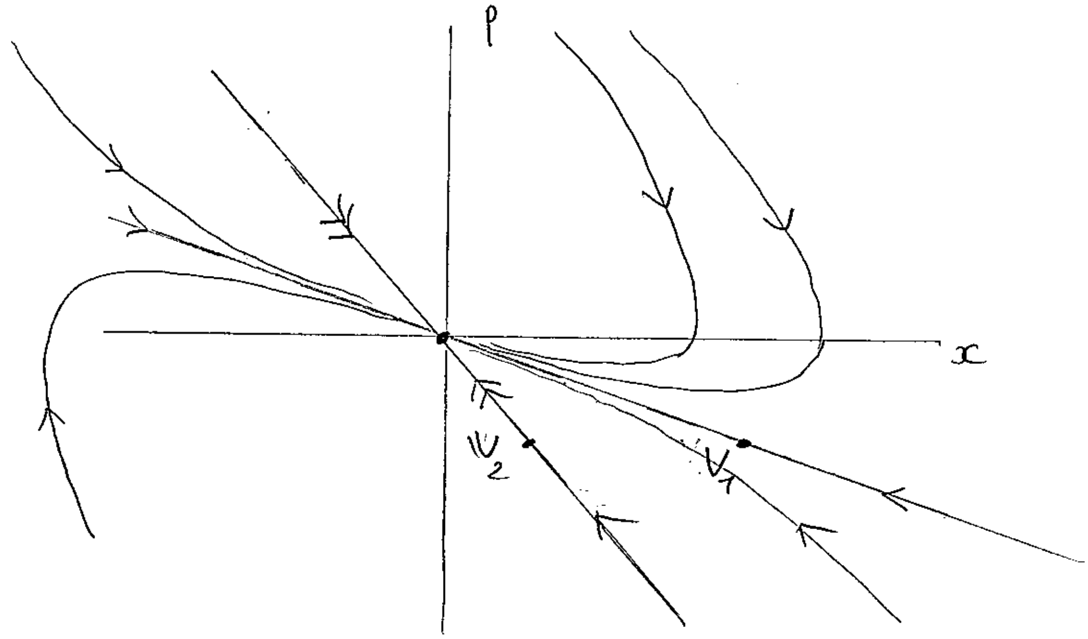
$$t = \frac{1}{d_1} \log \frac{X_1(t)}{X_1(0)} = \frac{1}{d_2} \log \frac{X_2(t)}{X_2(0)}$$

$$\longrightarrow \quad X_2(t) = c . \left(X_1(t)\right)^{(d_2/d_1)} \quad > 1$$

dans les axes (X_1, X_2) :



dans les coordonnées (x, p) de l'espace de phase :



$2^{\text{ème}}$ cas, $\Delta < 0$, (il faut $\gamma^2 \leq 4Km$)

alors $\sqrt{\Delta} = i\sqrt{|\Delta|} \in i\mathbb{R}$,

et $\frac{\gamma}{m} < \sqrt{|\Delta|}$

$$\begin{cases} d_1 = -\frac{1}{2}\left(\frac{\gamma}{m} - i\sqrt{|\Delta|}\right) = -\frac{\gamma}{2m} + i\omega, & \omega = \frac{1}{2}\sqrt{|\Delta|} \\ d_2 = -\frac{1}{2}\frac{\gamma}{m} - i\omega = d_1^* \end{cases}$$

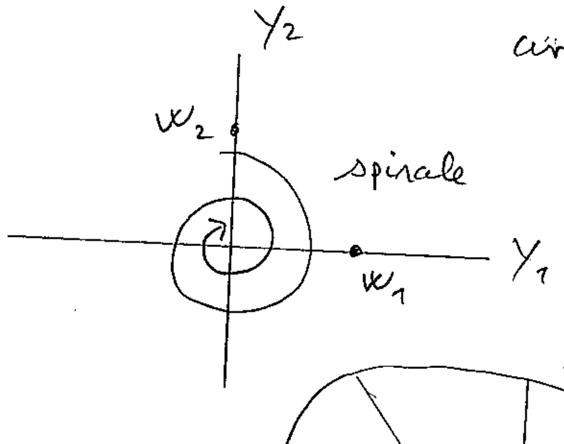
Les vecteurs propres V_1, V_2 sont complexes mais $V_1^* = V_2$

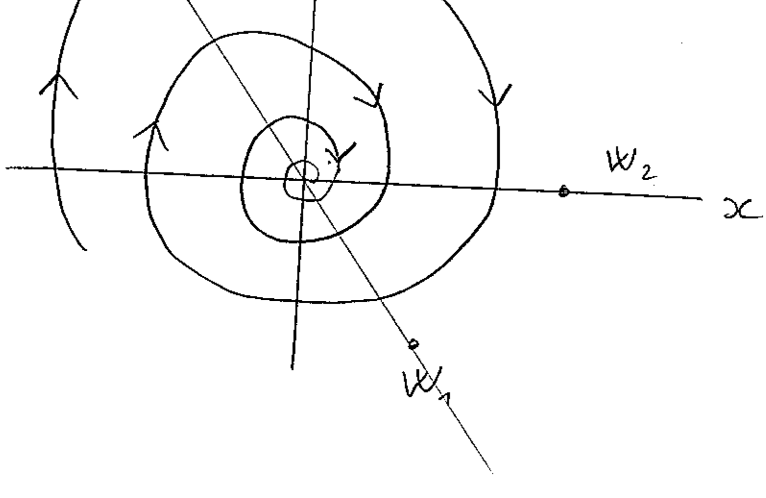
On considère
$$\begin{cases} W_1 := Re(V_1) = \begin{pmatrix} \frac{\gamma}{m} \\ -2K \end{pmatrix} \\ W_2 = Im(V_1) = \begin{pmatrix} \sqrt{|\Delta|} \\ 0 \end{pmatrix} \end{cases}$$

et les coordonnées dans la base (W_1, W_2): $\begin{cases} Y_1 = Re(X_1) \\ Y_2 = Im(X_1) \end{cases}$

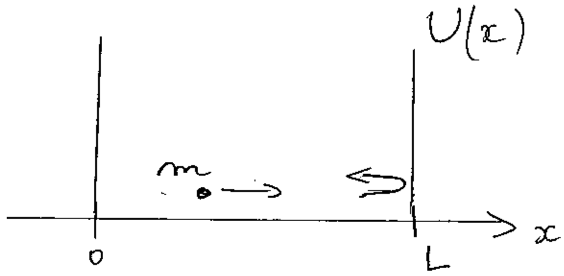
alors
$$\begin{cases} Y_1(t) = Re\left(X_1(0)\, e^{d_1 t}\right) = |X_1(0)| \cdot e^{-\frac{\gamma}{2m}t} \cdot \cos(\omega t + \underbrace{\varphi_0}_{\arg(X_1(0))}) \\ Y_2(t) = Im\left(\;''\; \right) = |X_1(0)|\, \underbrace{e^{-\frac{\gamma}{2m}t}}_{\text{amortissement}}\, \underbrace{\sin\left(\;''\; \right)}_{\text{oscillations}} \end{cases}$$

schéma:



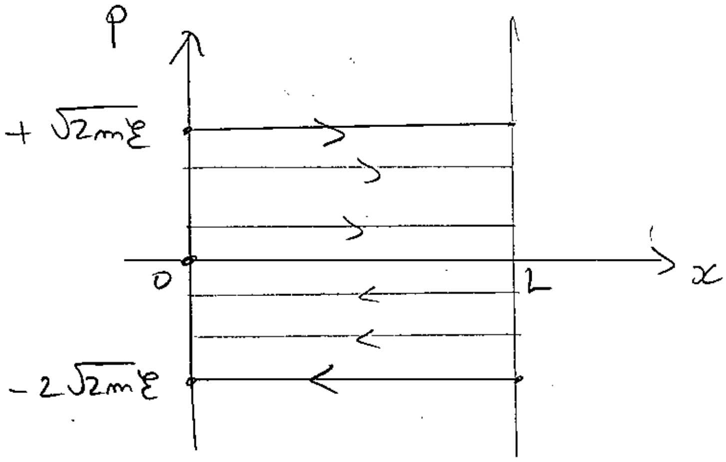


exercice 5



$$H(x,p) = \frac{p^2}{2m} = \mathcal{E}$$

$$\rightarrow p = \pm\sqrt{2m\mathcal{E}}$$

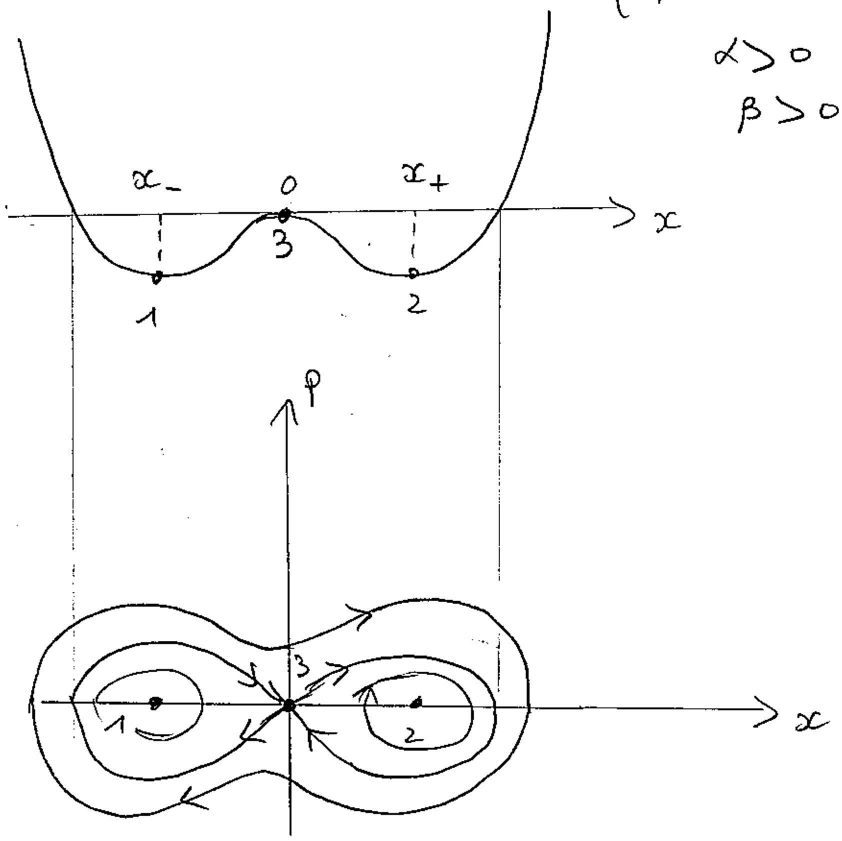


Exercice 6

$$U(x) = \alpha x^4 - \beta x^2$$

$\alpha > 0$

$\beta > 0$



$$H(x,p) = \frac{p^2}{2m} + U(x)$$

points fixes :

$U'(x) = 0$ et $p = 0$

$\Leftrightarrow 4\alpha x^3 - 2\beta x = 0$

$\Leftrightarrow \begin{cases} x = 0 \\ \text{ou } 4\alpha x^2 - 2\beta = 0 \end{cases}$

$\Leftrightarrow x_\pm = \pm\sqrt{\frac{\beta}{2\alpha}}$

$$U''(x) = 12\alpha x^2 - 2\beta$$

donc

$$\begin{cases} U''(x_\pm) = 12\alpha\frac{\beta}{2\alpha} - 2\beta = 4\beta \\ U''(0) = -2\beta \end{cases}$$

donc fréquences aux points stables 1 et 2 : $\omega_1 = \omega_2 = \sqrt{\frac{U''(x_\pm)}{m}} = 2\sqrt{\frac{\beta}{m}}$

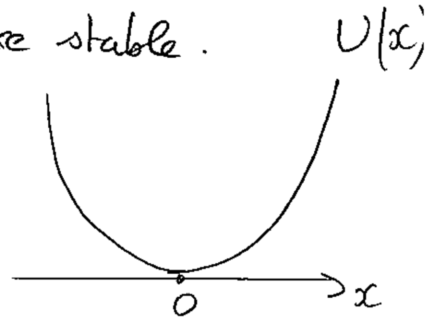
et coefficient d'instabilité en $x = 0$: $\lambda_3 = \sqrt{\frac{-U''(0)}{m}} = \sqrt{\frac{2\beta}{m}}$

pour $\beta > 0$ on a donc en résumé :

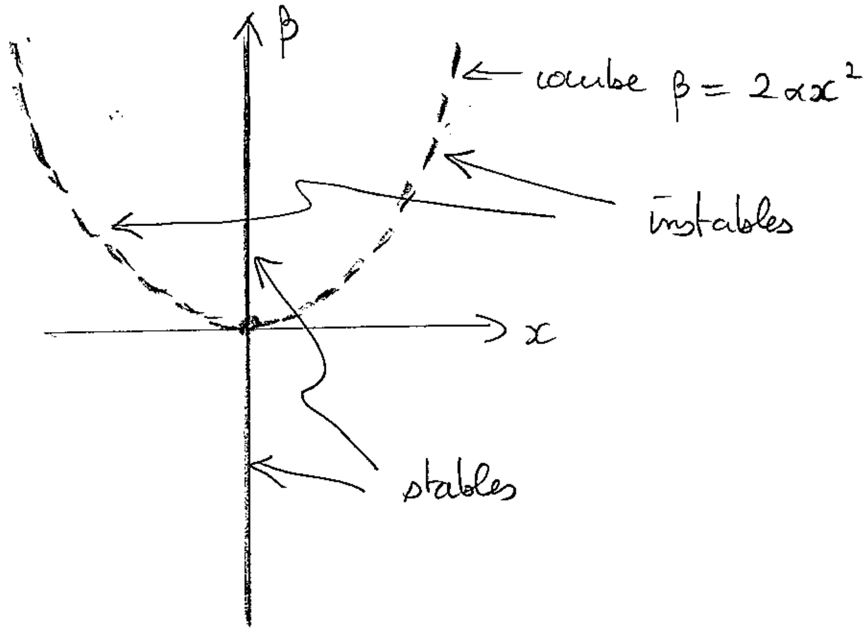
$$\left\{ \begin{array}{ll} x_1(\beta) = x_- = -\sqrt{\frac{\beta}{2\alpha}} , & \text{point instable} \\ x_2(\beta) = 0 & \text{point stable} \\ x_3(\beta) = x_+ = +\sqrt{\frac{\beta}{2\alpha}} & \text{point instable} \end{array} \right\}$$

Pour $\beta < 0$, seul $x = 0$ est un point fixe stable.

$\beta < 0$:



Donc diagramme des points fixes :



Exercice 7

1. $U(x) = \frac{1}{6}x^6 - \frac{1}{4}x^4 + \frac{1}{2}\beta x^2 \qquad \beta \in \mathbb{R}$

points fixes : $U'(x) = x^5 - x^3 + \beta x = 0$

$\Longleftrightarrow \quad x = 0$

(*) ou $x^4 - x^2 + \beta = 0 \quad \Longleftrightarrow X^2 - X + \beta = 0$ avec $X = x^2$

$\Longleftrightarrow X = x^2 = \dfrac{+1 \pm \sqrt{1-4\beta}}{2}$

- donc si $\begin{cases} 1-4\beta > 0 \\ \beta > 0 \end{cases} \Longleftrightarrow 0 < \beta < \frac{1}{4}$ il y a 5 points fixes :

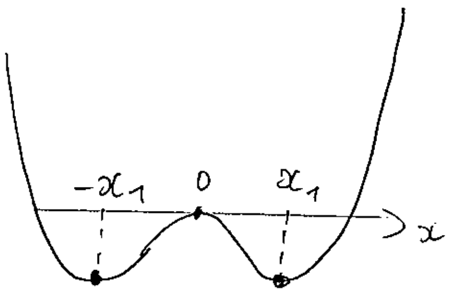
$x = -x_1, -x_2, 0, x_2, x_1$ avec $x_1 = \sqrt{\dfrac{1+\sqrt{1-4\beta}}{2}}$

$x_2 = \sqrt{\dfrac{1-\sqrt{1-4\beta}}{2}}$

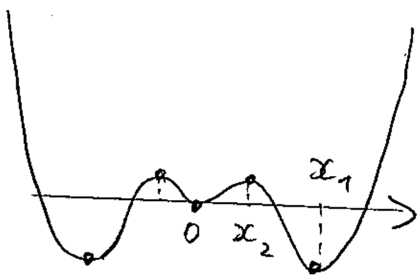
- pour $\beta < 0$, il y a 3 points fixes :

$x = -x_1, 0, +x_1$

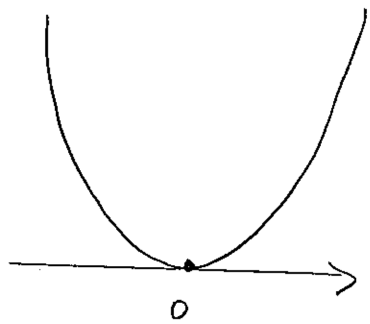
- pour $\beta > \frac{1}{4}$, il y a seulement 1 point fixe $x = 0$.



$\beta < 0$



$0 < \beta < \frac{1}{4}$

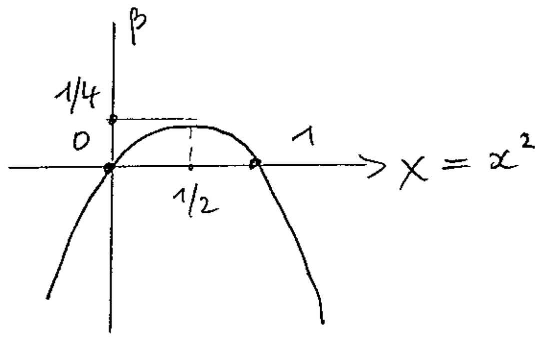


$\frac{1}{4} < \beta$

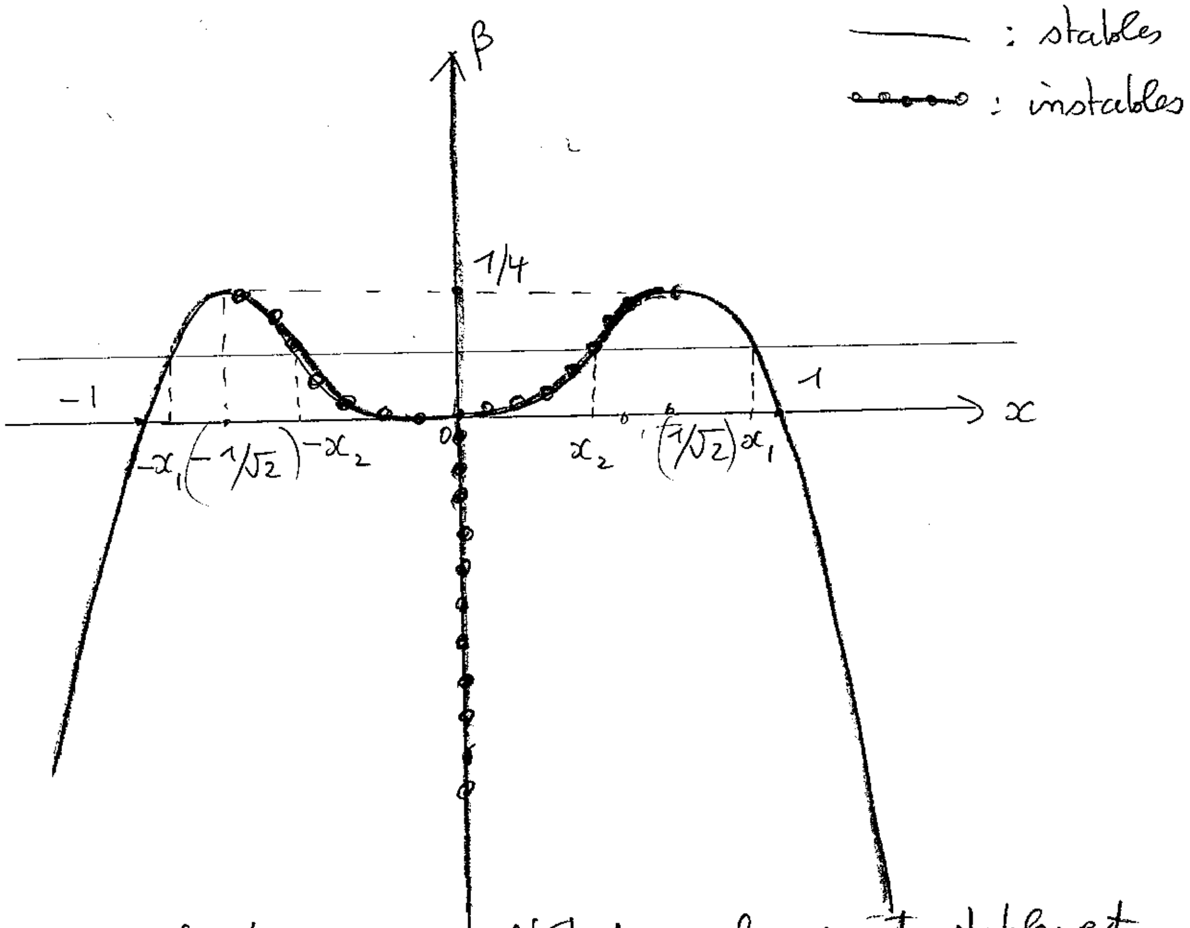
② D'après (*) on a $X^2 - X + \beta = 0$, $X = x^2$

$\Leftrightarrow \beta = X - X^2 = X(1-X)$

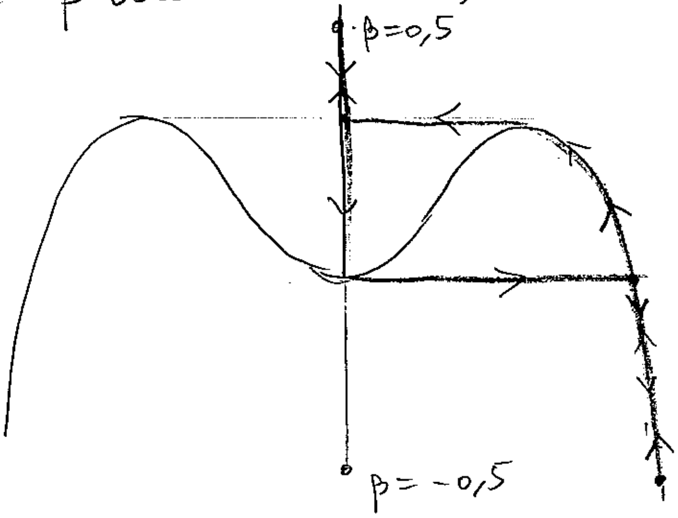
donc :



donc :



Si des frottements font converger l'état vers les points stables et si β oscille entre −0,5 et +0,5 alors l'état suit le chemin :



appelé cycle d'Hystérésis.

(le saut peut être aussi sur $-x_1$)