Some aspects of geometric quantization and quantum chaos

Slides are on my web-page.

F. Faure (Grenoble)

January 19, 2024, Marseille

Geometric quantization of a symplectic map (flow)

Let $\phi : (M, \Omega) \to (M, \Omega)$ a symplectic map.

Example (geodesic flow)

$M = T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d \ni$ (position q, momentum p) and $\Omega = \sum_j dq_j \wedge dp_j$.
Hamiltonian function $H(q,p)$, generates a (symplectic) flow ϕ^t by
$\frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}, \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}$.
If g is a metric on $\mathbb{R}^d_q$, $H(q,p) = \|p\|_{g_q}$ generates the **geodesic flow** (motion of a free particle or adhesive tape).

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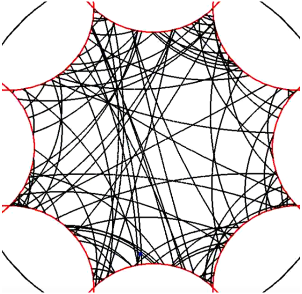
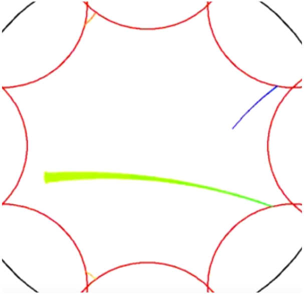
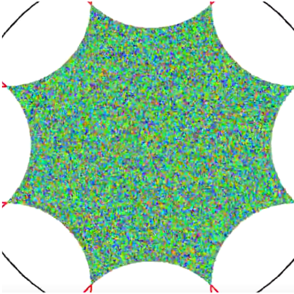
video bolza 1 particle, video bolza rays, video bolza 1e6 particles

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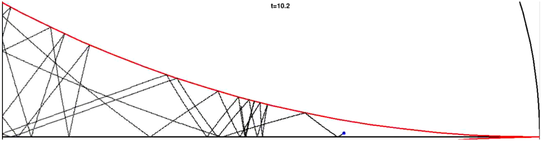
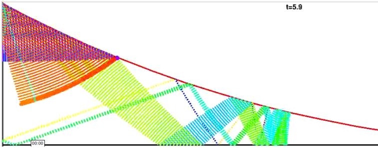
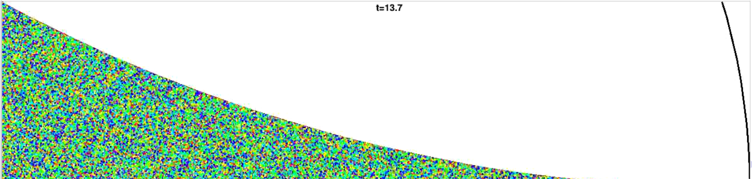
video modular billiard 1 particle, video 1e5 particles in modular billiard, video 1e5 particles on a closed horocycle.



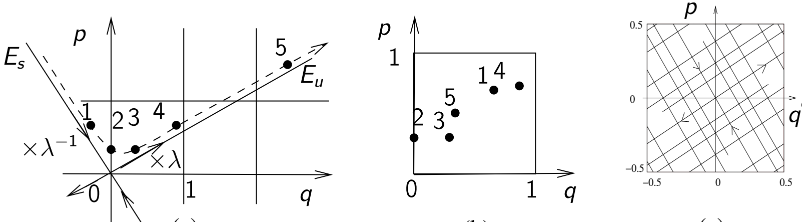




Cat Map

Example (Cat map)

On $M = T^*\mathbb{R} = \mathbb{R}^2 \ni (q, p)$, the hyperbolic symplectic map $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is the flow $\phi^{t=1}$ generated by $H(q, p) = \alpha q^2 + \beta p^2 + \gamma qp$ with some $\alpha, \beta, \gamma \in \mathbb{R}$. This gives a **mixing** map on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (that can be perturbed). See movie of 1 particle in cat map, movie of cat map, movie of pertubated cat map



(a) (b) (c)

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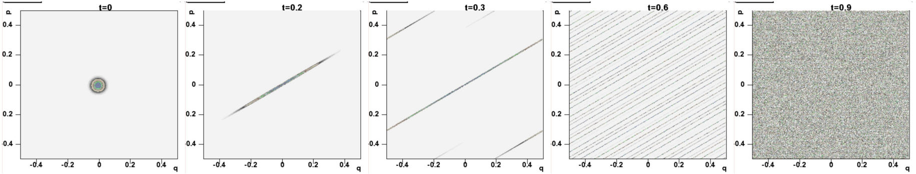
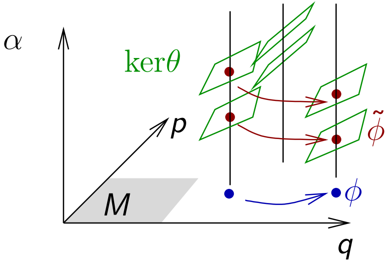


Pre-quantization of $\phi : M \to M$, :1st step (Kostant Souriau 70')

Definition

Let $P = M \times \mathbb{R}_\alpha$, with $M = \left(T^*\mathbb{R}^d\right)_{q,p}$ (no topological issues),
1-form $\theta = d\alpha - \sum_{j=1}^{d} p_j dq_j$ hence $d\theta = \Omega$.
The **prequantum map** is the equivariant extension $\tilde{\phi} : P \to P$ of $\phi : M \to M$ such that $\tilde{\phi}^*\theta = \theta$.



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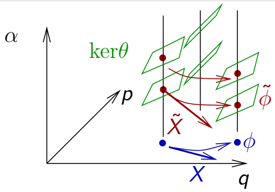
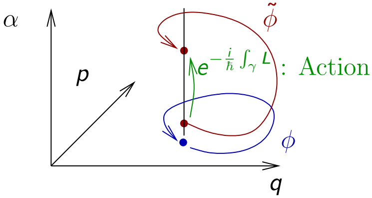


- Lemma: The lift of $X = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$ is $\tilde{X} = X + L\frac{\partial}{\partial \alpha}$ with Lagrangian $L = \sum_j p_j \frac{\partial q_j}{\partial t} - H$
- Proof: $\mathcal{L}_{\tilde{X}}\theta = d\iota_{\tilde{X}}\theta + \iota_{\tilde{X}}d\theta = d\left(L - A\left(X\right)\right) + dH = 0$ with $A = \sum_j p_j dq_j$.

- Consider **push forward of functions** $\tilde{\phi}_*^t : v \in L^2(P) \to v \circ \tilde{\phi}^{-t} \in L^2(P)$.



- Let $\omega \in \mathbb{R}$, and consider the reduced space invariant under $\tilde{\phi}_*$:

$$L^2_\omega(P) = \left\{ v \in L^2(P), v(q,p,\alpha) = e^{-i\omega\alpha} u(q,p), u \in L^2\left(T^*\mathbb{R}^d\right) \right\},$$
$$\tilde{X}_\omega = X - i\omega L$$

- The **prequantum Schrödinger equation** is, with $\hbar = \frac{1}{\omega}$,

$$\frac{\partial u_t}{\partial t} = -\tilde{X}_\omega u_t \quad \Leftrightarrow i\hbar \frac{\partial u_t}{\partial t} = \hat{H}_{\mathrm{preq.}} u_t, \quad \hat{H}_{\mathrm{preq.}} = -i\hbar \tilde{X}_\omega = -i\hbar X - L$$

- Ex: $H(q,p) = q_j$ gives $\hat{H}_{\mathrm{preq.}} = i\hbar \frac{\partial}{\partial p_j} + q_j$.
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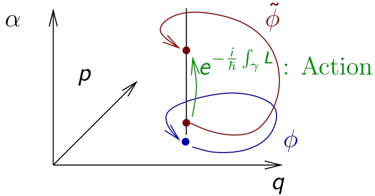

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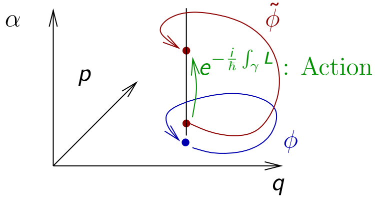

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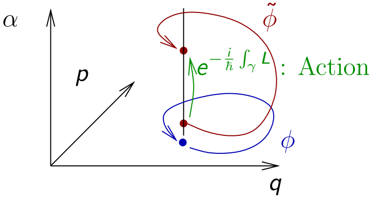

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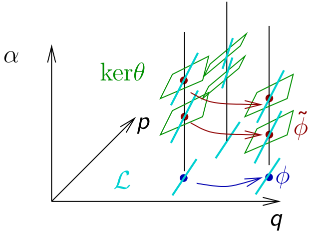
Quantization of ϕ (:2nd step, less intrinsic)

- Def: a **polarization** is an arbitrary integrable **Lagrangian distribution** $\mathcal{L}$ on TM lifted to $\ker\theta$. **Not invariant by the dynamics!**



- The **quantum space** $\mathcal{H} \subset L^2_\omega(P)$ is functions invariant along $\mathcal{L}$.
- Ex: **"vertical polarization"** $\mathrm{Span}\left(D_{\frac{\partial}{\partial p_j}}\right)_j$ (cov. deriv.) gives $\mathcal{H} \equiv L^2(\mathbb{R}^d)$.
- Let $\Pi : L^2_\omega(P) \to \mathcal{H}$ the "orthogonal projector" (averaging along $\mathcal{L}$). The **quantum Schrödinger equation** in $\mathcal{H}$ is

$$i\hbar \frac{\partial u_t}{\partial t} = \underbrace{\left(-i\hbar \Pi \tilde{X} \Pi\right)}_{\hat{H}} u_t \Leftrightarrow u_t = e^{-it\hat{H}/\hbar} u_0$$

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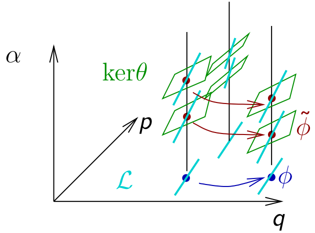


- Ex: **"vertical polarization"** $\mathrm{Span}\left(\widetilde{\frac{\partial}{\partial_{p_j}}}\right)_j$ gives $\mathcal{H} \equiv L^2\left(\mathbb{R}^d\right)$.

  $H(q,p) = q_j$ gives $\hat{H} = q_j$. $H(q,p) = p_j$ gives $\hat{H} = -i\hbar\frac{\partial}{\partial q_j}$.

- For **geodesic flow**, $H(q,p) = \|p\|$ gives $\hat{H} = -\hbar\sqrt{\Delta} + l.o.t$ hence $i\hbar\frac{\partial u_t}{\partial t} = \left(-\hbar\sqrt{\Delta}\right)u_t$ gives the **wave equation** $\frac{\partial^2 u_t}{\partial t^2} = -\Delta u_t$.

Complex polarization

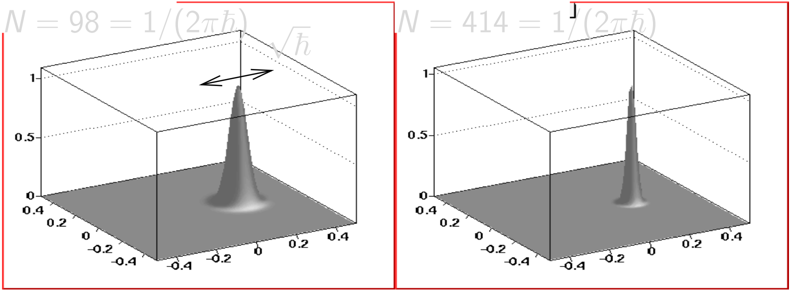
- Ex:**"complex polarization"** $\mathrm{Span}\left(\widetilde{\frac{\partial}{\partial \bar{z}_j}}\right)_j$ with $z_j = q_j + ip_j \in \mathbb{C}^d \equiv T^*\mathbb{R}^d$ gives

$$\mathcal{H} \underset{(16)}{=} \left\{ u(q,p) = w(z)\, e^{-\left(\frac{|z|}{2\sqrt{\hbar}}\right)^2} \in L^2\left(T^*\mathbb{R}^d\right) \right\} \qquad (1)$$

with holomorphic w.

- A **wave-packet** at $z_0 \in T^*\mathbb{R}^d$ is $\varphi_{z_0} := \Pi\delta_{z_0} \in \mathcal{H}$, gives $|\varphi_{z_0}(z)| = Ce^{-\left(\frac{|z-z_0|}{2\sqrt{\hbar}}\right)^2}$ and

$$\mathrm{Id}_{\mathcal{H}} = \int_{T^*\mathbb{R}^d} \varphi_z \langle \varphi_z | . \rangle \frac{dqdp}{\hbar} \quad : \text{"}\textbf{resolution of identity}\text{"}$$



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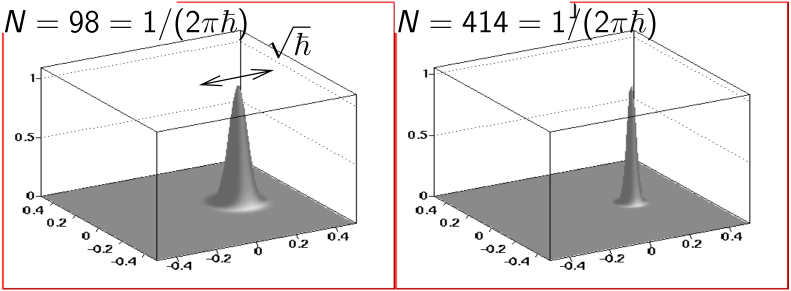


Quantum cat map (or perturbed)

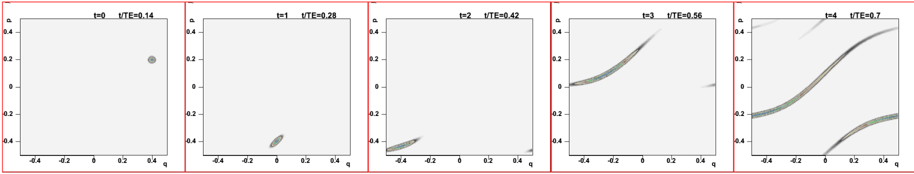
Recall the cat map $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$. By geometric quantization we get a **quantum space** $\mathcal{H}$ with

$$N = \dim \mathcal{H} = \frac{\omega}{2\pi} = \frac{1}{2\pi\hbar} \in \mathbb{N} \qquad \text{(Riemann-Roch thm)}$$

and a **unitary operator**

$$\hat{U} := e^{-i\hat{H}/\hbar} : \mathcal{H} \to \mathcal{H}.$$

We will use complex polarization, denote $z = q + ip \in \mathbb{T}^2$ and consider the **semi-classical limit** $\hbar \ll 1$.



Propagation of singularities

Theorem

"***propagation of singularities at finite time***". $\forall t \in \mathbb{Z}$, $\forall n, \exists C_{n,t}$, $\forall z_0 \in \mathbb{T}^2$, $\forall z \in \mathbb{T}^2$,

$$\left| \left(\hat{U}^t \varphi_{z_0} \right)(z) \right| \leq C_{n,t} \left(\frac{\mathrm{dist}\left(z, \phi^t(z_0)\right)}{\sqrt{\hbar}} \right)^{-n}$$

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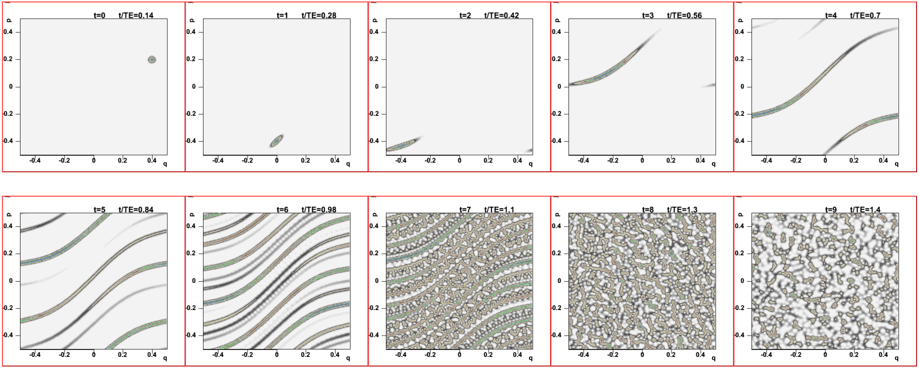
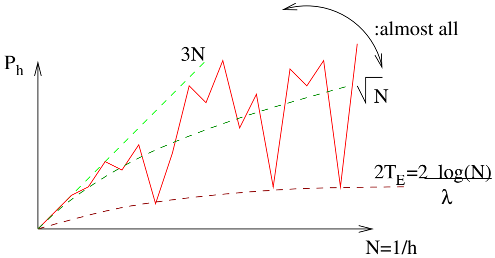


After the Ehrenfest time?

Question (Open in general)

What happens to $\hat{U}^t \varphi_{z_0}$ for $e^{\lambda t}\hbar \gg 1 \Leftrightarrow t \gg T_E := \frac{1}{\lambda}\log\frac{1}{\hbar}$ ***i.e. after the Ehrenfest time****? ($\lambda > 1$ is Lyapunov exponent)*

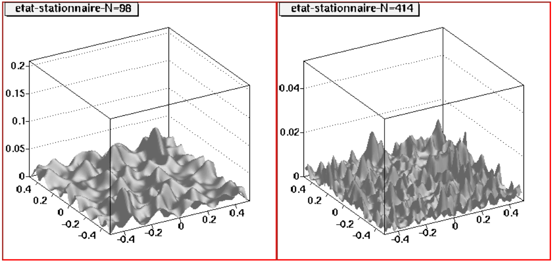
- For the cat map, we have some **exceptional phenomena** of **quantum period** (Hannay-Berry 1980): $\forall \hbar, \exists P_\hbar, \exists \alpha, \hat{U}^{P_\hbar} = e^{i\alpha}\mathrm{Id}$, and $P_\hbar = 2\left(\frac{1}{\lambda}\log\frac{1}{\hbar}\right)$ for infinite sequences of $\hbar$. See **videos**.



Quantum ergodicity

Ref: S. Dyatlov-review arxiv 2103.08093

It concerns eigenvectors $\hat{U}\psi_j = e^{i\theta_j}\psi_j$, $\psi_j \in \mathcal{H}$, $\|\psi_j\|_{\mathcal{H}} = 1$, $j = 1 \to N$.



Theorem (Quantum ergodicity (Schnirelmann 74, Zelditch 87, Colin de Verdière 85))

Suppose that the dynamics ϕ is ergodic. For any "observable" $a \in C^\infty(\mathbb{T}^2)$, define $\mathrm{Op}_N(a) = \Pi \mathcal{M}_a \Pi : \mathcal{H} \to \mathcal{H}$. *We have*

$$\text{"quantum variance": } V_{a,N} := \frac{1}{N}\sum_{j=1}^{N} \left|\langle \psi_j | \mathrm{Op}_N(a)\, \psi_j\rangle - \langle a \rangle_{\mathbb{T}^2}\right|^2 \underset{N\to\infty}{\to} 0$$

*means that **almost all** eigenfunctions ψ_j get **equidistributed** on $\mathbb{T}^2$ in the limit $N \to \infty$.*

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means that ***almost all*** *eigenfunctions ψ_j get* ***equidistributed*** *on $\mathbb{T}^2$ in the limit $N \to \infty$.*

Proof: suppose $\langle a\rangle_{\mathbb{T}^2} = 0$. Let $T \geq 1$.

$$\langle\psi_j|\mathrm{Op}_N(a)\,\psi_j\rangle = \frac{1}{T}\sum_{t=0}^{T-1}\langle\psi_j|\hat{U}^{-t}\mathrm{Op}_N(a)\,\hat{U}^t\psi_j\rangle \underset{(\mathrm{Egorov})}{=} \frac{1}{T}\sum_{t=0}^{T-1}\langle\psi_j|\mathrm{Op}_N\left(a\circ\phi^t\right)\psi_j\rangle + O_T\left(\frac{1}{N}\right)$$

$$= \langle\psi_j|\mathrm{Op}_N\left(\underbrace{\frac{1}{T}\sum_{t=0}^{T-1} a\circ\phi^t}_{a_T}\right)\psi_j\rangle + O_T\left(\frac{1}{N}\right)$$

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Proof (cont.)

$$|\langle \psi_j | \mathrm{Op}_N(a_T)\, \psi_j \rangle|^2 \underset{(\mathrm{C.S.})}{\leq} \|\mathrm{Op}_N(a_T)\, \psi_j\|^2 = \langle \psi_j | \mathrm{Op}_N(a_T)^2\, \psi_j \rangle \underset{(\mathrm{Compos.})}{=} \langle \psi_j | \mathrm{Op}_N\left(a_T^2\right) \psi_j \rangle + O_T(1/N)$$

$$\begin{aligned} V_{a,N} &= \frac{1}{N}\sum_{j=1}^{N} \langle \psi_j | \mathrm{Op}_N\left(a_T^2\right) \psi_j \rangle + O_T\left(\frac{1}{N}\right) = \frac{1}{N}\mathrm{Tr}\left(\mathrm{Op}_N\left(a_T^2\right)\right) + O_T\left(\frac{1}{N}\right) \\ &\underset{(\mathrm{Trace})}{=} \int_{\mathbb{T}^2} a_T^2\, dqdp + O_T\left(\frac{1}{N}\right) \end{aligned}$$

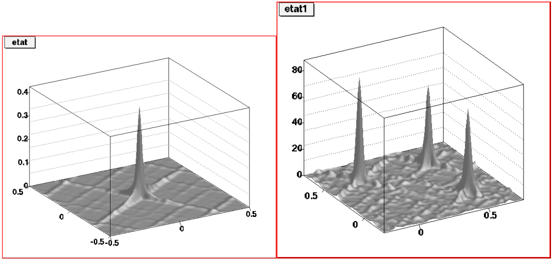
Ergodicity implies $\int_{\mathbb{T}^2} a_T^2\, dqdp \underset{T\to\infty}{\to} 0$. Conclude by taking T large, then N large.

Conjecture of unique quantum ergodicity

Conjecture (Rudnik Sarnak 1994)

*For a **generic uniformly hyperbolic dynamics**, for every sequence* $k \in \mathbb{N} \rightarrow \psi_{j_k} \in \mathcal{H}_{N_k}$ *with* $N_k \underset{k\rightarrow\infty}{\rightarrow} \infty$, $\forall a \in C^{\infty}\left(\mathbb{T}^2\right)$,
$\langle \psi_{j_k} | \mathrm{Op}_{N_k}(a)\, \psi_{j_k} \rangle \underset{k\rightarrow\infty}{\rightarrow} \langle a \rangle_{\mathbb{T}^2}$, *i.e.* ***all eigenfunctions get equidistributed***.

- **Counter-example** with the cat map (DB,F,N 2003): the **semiclassical measure** can be: $\mu_{sc} = \frac{1}{2}\delta_{\mathrm{periodic-orbit}} + \frac{1}{2} dqdp$,
  i.e. $\langle \psi_{j_k} | \mathrm{Op}_{N_k}(a)\, \psi_{j_k} \rangle \underset{k\rightarrow\infty}{\rightarrow} \int_{\mathbb{T}^2} a\mu_{sc}$. See video of revival, video generic evolution



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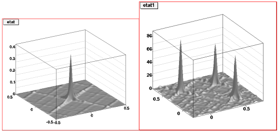
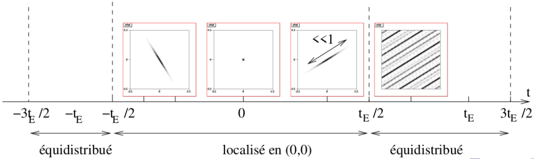
- **Counter-example** : $\mu_{sc} = \frac{1}{2}\delta_{\mathrm{periodic-orbit}} + \frac{1}{2}dqdp$,



- proof: direct consequence of the short quantum period $P_\hbar = 2T_E$:



Conjecture of unique quantum ergodicity

Conjecture (Rudnik Sarnak 1994)

For a ***generic uniformly hyperbolic dynamics****, for every sequence* $k \in \mathbb{N} \to \psi_{j_k} \in \mathcal{H}_{N_k}$ *with* $N_k \underset{k\to\infty}{\to} \infty$, $\forall a \in C^\infty\left(\mathbb{T}^2\right)$,
$\langle \psi_{j_k} | \mathrm{Op}_{N_k}(a)\, \psi_{j_k} \rangle \underset{k\to\infty}{\to} \langle a \rangle_{\mathbb{T}^2}$, *i.e.* ***all eigenfunctions get equidistributed****.*

For any semiclassical measure μ_{sc} defined by $\forall a, \langle \psi_{j_k} | \mathrm{Op}_{N_k}(a)\, \psi_{j_k} \rangle \underset{k\to\infty}{\to} \int_{\mathbb{T}^2} a\mu_{sc}$,

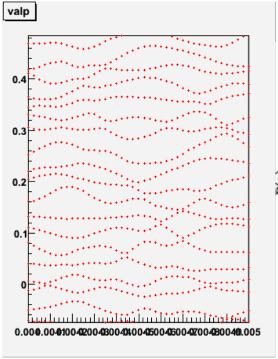
- Results from N Anantharaman et al. (2007): for hyperbolic surfaces the Kolmogorov Sinai entropy $h_{\mathrm{KS}}(\mu_{sc}) \geq \frac{1}{2}$.
- Results from S. Dyatlov et al. (2018): μ_{sc} has full support.

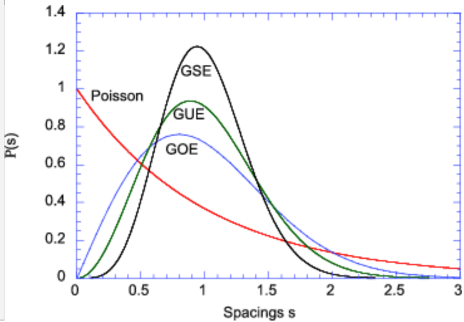
Random matrix conjecture

It concerns eigenvalues $\hat{U}\psi_j = e^{i\theta_j}\psi_j$, $\psi_j \in \mathcal{H}$, $j = 1 \to N$.

Conjecture (Universal random matrix conjecture (Bohigas et al. 1984))

*For a **generic uniformly hyperbolic dynamics**, at scale $\Delta\theta \ll T_E^{-1} = 1/\log(1/\hbar)$, eigenvalues $(\theta_j)_j$ have the same statistical distributions as eigenvalues of a **random unitary matrix** (GUE). (the same for eigenvectors).*



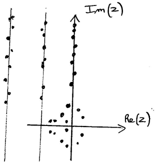


Resonance spectrum of the prequantum dynamics

Remark that discrete eigenvalues of $\hat{H}$ are poles of

$$z \to \langle v| \left(z - \hat{H}\right)^{-1} u\rangle = -\frac{i}{\hbar}\int_0^{\infty} e^{itz/\hbar} \underbrace{\langle v|e^{-it\hat{H}/\hbar}u\rangle}_{\text{correlations}} dt,\ u, v \in C^{\infty}.$$

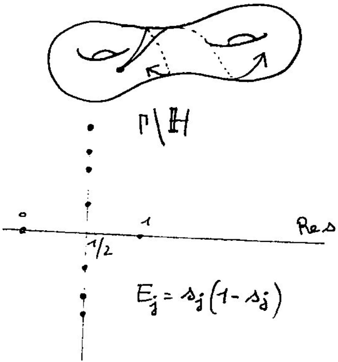
Thm: If the flow ϕ^t is Anosov, for the prequantum dynamics, $z \to \langle v| \left(z - \tilde{X}\right)^{-1} u\rangle$ has discrete poles called Ruelle resonances that contain the spectrum of a quantum operator:

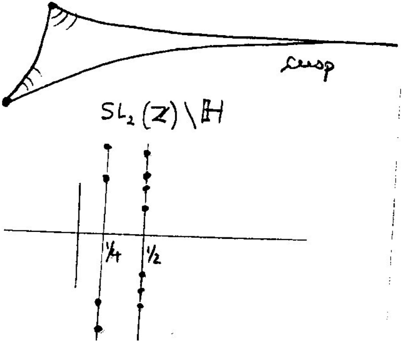


- This provides a natural or "**dynamical quantization**".
- **Interpretation:** quantum dynamics emerges from classical dynamics after long time.

Resonances of the dynamics on the modular surface

Observation: (ref: Lax-Phillips): The Laplacian Δ on the modular surface $SL_2(\mathbb{Z})\backslash\mathbb{H}$ has a discrete spectrum (resonances $E_j = s_j(1-s_j)$) containing the Riemann zeroes conjectured at $\mathrm{Re}(s) = 1/4$.





Appendix: proof of (1)

Let

$$z = q + ip$$

$$q = \frac{1}{2}(z + \overline{z}), \qquad p = \frac{i}{2}(\overline{z} - z)$$

Let

$$\theta = d\alpha - \frac{1}{2}pdq + \frac{1}{2}qdp = d\alpha - \frac{1}{8}i(\overline{z} - z)(dz + d\overline{z}) + \frac{1}{8}i(z + \overline{z})(d\overline{z} - dz)$$
$$= d\alpha - \frac{i}{4}\overline{z}dz + \frac{i}{4}zd\overline{z}$$

that satisfies $d\theta = dq \wedge dp = \Omega$. Let

$$\frac{\widetilde{\partial}}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}} + h(z, \overline{z})\frac{\partial}{\partial \alpha}$$

with h such that

$$0 = \theta\left(\frac{\widetilde{\partial}}{\partial \overline{z}}\right) = h(z, \overline{z}) + i\frac{z}{4}$$

hence

$$h(q, p) = -\frac{i}{4}(q + ip) = -\frac{i}{4}z$$

So $v(q, p, \alpha) = e^{i\omega\alpha}u(q, p)$ with $\frac{\widetilde{\partial}}{\partial \overline{z}}v = 0$ gives

$$\left(\frac{\partial}{\partial \overline{z}} - \frac{i}{4}z\frac{\partial}{\partial \alpha}\right)\left(e^{i\omega\alpha}u(z, \overline{z})\right) = 0 \Leftrightarrow \frac{\partial u}{\partial \overline{z}} + \frac{1}{4}z\omega u = 0$$

Solution is $u = w(z)e^{-\frac{1}{4}\omega\overline{z}z} = w(z)e^{-\frac{1}{4\hbar}\overline{z}z}$, indeed $\frac{\partial u}{\partial \overline{z}} = -\frac{1}{4}\omega z u$.