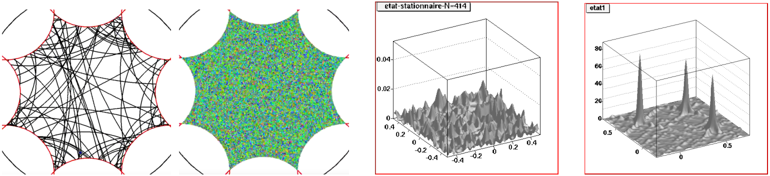
Equidistribution in classical and quantum chaos

Slides are on my web-page.



Journées inter-thèmes de l' institut Fourier, juin 2024.

Classical dynamics (Newton 1685, Hamilton 1834)

position $q \in \mathbb{R}^d$ (local coord. on a smooth closed manifold $\mathcal{N}$)

Definition

On $\mathbb{R}^d \times \mathbb{R}^d$ (local. coord. on $T^*\mathcal{N}$), a function

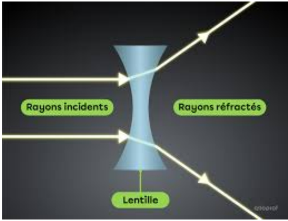
$$H : (\underbrace{q}_{\text{position}}, \underbrace{p}_{\text{impulsion}}) \in T^*\mathcal{N} \to \underbrace{H(q,p)}_{\text{energie}} \in \mathbb{R}$$

generates **Hamiltonian flow** $\phi^t : (q(0), p(0)) \to (q(t), p(t))$,

$$\frac{dq_j(t)}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j(t)}{dt} = -\frac{\partial H}{\partial q_j}, \quad j = 1 \ldots d.$$

Ex: **Geodesic flow:** metric g on $\mathcal{N}$, $H(q,p) = \|p\|_{g_q^*}$,

- Adhesive tape
- Free particle, **Newton eq.** $\frac{D\dot{q}}{dt} = 0$
- Light beam in medium with velocity $c(q) > 0$, $g_q^* = c(q)\, g^*_{\mathrm{Eucl}}$.



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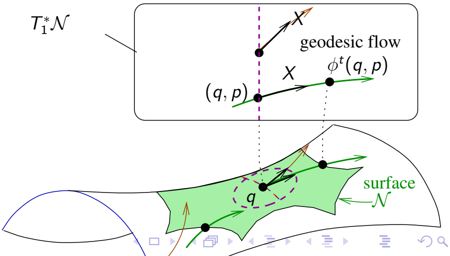
- **«Energy conservation»**: $T_1^*\mathcal{N} = \left\{ (q,p), H(q,p) = \|p\|_{g_q^*} = 1 \right\}$ is invariant.
- proof: $\frac{dH(q(t),p(t))}{dt} = \sum_j \frac{\partial H}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial H}{\partial p_j} \frac{dp_j}{dt} = 0.$



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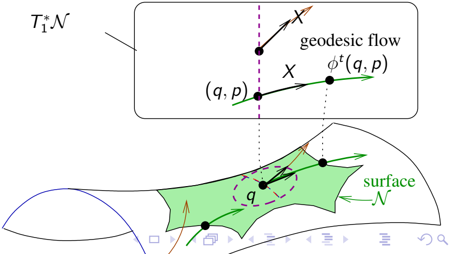
- **Rem: geometric formulation on $T^*\mathcal{N}$**

  $\Omega = \sum_{j=1}^{d} dq_j \wedge dp_j : TT^*\mathcal{N} \to T^*T^*\mathcal{N}$.
  $X = \Omega^{-1}(dH)$: vector field
  $\phi^t = e^{tX}$



Chaotic classical dynamics

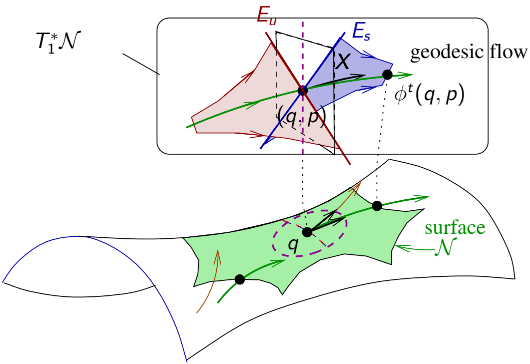
Theorem (Anosov (1966))

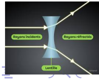
If $(\mathcal{N}, g)$ *has curvature* $\kappa < 0$ *then the geodesic flow is* ***unif. hyperbolic****:*

$$T\left(T_1^*\mathcal{N}\right) = \mathbb{R}X \oplus E_{\text{unstable}} \oplus E_{\text{stable}}$$

.

called "sensitivity to initial conditions". Each trajectory has a "unique story".





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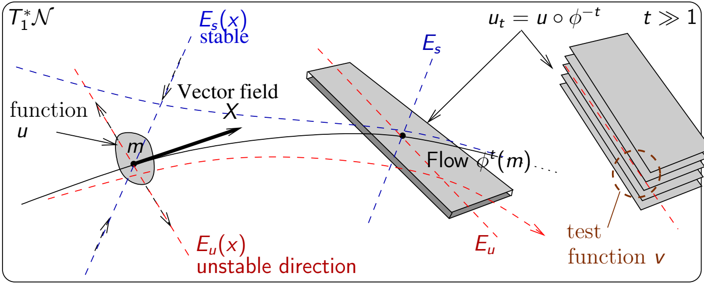
Evolution of a cloud of points (or function or proba density) $u \circ \phi^{-t}$:



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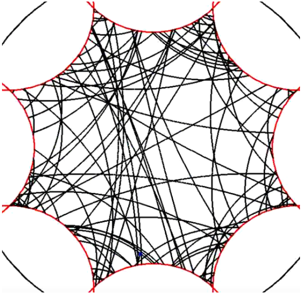
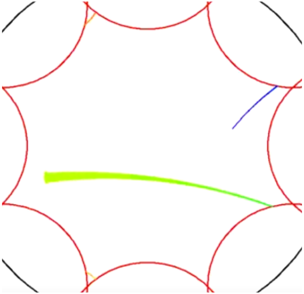
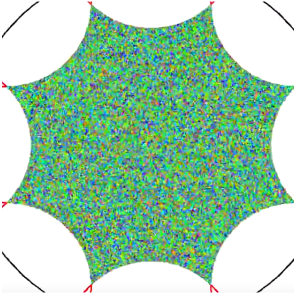
video bolza 1 particle, video bolza rays, video bolza 1e6 particles







Dynamique classique chaotique: mélange, équidistribution.

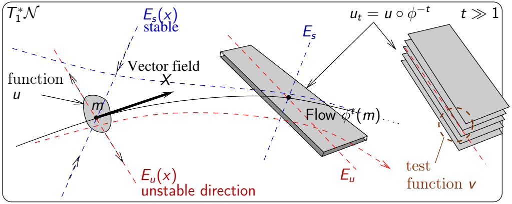
Theorem (Anosov (1967))

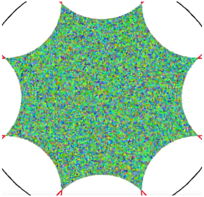
If $(\mathcal{N}, g)$ *has curvature* $\kappa < 0$ *then the geodesic flow is* ***unif. hyperbolic***:

$$T\left(T_1^* N\right) = \mathbb{R}X \oplus E_{\text{unstable}} \oplus E_{\text{stable}}$$

and ***mixing (equidistribution):*** *(Anosov 67, Liverani 04, Tsujii 08)* $\forall u, v \in C^\infty\left(T_1^*\mathcal{N}\right)$,

$$\langle v | u \circ \phi^{-t} \rangle_{L^2} \underset{t\to+\infty}{\rightarrow} \langle v|1\rangle_{L^2} \langle \frac{1}{\text{Vol}(M)} | u \rangle_{L^2} \quad \left(+O_{u,v}\left(e^{-t/2}\right) \text{ for Bolza}\right)$$





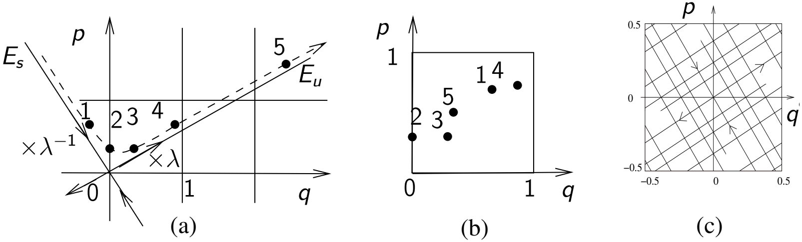
Simpler model: Cat Map ϕ

Definition (Cat map ϕ)

On $T^*\mathbb{R} = \mathbb{R}^2 \ni (q, p)$, the hyperbolic symplectic map $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

- is the flow $\phi^{t=1}$ generated by $H(q,p) = \alpha q^2 + \beta p^2 + \gamma qp$ with some $\alpha, \beta, \gamma \in \mathbb{R}$.
- This gives a **mixing** map on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (that can be perturbed).

See movie of 1 particle in cat map, movie of cat map, movie of pertubated cat map



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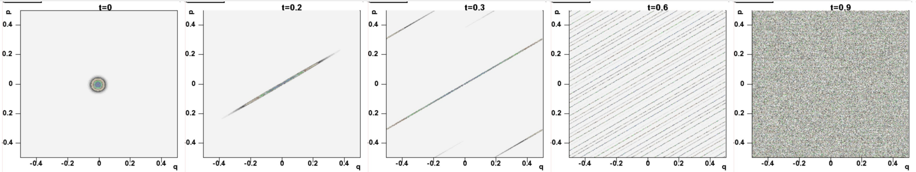

Proposition (Mixing, equidistribution)

$$\forall u, v \in C^\infty\left(\mathbb{T}^2\right), \langle v | u \circ \phi^{-t} \rangle_{L^2} \underset{t \to +\infty}{\to} \langle v | 1 \rangle_{L^2} \langle 1 | u \rangle_{L^2} + O_{u,v}\left(e^{-^\forall \alpha t}\right)$$

Proof: use Fourier series $x = (q,p)$, $n \in \mathbb{Z}^2$, $\varphi_n(x) = e^{i2\pi n.x}$ and $\varphi_n \circ \phi^{-t} = \varphi_{\phi^{-t} n}$.

Quantum dynamics

Question

Why classical dynamics is governed by the
***«strange looking» Hamilton/Newton equations** on $T^*\mathcal{N}$?*

$$\frac{dq_j(t)}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j(t)}{dt} = -\frac{\partial H}{\partial q_j}$$

$$X = \Omega^{-1}(dH)$$

Answer (as we will see): because an **Hamiltonian dynamics emerges from linear EDP(s)** in the limit of small wave lenght.

Hamiltonian particle $\Leftrightarrow$ wave-packet

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Hamiltonian particle ⇔ wave-packet

Quantum dynamics. Microlocal Analysis

- Fix $0 < \hbar \ll 1$. **Wave paquets** $(q,p) = T^*\mathbb{R}^d \to \varphi_{q,p} \in L^2\left(\mathbb{R}^d\right)$:

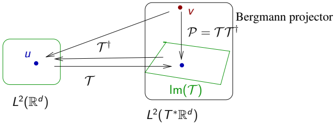
$$\varphi_{q,p}\left(q'\right) := C e^{i\left(\frac{p}{\hbar}\right)q'} e^{-\left|\frac{q'-q}{\sqrt{\hbar}}\right|^2}, \qquad \left\|\varphi_{q,p}\right\|_{L^2\left(\mathbb{R}^d\right)} = 1.$$

- **Wave packet transform:**

$$\mathcal{T} : \begin{cases} L^2\left(\mathbb{R}^d\right) & \to L^2\left(T^*\mathbb{R}^d\right) \\ u & \to \left(\mathcal{T}u\right)(q,p) := \left\langle \varphi_{q,p} | u \right\rangle_{L^2\left(\mathbb{R}^d\right)} \end{cases}$$

Lemma (fundamental 1. "Resolution of identity")

$$\mathcal{T}^\dagger \circ \mathcal{T} = \mathrm{Id}$$



- $\forall v \in L^2\left(T^*\mathbb{R}^d\right), \quad \left(\mathcal{T}^\dagger v\right)\left(q'\right) = \int_{T^*\mathbb{R}^d} \varphi_{q,p}\left(y'\right) v\left(q,p\right) \frac{dqdp}{(2\pi)^{d+1}}.$
- Rem: $\mathcal{T} : L^2(M) \to \mathrm{Im}(\mathcal{T}) \subset L^2\left(T^*M\right)$ is an isomorphism.

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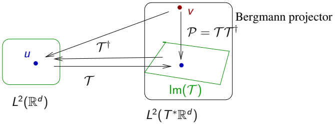

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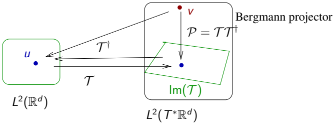

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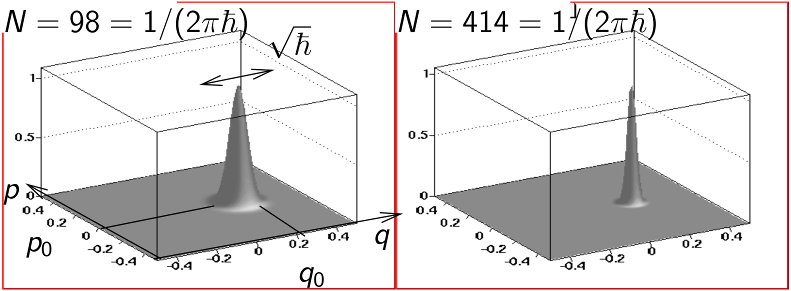
Wave packets on $T^*\mathbb{R} \equiv \mathbb{R}^2$

ex: $\left|\left(\mathcal{T}\varphi_{q_0,p_0}\right)(q,p)\right|^2$:



Quantum dynamics

Consider a **linear EDP** for $t \in \mathbb{R} \to u_t \in L^2\left(\mathbb{R}_q^d\right)$ of the form

$$u_{t=0} = \varphi_{q_0,p_0}, \qquad \frac{\partial u_t}{\partial t} = -\frac{i}{\hbar} H\left(q, -i\hbar\frac{\partial .}{\partial q}\right) u_t,$$

where $\hat{H} = H\left(q, -i\hbar\frac{\partial .}{\partial q}\right)$ is self-adjoint hence $u_t = \hat{U}^t u_0$ with $\hat{U}^t = e^{-t\frac{i}{\hbar}\hat{H}}$ unitary (preserves L^2 norm).

Ex: for the wave equation:

$$\frac{\partial^2 u_t}{\partial t^2} = -\Delta u_t \quad \Longleftrightarrow \quad \frac{\partial u_t}{\partial t} = \pm i\sqrt{\Delta} u_t = \pm\frac{i}{\hbar}\sqrt{\sum_{j=1}^{d}\left(-i\hbar\frac{\partial}{\partial q_j}\right)^2} u_t$$

hence $H(q,p) = \mp\|p\|$ (geodesic flow),(movie of wave packet)

Theorem (Fundamental 2. "Propagation of singularities", Duistermaat-Hörmander 1972)

$\forall t \in \mathbb{R}, \forall N, \exists C_{t,N} > 0, \forall \hbar < 1, \forall q, p$

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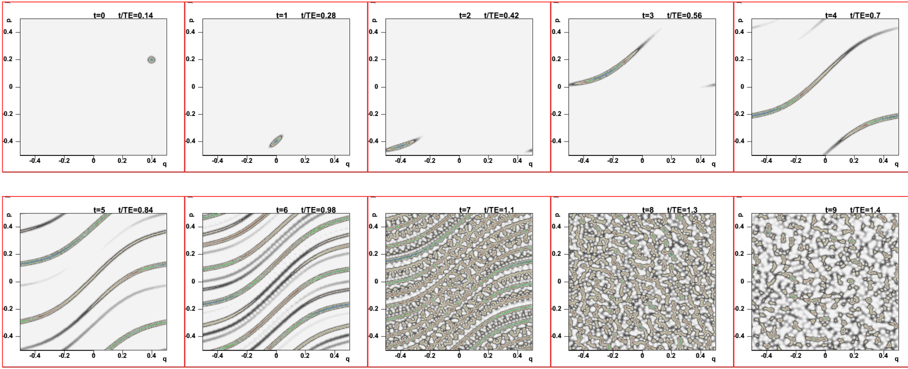
with ϕ^t *:Hamiltonian flow generated by* $H(q,p)$*.*

Convincing check: with linear symbol $H(q,p) = \alpha q + \beta p$.



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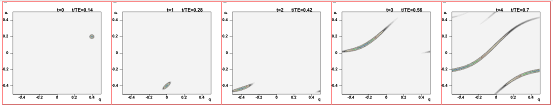


- So **Hamiltonian flow ϕ^t emerges from linear EDP(s)** u_t in the limit $\hbar \ll 1$:
  - Ex: Light:
    - ⋆ Newton 1666: geodesic flow (Hamiltonian) for light particles.
    - ⋆ Huygens 1690, Fresnel 1810, Maxwell 1865: wave aspect of light. $\lambda \approx 10^{-6} m$.
  - Ex: Matter:
    - ⋆ Newton 1685, Hamilton 1830: Hamiltonian flow for matter
    - ⋆ Schrödinger et al 1920: "quantum matter waves" $\lambda = \frac{2\pi\hbar}{p} \approx 10^{-15} m$, $\hbar \approx 10^{-34} J/s^{-1}$ (chosen in 2019).

Quantum cat map (or perturbed)

Recall the cat map $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$, $H(q,p) = \alpha q^2 + \beta p^2 + \gamma qp$.

Quantization gives a **quantum space**

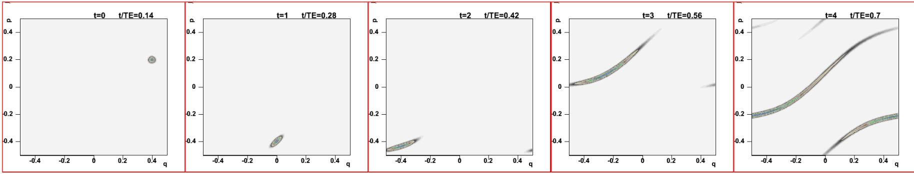
$$\mathcal{H} = \left\{ u \in \mathcal{S}'(\mathbb{R}), u(q+1) = u(q), (\mathcal{F}_\hbar u)(p) = (\mathcal{F}_\hbar u)(p+1) \right\},$$

with

$$N = \dim \mathcal{H} = \frac{\omega}{2\pi} = \frac{1}{2\pi\hbar} \in \mathbb{N} \qquad \text{(Riemann-Roch thm)}$$

and a **unitary operator**

$$\hat{U} := e^{-i\hat{H}/\hbar} : \mathcal{H} \to \mathcal{H}.$$

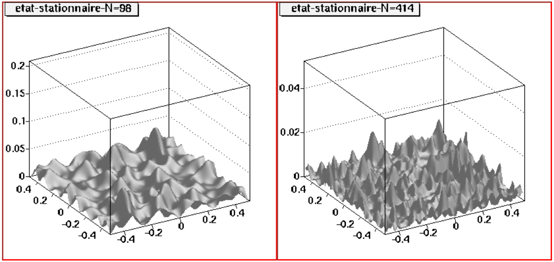


Quantum ergodicity (Equidistribution)

Ref: S. Dyatlov-review arxiv 2103.08093

It concerns eigenvectors $\hat{U}\psi_j = e^{i\theta_j}\psi_j$, $\psi_j \in \mathcal{H}$, $\|\psi_j\|_{\mathcal{H}} = 1$, $j = 1 \to N$.

$\left|\left(\mathcal{T}\psi_j\right)(\rho)\right|^2$:



Theorem (Quantum ergodicity (Schnirelmann 74, Zelditch 87, Colin de Verdière 85))

Suppose that the dynamics ϕ is ergodic. For any $a \in C^\infty\left(\mathbb{T}^2\right)$,

$$\text{"quantum variance": } V_{a,N} := \frac{1}{N}\sum_{j=1}^{N}\left|\int_{\mathbb{T}^2} a(\rho)\left|\left(\mathcal{T}\psi_j\right)(\rho)\right|^2 d\rho - \int_{\mathbb{T}^2} a(\rho)\, d\rho\right|^2 \underset{N\to\infty}{\to} 0$$

*means that **almost all** eigenfunctions $\mathcal{T}\psi_j$ get **equidistributed** on $\mathbb{T}^2$ in the limit $N \to \infty$.*

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Proof: suppose $\int_{\mathbb{T}^2} a(\rho)\, d\rho = 0$. Let $T \geq 1$.

$$\int_{\mathbb{T}^2} a(\rho)\left|(\mathcal{T}\psi_j)(\rho)\right|^2 d\rho = \langle \mathcal{T}\psi_j | a\mathcal{T}\psi_j\rangle_{L^2(\mathbb{T}^2)} = \langle \psi_j | \mathcal{T}^\dagger a \mathcal{T}\psi_j\rangle_{\mathcal{H}} = \frac{1}{T}\sum_{t=0}^{T-1}\langle \psi_j | \hat{U}^{-t}\mathcal{T}^\dagger a \mathcal{T}\hat{U}^t\psi_j\rangle$$

$$\underset{\text{(prop.sing.,Egorov)}}{=} \frac{1}{T}\sum_{t=0}^{T-1}\langle \psi_j | \mathcal{T}^\dagger\left(a\circ\phi^t\right)\mathcal{T}\psi_j\rangle + O_T\left(\frac{1}{N}\right) = \langle \mathcal{T}\psi_j | \left(\underbrace{\frac{1}{T}\sum_{t=0}^{T-1} a\circ\phi^t}_{a_T}\right)\mathcal{T}\psi_j\rangle + O_T\left(\right.$$

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Proof (cont.)

$$\left|\langle \mathcal{T}\psi_j | a_T \mathcal{T}\psi_j\rangle\right|^2 \underset{\text{(C.S.)}}{\leq} \left\|a_T \mathcal{T}\psi_j\right\|^2 = \langle \mathcal{T}\psi_j | a_T^2 \mathcal{T}\psi_j\rangle$$

$$V_{a,N} \leq \frac{1}{N}\sum_{j=1}^{N}\langle \mathcal{T}\psi_j | a_T^2 \mathcal{T}\psi_j\rangle + O_T\left(\frac{1}{N}\right) \underset{\text{(Trace)}}{=} \int_{\mathbb{T}^2} a_T^2\, dq dp + O_T\left(\frac{1}{N}\right)$$

Ergodicity implies $\int_{\mathbb{T}^2} a_T^2\, dqdp \underset{T\to\infty}{\to} 0$. Conclude by taking T large, then N large.

Conjecture (**Unique Ergodicity,** Rudnik Sarnak 1994)

*For a **generic unif. hyperbolic dynamics**, for every sequence $k \in \mathbb{N} \rightarrow \psi_{j_k} \in \mathcal{H}_{N_k}$ with $N_k \underset{k\to\infty}{\rightarrow} \infty$, $\forall a \in C^\infty(\mathbb{T}^2)$,*

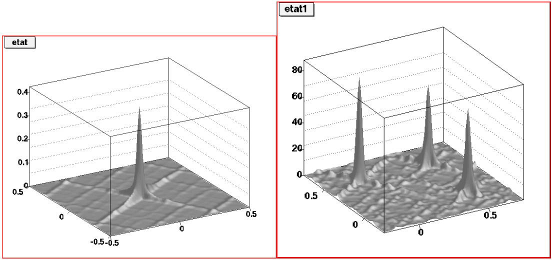
$$\int_{\mathbb{T}^2} a(\rho) \left|(\mathcal{T}\psi_{j_k})(\rho)\right|^2 d\rho \underset{k\to\infty}{\rightarrow} \int_{\mathbb{T}^2} a(\rho)\, d\rho$$

*i.e. **all eigenfunctions get equidistributed**.*

- **Counter-example** with the cat map (DB,F,N 2003):

$$\left|\mathcal{T}\psi_{j_k}\right|^2 \underset{k\to\infty}{\rightarrow^*} \mu_{sc} = \frac{1}{2}\delta_{\mathrm{periodic-orbit}} + \frac{1}{2}dqdp$$

See video of revival, video generic evolution



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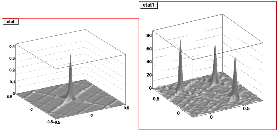
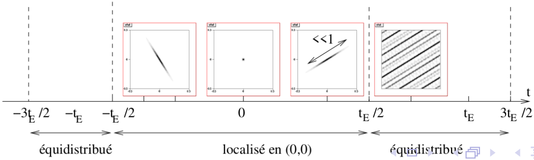


- proof: use short quantum period



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i.e. ***all eigenfunctions get equidistributed****.*

For any semiclassical measure $\mu_{sc} = \lim_{k\to\infty} \left|\mathcal{T}\psi_{j_k}\right|^2$,

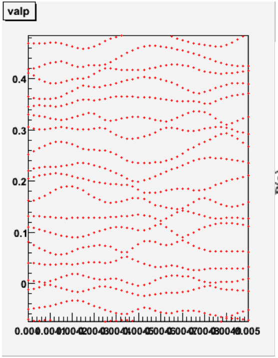
- Results from N Anantharaman et al. (2007): for hyperbolic surfaces the Kolmogorov Sinai entropy $h_{\mathrm{KS}}\left(\mu_{sc}\right) \geq \frac{1}{2}$.
- Results from S. Dyatlov et al. (2018) for hyperbolic surfaces: μ_{sc} has full support.

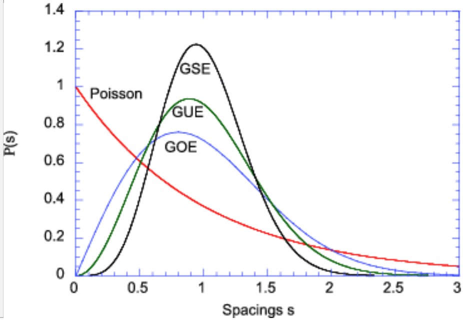
Random matrix conjecture

It concerns eigenvalues $\hat{U}\psi_j = e^{i\theta_j}\psi_j$, $\psi_j \in \mathcal{H}$, $j = 1 \to N$.

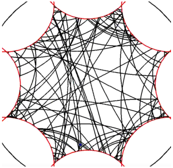
Conjecture (Universal random matrix conjecture (Bohigas et al. 1984))

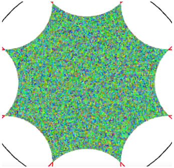
*For a **generic uniformly hyperbolic dynamics**, at scale $\Delta\theta \ll T_E^{-1} = 1/\log(1/\hbar)$, eigenvalues $(\theta_j)_j$ have the same statistical distributions as eigenvalues of a **random unitary matrix** (GUE). (the same for eigenvectors).*

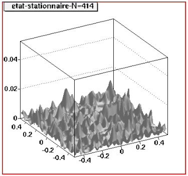


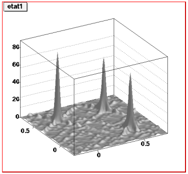


Thank you for your attention!



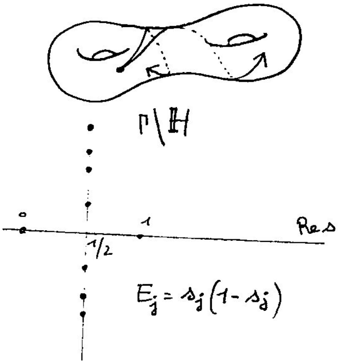


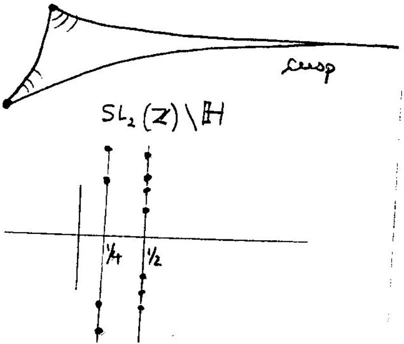




Resonances of the dynamics on the modular surface

Observation: (ref: Lax-Phillips): The Laplacian Δ on the modular surface $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ has a discrete spectrum (resonances $E_j = s_j(1 - s_j)$) containing the Riemann zeroes conjectured at $\mathrm{Re}(s) = 1/4$.





After the Ehrenfest time?

Question (Open in general)

What happens to $\hat{U}^t \varphi_{z_0}$ for $e^{\lambda t}\hbar \gg 1 \Leftrightarrow t \gg T_E := \frac{1}{\lambda}\log\frac{1}{\hbar}$ ***i.e. after the Ehrenfest time****? ($\lambda > 1$ is Lyapunov exponent)*

- For the cat map, we have some **exceptional phenomena** of **quantum period** (Hannay-Berry 1980): $\forall \hbar, \exists P_\hbar, \exists \alpha, \hat{U}^{P_\hbar} = e^{i\alpha}\mathrm{Id}$, and $P_\hbar = 2\left(\frac{1}{\lambda}\log\frac{1}{\hbar}\right)$ for infinite sequences of $\hbar$. See **videos**.