Micro-local analysis of hyperbolic dynamics. Part I: Anosov flows

F. Faure (Grenoble) with M. Tsujii (Kyushu),

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(this notes, based on arXiv:1706.09307 can be downloaded on the web page of frédéric faure)
Part II will be "Geodesic (or contact) Anosov flows".

Outline

1. **Anosov vector field** $X : C^\infty(M) \to C^\infty(M)$, **transfer operator (pull-back)** $e^{tX}u = u \circ \phi^t$ and examples of **Ruelle-Pollicott discrete spectra**.
2. **Micro-local analysis** with wave-packets transform $\mathcal{T}$ (an isometry, i.e. $\mathcal{T}^*\mathcal{T} = \mathrm{Id}$):

$$\begin{array}{ccc} L^2(M) & \xrightarrow{e^{tX}} & L^2(M) \\ \mathcal{T}\downarrow & & \mathcal{T}\downarrow \\ L^2(T^*M) & \xrightarrow{\mathcal{T}e^{tX}\mathcal{T}^*} & L^2(T^*M) \end{array} \qquad (1)$$

   1. **Trace formula** for $A \subset T^*M$: $\mathrm{Tr}(\mathcal{T}^*\chi_A\mathcal{T}) \asymp \frac{1}{(2\pi)^{\dim M}}\mathrm{Vol}(A)$. :"**uncertainty principle**", "density of information is finite".
   1. **Theorem of propagation of singularities**: The Schwartz kernel of $\mathcal{T}e^{tX}\mathcal{T}^*$ decays very fast outside the graph of $(D\phi^t)^* : T^*M \to T^*M$.
   2. **Anisotropic Sobolev space** $\mathcal{H}_W(M)$: for $u \in C^\infty(M)$, define

$$\|u\|_{\mathcal{H}_W(M)} := \|W\mathcal{T}u\|_{L^2(T^*M)}$$

   with a **weight function** $W : T^*M \to \mathbb{R}^+$, that is (Lyapunov-escape function for $(D\phi^t)^*$)

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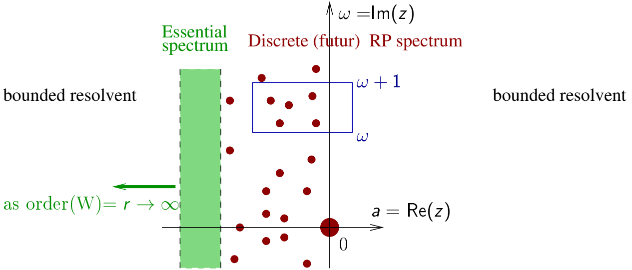

Rem: eigenvalue $z_j = a_j + i\omega_j \Rightarrow e^{tz_j} = \underbrace{e^{ta_j}}_{\text{decay if } a_j<0,\, t\to+\infty,} \underbrace{e^{it\omega_j}}_{\text{oscillation}}$.

2. "**Fractal Weyl law**": upper bound for the density of eigenvalues $\leq C\omega^{\frac{\dim M - 1}{1+\beta_0}}$, with β_0: Hölder exp. of $E_u \oplus E_s$.
3. "**Wave front set**": where $(\mathcal{T}u)(\rho), \rho \in T^*M$ is non negligible for an eigenfunction u of X.
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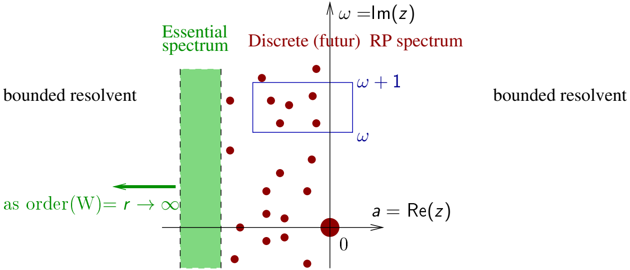

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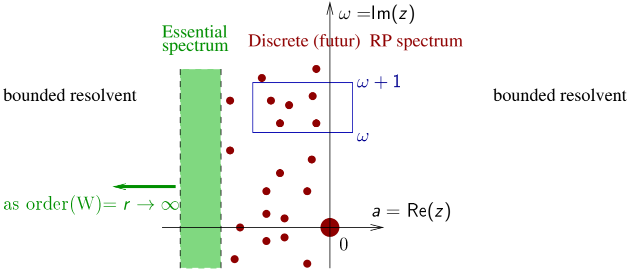

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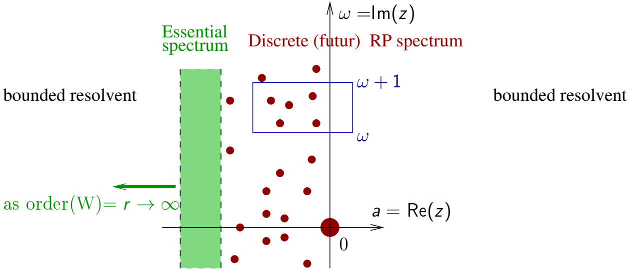

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- **Series of work and interesting recent activity around Ruelle-Pollicott resonances in hyperbolic dynamics:** Ruelle, Bowen 70', Pollicott 86, Rugh 90', Blank, Keller, Liverani 2002, Gouëzel, Liverani 2005, Naud 2005, Baladi, Tsujii 2005,2008, Butterley-Liverani 2007, Roy-Sjöstrand-F. 2008, Datchev-Dyatlov-Zworski, Dyatlov-Guillarmou 2014 for Axiom A flows. Dang-Rivière 2016 for Morse-Smale flows, Bonthonneau-Weich 2017 for cusps, Guillarmou-Hilgert-Weich 2018 for symmetric spaces, Baladi-Demers-Liverani 2018 for Sinaï billiards.

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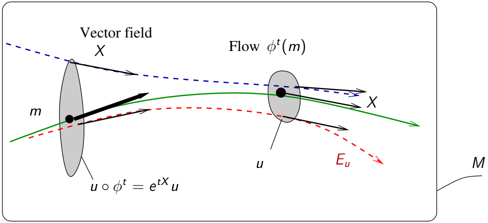

On a closed manifold M, let $X \in C^\infty(M; TM)$ be a **smooth vector field** i.e. first order diff. operator

$$X \equiv \sum_{j=1}^{\dim M} X_j(y) \frac{\partial}{\partial y_j}$$

Let

$$\phi^t : M \to M, \qquad t \in \mathbb{R} \qquad : \textbf{flow map}.$$

defined by $\frac{d(u \circ \phi^t)}{dt} = Xu$, $\forall u \in C^\infty(M)$, i.e.

$$\boxed{e^{tX} u := u \circ \phi^t, \qquad : \textbf{pull-back} \text{ (or } \textbf{transfer}\text{) operator on functions}}$$

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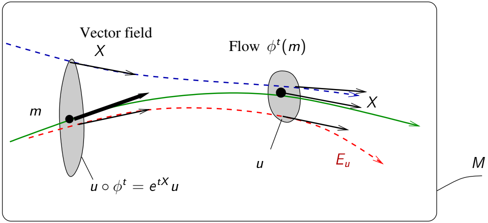

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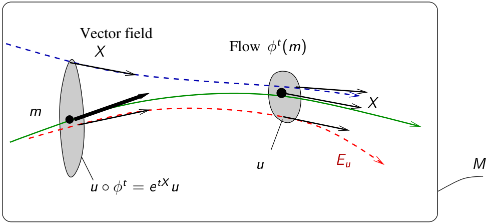

- For any smooth measure μ on M, the $L^2(M,\mu)$-adjoint is

$$\left(e^{tX}\right)^* u = \left|\det D\phi^{-t}\right| . u \circ \phi^{-t} = e^{t(-X - \mathrm{div}_\mu X)} u$$: **Perron-Frobenius** op.

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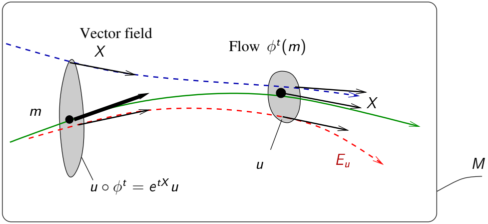

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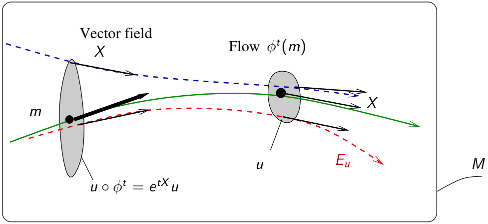

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s.t. **Leibniz condition** holds (i.e. X_F is an "extension" of X):

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- Example: on the trivial bundle $F = M \times \mathbb{C}$, $X_F = X + V$ with $V \in C^\infty(M)$: "**Gibbs potential**", then

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Anosov flow (or uniformly hyperbolic flow)

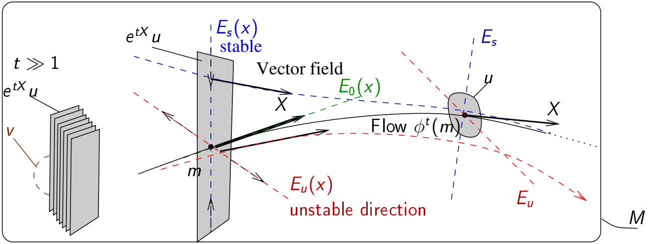
Definition

Vector field X is **Anosov** if

$$\forall m \in M, \quad T_m M = E_u(m) \oplus E_s(m) \oplus \underbrace{E_0(m)}_{\mathbb{R}X}$$

is $D\phi^t$-invariant, continuous, s.t. $\exists g, \exists C > 0, \lambda > 0, \forall t \geq 0, m \in M$,

$$\left\| D\phi^t_{/E_s(m)} \right\|_g \leq C e^{-\lambda t}, \quad \left\| D\phi^{-t}_{/E_u(m)} \right\|_g \leq C e^{-\lambda t},$$



Remarks

- Anosov property is **stable** under any (small C^1) perturbation of X.
- maps $m \to E_u(m), E_s(m), E_u(m) \oplus E_s(m)$ are **Hölder-continues** with some respective exponents $0 < \beta_u, \beta_s, \beta_0 \leq 1$.

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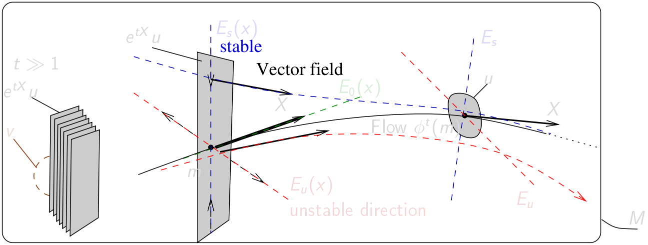
Description of long time behavior of the dynamics $\langle v | e^{tX} u \rangle$? i.e. ***discrete spectrum*** *of X (or X_F)? i.e. See movie2, movie3.*



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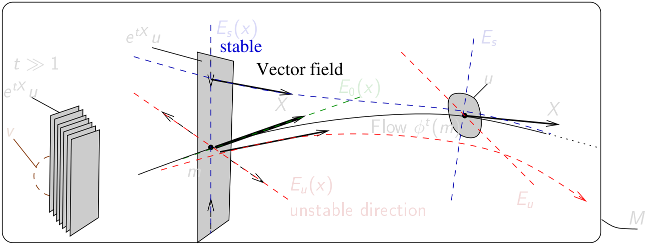

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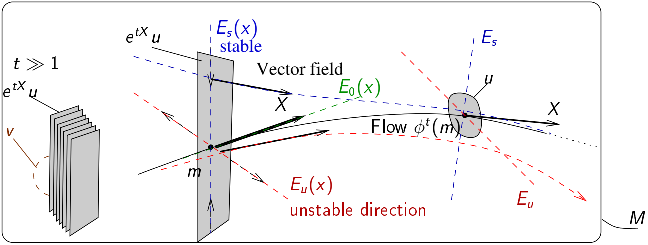

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$$X = -x\partial_x + \partial_z. \qquad \text{Flow: } \phi^t(x,z) = \left(e^{-t}x, z+t\right).$$

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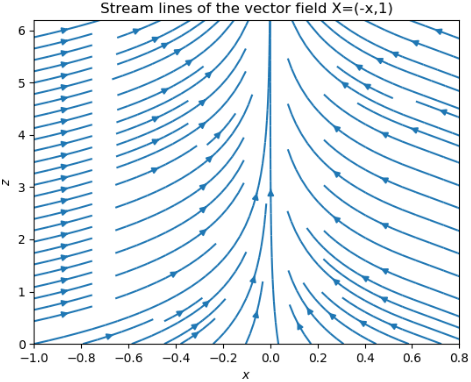

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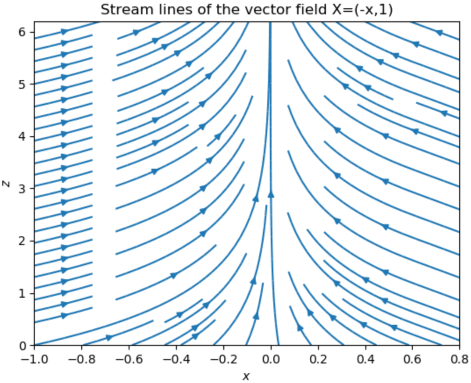

Example 1 (cont)

$M=\mathbb{R}_x\times(\mathbb{R}_z/(2\pi\mathbb{Z}))$: cylinder

$$X=-x\partial_x+\partial_z.$$

- in $L^2(M)$, $X^*=-X+1 \Leftrightarrow \left(X-\frac{1}{2}\right)=-\left(X-\frac{1}{2}\right)^*$, $\mathrm{Spec}_{L^2}(X)=i\mathbb{R}+\frac{1}{2}$.
  Not good: we are looking for discrete spectrum...
- The **induced flow in** T^*M is

$$\tilde{\phi}^t(x,z,\xi,\omega)=\left(\phi^{-t}(x,z),\left(D\phi^t\right)^*(\xi,\omega)\right)=\left(e^tx,z-t,e^{-t}\xi,\omega\right).$$

- Let the **Lyapunov (escape) function**

$$W(x,\xi)=\frac{\left\langle\sqrt{h}\xi\right\rangle^r}{\left\langle\sqrt{h}x\right\rangle^r}$$

with $0<h\ll 1$, $r\geq 0$, (Notation: $\langle x\rangle:=|x|$ if $|x|\geq 1$, otherwise $\langle x\rangle=1$.) and let

$$\|u\|_{\mathcal{H}_W(M)}:=\|W\mathcal{T}u\|_{L^2(T^*M)}$$

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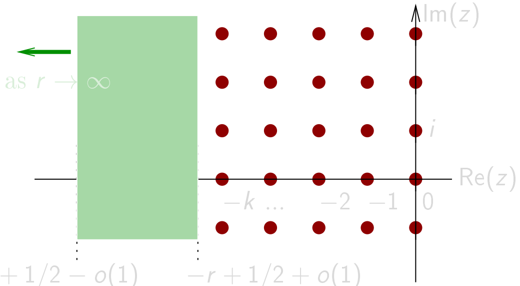
$$X\left(x^k e^{ilz}\right) = (-k + il)\left(x^k e^{ilz}\right), \quad k \in \mathbb{N}, l \in \mathbb{Z},$$

This is the **Ruelle-Pollicott spectrum** of X in $\mathcal{H}_W(M)$ on $\mathrm{Re}(z) > -r + \frac{1}{2} + o(1)$,

$$\Pi_{k,l} = \left(|x^k\rangle\langle\frac{1}{k!}\delta^{(k)}|.\rangle\right) \otimes \left(|e^{ilz}\rangle\langle e^{ilz}|.\rangle\right) \text{ is the bounded spectral proj.}$$



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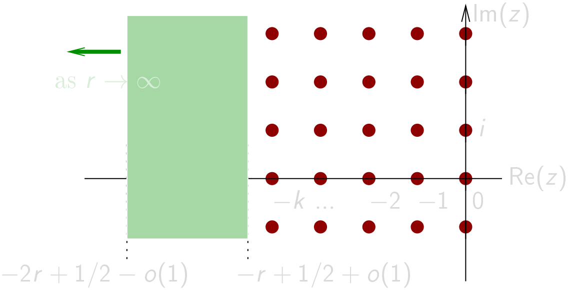

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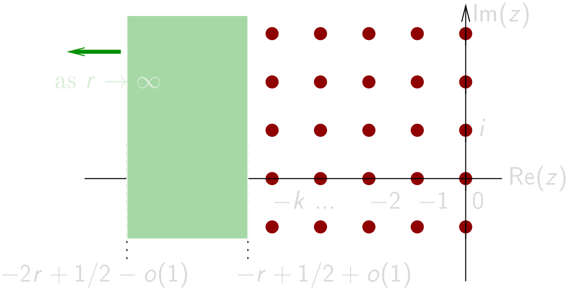

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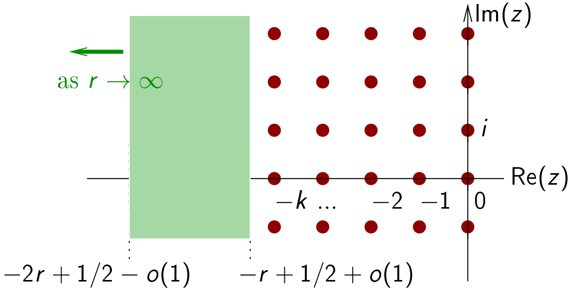
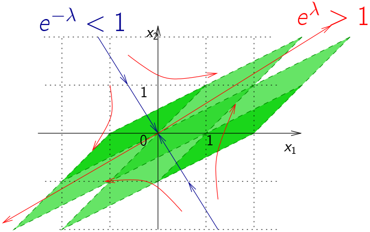
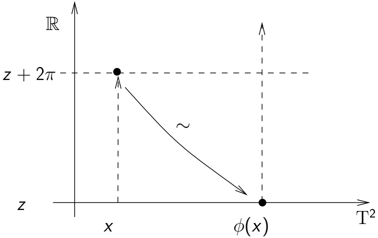


Example 2: suspension of a cat map

Let $M := \left(\mathbb{T}^2 \times \mathbb{R}\right) / \sim$ with $(x, z + 2\pi) \sim (\phi(x), z)$ with

$$\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2 \qquad : \text{cat map}$$





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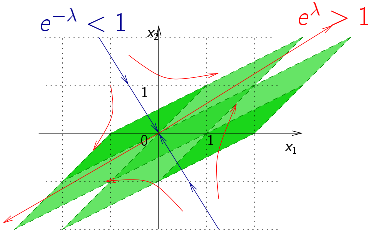

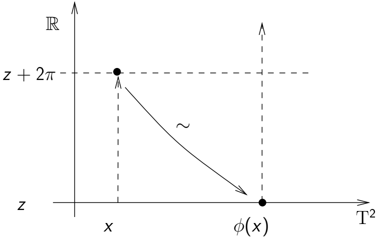

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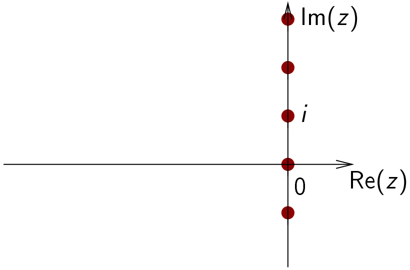

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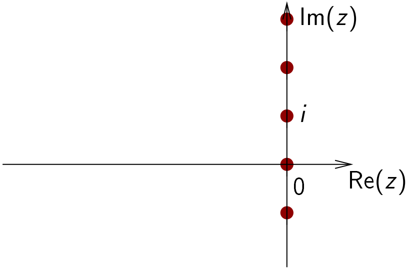

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Example 3.Geodesic flow on hyperbolic surface (still special but less trivial)

(ref: Flaminio-Forni 02, Dyatlov-F-Guillarmou 14 for $\Gamma\backslash SO_{1,n}/SO_{n-1}$).
$M = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ smooth compact, with Γ cocompact subgroup.
$\mathrm{sl}_2(\mathbb{R})$ algebra:

$$X = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[U, X] = U, \quad [S, X] = -S, \quad [S, U] = 2X$$

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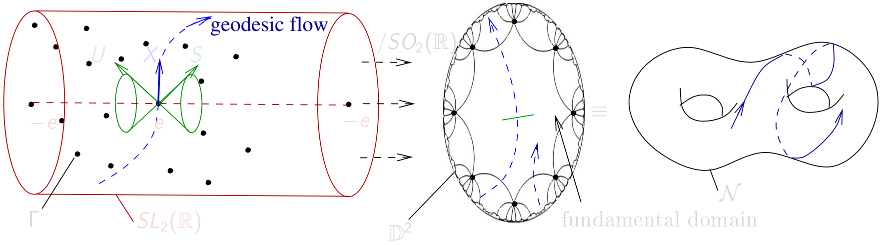

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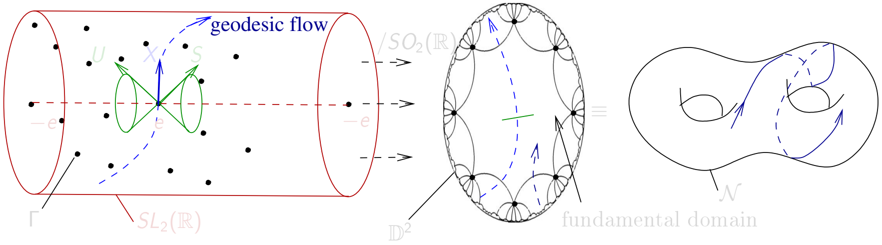

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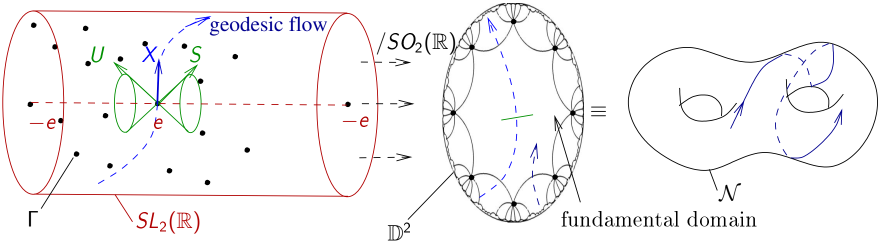

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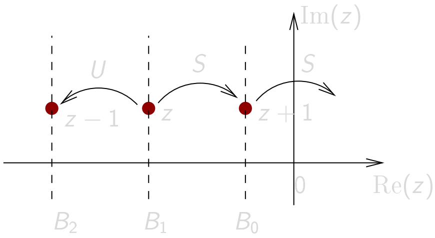

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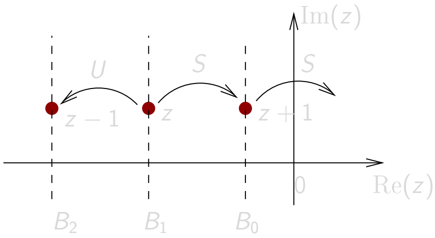

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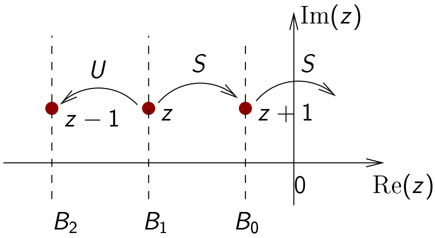

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thus $\langle u \rangle_{SO_2} \in C^\infty(\mathcal{N})$ is an **eigenfunction** of $\Delta \equiv -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$:

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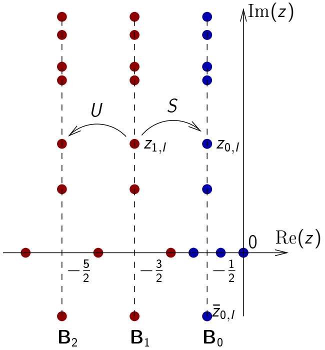

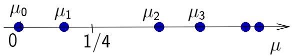

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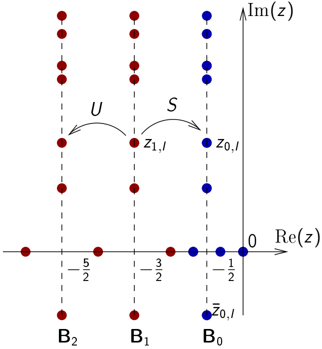

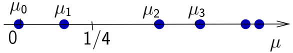

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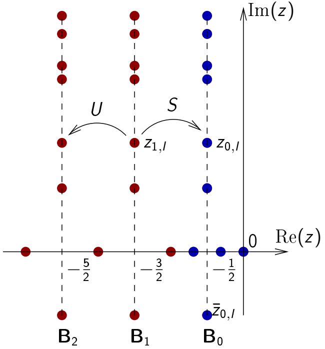

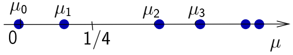

- Rem: if $\mu_1 < \frac{1}{4}$, the exponential rate for mixing is $e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\mu_1}\right)t}$.
- Rem: then the fluctuations are governed by B_0, i.e. the **spectrum of the wave operator** $\sqrt{\Delta - \frac{1}{4}}$.

Example 4, from numerical computation.

Consider the partially expanding map on $(\mathbb{R}/\mathbb{Z})_x \times \mathbb{R}_z$

$$\phi : \begin{cases} x & \to 2x \quad \mod 1 \\ z & \to z + \sin(2\pi x) \end{cases}$$

If $u(x,z) = v(x) e^{i\omega z}$ with $\omega \in \mathbb{R}$, then

$$(u \circ \phi)(x,z) = (\mathcal{L}_\omega v)(x) e^{i\omega z}$$

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$$\boxed{(\mathcal{L}_\omega v)(x) = e^{i\omega \sin(2\pi x)} v(2x).}$$

Discrete RP spectrum in $H^{-r}(S^1)$, $r \gg 1$:

$$\mathcal{L}_\omega v_{j,\omega} = e^{z_j(\omega)} v_{j,\omega}, \quad j \in \mathbb{N}, \ z_j(\omega) \in \mathbb{C}.$$

The only obvious is $z_0(0) = 0$. See **movie** of $(z_j(\omega))_j$, movie of $e^{z_j(\omega)}$, for $\omega \in \mathbb{R}$.

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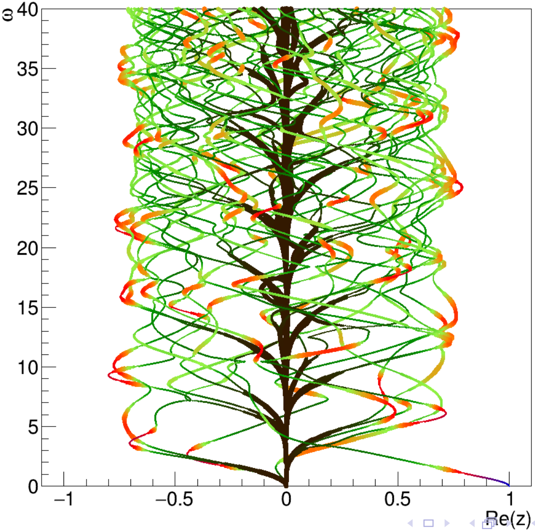
Tree of Ruelle-Pollicott resonances

For example 4, here is the spectrum of $\mathcal{L}_\omega$, $\mathrm{Re}\left(e^{z_j(\omega)}\right)$ for $j \in \mathbb{N}$ and $\omega \in [0, 40]$.
(colors are related to $\left|e^{z_j(\omega)}\right|$)



PART 2: Microlocal analysis of $X : C^\infty(M) \circlearrowleft$

- Objective: **understand the discrete spectrum of an Anosov vector field** $X = -\sum_{j=1}^{\dim M} X_j(x) \frac{\partial}{\partial x^j}$ **on** M
- We will use **wave-packet transform**, quantization, also called "FBI, wavelet, Bargmann, Anti-Wick, Wick, Toeplitz, Coherent-states" quantization. Wave-packet calculus is equivalent to the usual Weyl quantization and PDO calculus but (more) convenient for Hölder regularity.
- We will observe **"quantum scattering on a compact trapped set** E_0^***"** in T^*M. From Helffer-Sjöstrand like analysis (86), we will obtain a **discrete spectrum of "Ruelle resonances" in suitable anisotropic Sobolev spaces**.

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Wave packets

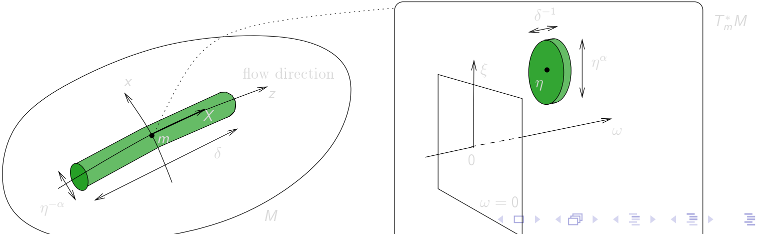
- Let $X \in C^{\infty}(M; TM)$, $X(m) \neq 0, \forall m \in M$.
- Local flow box coordinates on M: $y = (x, z) \in \mathbb{R}^n_x \times \mathbb{R}_z$ s.t. $X = \frac{\partial}{\partial z}$.
- Dual coordinates $\eta = (\xi, \omega) \in \mathbb{R}^n \times \mathbb{R}$ on $T^*_y M$ and write

$$\rho := (y, \eta) \in T^* M$$

- Let $\frac{1}{2} \leq \alpha < 1$ and $0 < \delta \ll 1$.
- **Wave packet function** (see paper) is:

$$\varphi_\rho(y') \underset{|\eta| \gg 1}{\approx} a \exp\left(i\eta.y' - \left| \frac{x' - x}{\langle \eta \rangle^{-\alpha}} \right|^2 - \left| \frac{z' - z}{\delta} \right|^2 \right),$$

$$\left\| \varphi_{(y,\eta)} \right\|_{L^2(M)} \underset{|\eta| \gg 1}{\approx} 1$$



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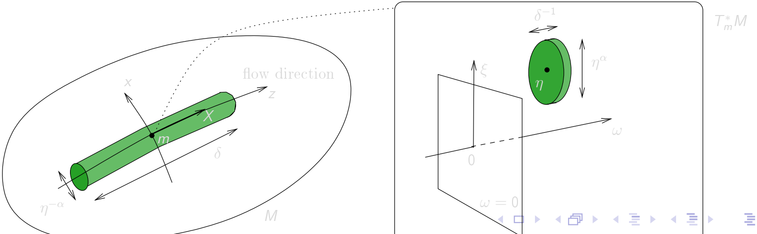

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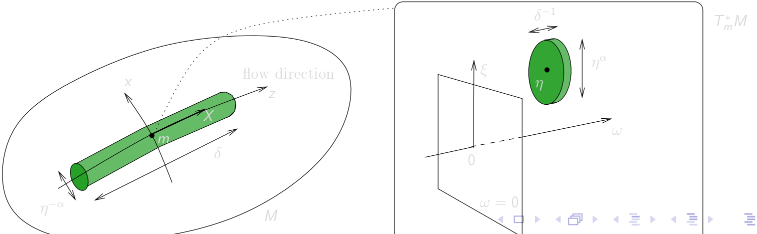

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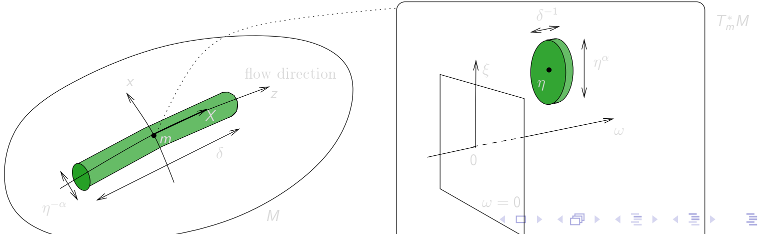

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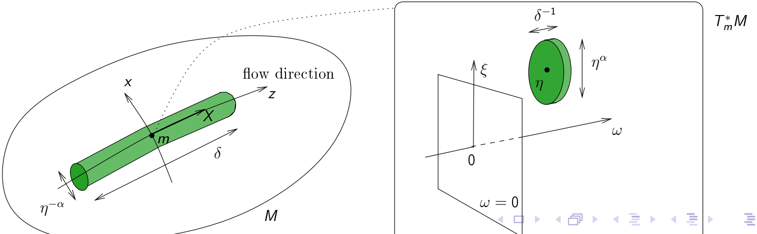

Wave packets and phase space metric g

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Lemma (Almost orthogonality)

$\forall N, \exists C_N > 0,$

$$|\langle \varphi_{\rho'} | \varphi_\rho \rangle| \leq C_N \langle \mathrm{dist}_g(\rho', \rho) \rangle^{-N}. \quad (2)$$

with the **metric** g **on** T^*M, compatible with $\Omega = dy \wedge d\eta$:

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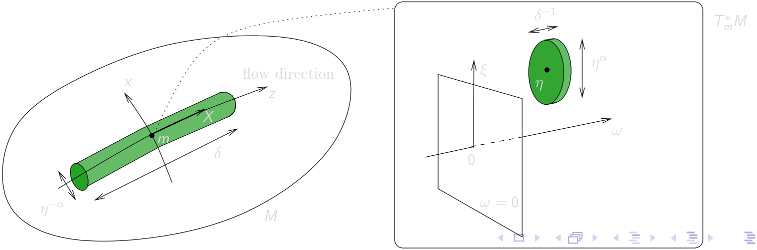

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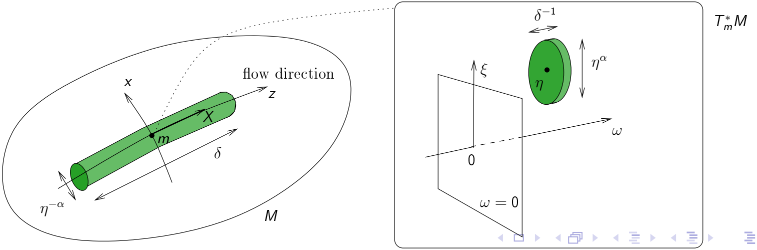

Phase space metric g

Lemma (for microlocal analysis on manifolds)

*If $\varphi : M \to M$ is a **local** "**flow-diffeomorphism**" i.e.*

$$\varphi : x' = f(x), z' = z + g(x) \Leftrightarrow X = \partial_z = \partial_{z'}$$

and $\tilde{\varphi} : T^*M \to T^*M$ is the induced map, then

$$\tilde{\varphi}^* g \underbrace{\asymp}_{\mathrm{unift}/\rho} g \Leftrightarrow \alpha \geq \frac{1}{2}.$$

Rem: Sasaki metric on T^*M,has $\alpha = 0$, is not uniformly invariant by the flow.

Phase space metric g

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Proof: Want to show $\exists C > 0, \forall \rho \in T^*M, \|D\tilde{\varphi}\|_g \leq C$.
Consider $\tilde{f}(x, \xi) = \left(f^{-1}(x), (Df_x)^* \xi\right)$.

$$D\tilde{f} \equiv \begin{pmatrix} Df_x^{-1} & 0 \\ D(Df_x)^* \xi & (Df_x)^* \end{pmatrix}$$

$$\left\|D\tilde{f}\right\|_g \leq C + C\left(\langle\xi\rangle^\alpha\right)^{-1} \xi \langle\xi\rangle^{-\alpha} \leq C \text{ if } \alpha \geq \tfrac{1}{2}.$$

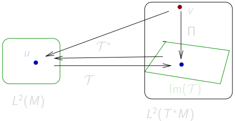
Wave packet transform (or FBI, wavelet, Bargmann, Anti-Wick... transform)

(Abuse of notations that forget charts and partitions of unity.)

$$\mathcal{T}:\begin{cases} C^{\infty}(M) & \to \mathcal{S}(T^*M) \\ u(y') & \to (\mathcal{T}u)(\rho) := \langle \varphi_\rho | u \rangle_{L^2(M)} \end{cases}$$

Lemma (fundamental 1. "Resolution of identity")

$$\mathcal{T}^* \circ \mathcal{T} = \mathrm{Id}_{C^\infty(M)}$$



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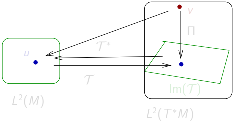

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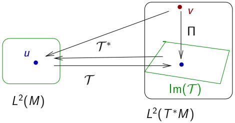

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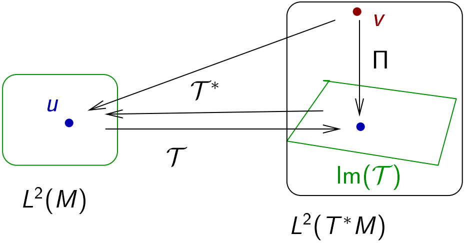

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$$\mathcal{T} : L^2(M) \to \mathrm{Im}(\mathcal{T}) \subset L^2(T^*M)$$

is an **isomorphism** that "lifts the analysis to T^*M".

- $\Pi = \mathcal{T} \circ \mathcal{T}^* : L^2(T^*M) \to \mathrm{Im}(\mathcal{T})$ is an **orthogonal projector**.

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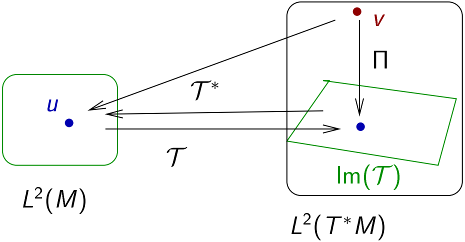

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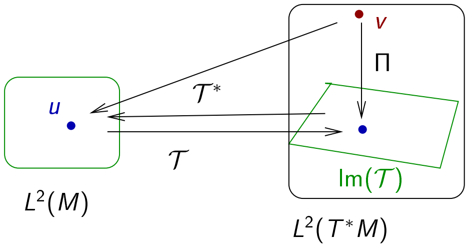

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Proof of $\mathcal{T}^*\mathcal{T} = \mathrm{Id}_{\mathcal{S}(\mathbb{R})}$ for example 1. "Bargman transform"

Let $\rho = (x, \xi) \in \mathbb{R}^2$, metric $g = dx^2 + d\xi^2$ (with $\alpha = 0$), and **wave-packet is the Gaussian function**

$$\varphi_\rho (y) = a \exp (i\xi y) \exp \left(-\frac{1}{2} |y - x|^2 \right), \quad a = \pi^{-1/4}. \tag{3}$$

Recall that $(\mathcal{T}u)(\rho) := \langle \varphi_\rho | u \rangle_{L^2(\mathbb{R})}$ and $(\mathcal{T}^* v)(y) = \int v(x, \psi) \varphi_{x,\xi}(y) \frac{dx d\xi}{2\pi}$. Then

$$\begin{aligned}
\langle \delta_{y'} | \mathcal{T}^*\mathcal{T} \delta_y \rangle &= \int_{\mathbb{R}^2} \langle \delta_{y'} | \mathcal{T}^* \delta_{x,\xi} \rangle \langle \delta_{x,\xi} | \mathcal{T} \delta_y \rangle dx d\xi \\
&= \int \varphi_{x\xi}(y') \overline{\varphi_{x\xi}(y)} \frac{dx d\xi}{2\pi} \\
&= \frac{a^2}{2\pi} \int e^{i\xi(y'-y) - \frac{1}{2}|x-y'|^2 - \frac{1}{2}|x-y|^2} dx d\xi
\end{aligned}$$

Since $\int_{\mathbb{R}} e^{i\xi(y'-y)} d\xi = (2\pi)\delta(y'-y)$ and $\int_{\mathbb{R}} e^{-|x|^2} dx = \pi^{1/2}$ we get

$$\begin{aligned}
\langle \delta_{y'} | \mathcal{T}^*\mathcal{T} \delta_y \rangle &= a^2 \delta(y'-y) \int e^{-|x-y|^2} dx \\
&= a^2 \pi^{1/2} \delta(y'-y) \\
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\end{aligned}$$

Proof of $\mathcal{T}^*\mathcal{T} = \mathrm{Id}_{\mathcal{S}(\mathbb{R})}$ for example 1. "Bargman transform"

Let $\rho = (x, \xi) \in \mathbb{R}^2$, metric $g = dx^2 + d\xi^2$ (with $\alpha = 0$), and **wave-packet is the Gaussian function**

$$\varphi_\rho(y) = a \exp(i\xi y) \exp\left(-\frac{1}{2}|y-x|^2\right), \quad a = \pi^{-1/4}. \tag{3}$$

Recall that $(\mathcal{T}u)(\rho) := \langle \varphi_\rho | u \rangle_{L^2(\mathbb{R})}$ and $(\mathcal{T}^* v)(y) = \int v(x, \psi) \varphi_{x,\xi}(y) \frac{dx d\xi}{2\pi}$.

Then

$$\begin{aligned} \langle \delta_{y'} | \mathcal{T}^*\mathcal{T}\delta_y \rangle &= \int_{\mathbb{R}^2} \langle \delta_{y'} | \mathcal{T}^* \delta_{x,\xi} \rangle \langle \delta_{x,\xi} | \mathcal{T}\delta_y \rangle dx d\xi \\ &= \int \varphi_{x\xi}(y') \overline{\varphi_{x\xi}(y)} \frac{dx d\xi}{2\pi} \\ &= \frac{a^2}{2\pi} \int e^{i\xi(y'-y) - \frac{1}{2}|x-y'|^2 - \frac{1}{2}|x-y|^2} dx d\xi \end{aligned}$$

Since $\int_{\mathbb{R}} e^{i\xi(y'-y)} d\xi = (2\pi)\delta(y'-y)$ and $\int_{\mathbb{R}} e^{-|x|^2} dx = \pi^{1/2}$ we get

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Trace formula, "uncertainty principle"

Let $A \subset T^*M$, measurable set and χ_A its characteristic function. Let

$$\mathrm{Op}\left(\chi_A\right) := \mathcal{T}^* \chi_A \mathcal{T} \qquad : L^2\left(M\right) \to L^2\left(M\right)$$

called "**Toeplitz quantization of** χ_A" that restricts functions to "their components on A".

One has $\left\|\mathrm{Op}\left(\chi_A\right)\right\|_{L^2} \leq 1$, $\mathrm{Op}\left(\chi_A\right) \geq 0$ and

$$\begin{aligned} \mathrm{Tr}\left(\mathrm{Op}\left(\chi_A\right)\right) = \mathrm{Tr}\left(\mathcal{T}\mathcal{T}^*\chi_A\right) &= \int_{T^*M} \langle \delta_\rho | \mathcal{T}\mathcal{T}^* \delta_\rho \rangle \chi_A\left(\rho\right) d\rho \\ &= \int_{T^*M} \underbrace{\langle \varphi_\rho | \varphi_\rho \rangle}_{\approx 1 \text{ if } |\eta| \gg 1} \frac{1}{(2\pi)^{\dim M}} \chi_A\left(\rho\right) d\rho \\ &\approx \frac{1}{(2\pi)^{\dim M}} \mathrm{Vol}\left(A\right). \end{aligned}$$

Interpretation: the number of independent functions $u \in C^\infty\left(M\right)$ that "can live" on $A \subset T^*M$ is $\approx \frac{1}{(2\pi)^{\dim M}} \mathrm{Vol}\left(A\right)$.

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Propagation of singularities

Consider the lifted transfer operator:

$$\begin{array}{ccc} L^2(M) & \xrightarrow{e^{tX}} & L^2(M) \\ \mathcal{T}\downarrow & & \mathcal{T}\downarrow \\ L^2(T^*M) & \xrightarrow{\mathcal{T}e^{tX}\mathcal{T}^*} & L^2(T^*M) \end{array}$$

The flow ϕ^t induces the flow $\tilde{\phi}^t(y,\eta) = \left(\phi^{-t}(y), (D\phi^t)^*_y \eta\right)$ on T^*M.

Theorem (fundamental 2. "Propagation of singularities", for any vector field X)

$\forall t \geq 0, \forall N > 0, \exists C_{N,t} > 0, \forall \rho, \rho' \in T^*M,$

$$\left|\langle \delta_{\rho'} | \mathcal{T} e^{tX} \mathcal{T}^* \delta_\rho \rangle_{L^2(T^*M)}\right| = \left|\langle \varphi_{\rho'} | e^{tX} \varphi_\rho \rangle_{L^2(M)}\right| \leq C_{N,t} \left\langle \mathrm{dist}_g\left(\rho', \tilde{\phi}^t(\rho)\right)\right\rangle^{-N} \quad (4)$$

- i.e. e^{tX} is a **Fourier Integral Operator** (FIO). This is **the fundamental property** for microlocal analysis, i.e. e^{tX} is a quasi orthogonal sum of finite rank operators parametrized by T^*M, but for a metric g well chosen.
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Proof of propagation of singularities for example 1

Recall $X = -x\partial_x$ on $\mathbb{R}$. Lifted flow is $\tilde{\phi}^t(x,\xi) = (e^t x, e^{-t}\xi)$.
Metric $g = dx^2 + d\xi^2$.
We will show that $\forall N > 0, \exists C_N, \forall t \in \mathbb{R}, \forall \rho, \rho' \in T^*\mathbb{R}$,

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Objective: understand $\tilde{\phi}^t : T^*M \to T^*M$

Recall: for X Anosov vector field,

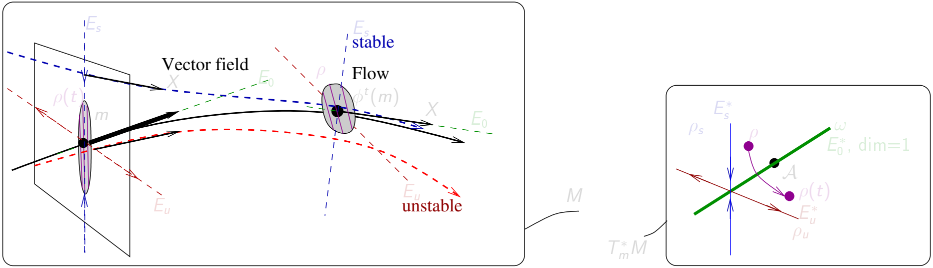
$$TM = E_u \oplus E_s \oplus \underbrace{E_0}_{\mathbb{R}X}$$

Let

$$E_0^* := (E_u \oplus E_s)^\perp = \{\eta \in T^*M, \text{ s.t. } \mathrm{Ker}(\eta) \supset (E_u \oplus E_s)\} = \mathbb{R}\mathcal{A}$$

with Anosov one-form $\mathcal{A}(X) = 1, \mathrm{Ker}\mathcal{A} = E_u \oplus E_s$ that is C^{β_0}.

$$E_s^* := (E_u \oplus E_0)^\perp, \qquad E_u^* := (E_s \oplus E_0)^\perp.$$



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$$E_0^* := (E_u \oplus E_s)^\perp = \{\eta \in T^*M, \text{ s.t. } \mathrm{Ker}(\eta) \supset (E_u \oplus E_s)\} = \mathbb{R}\mathcal{A}$$

with Anosov one-form $\mathcal{A}(X) = 1, \mathrm{Ker}\mathcal{A} = E_u \oplus E_s$ that is C^{β_0}.

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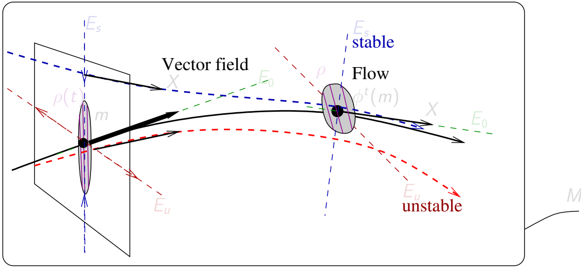

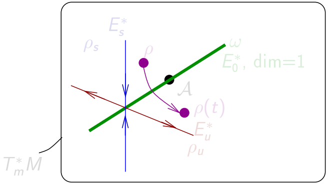

Objective: understand $\tilde{\phi}^t : T^*M \to T^*M$

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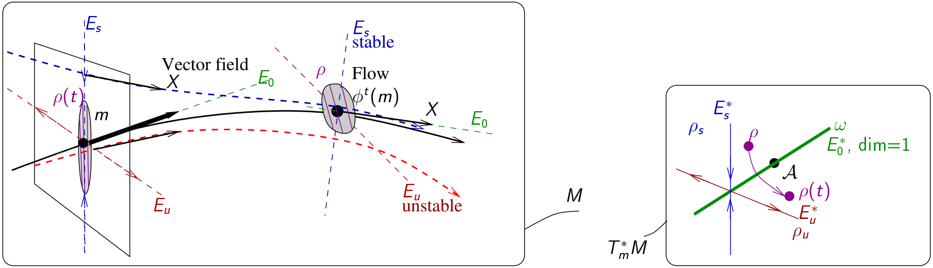


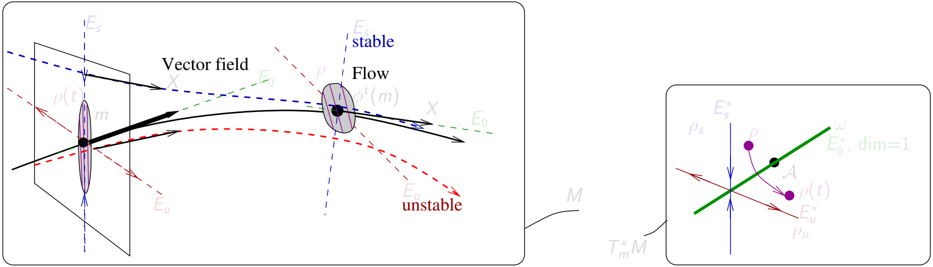
Scattering in T^*M on the trapped set E_0^*

$$T^*M = E_u^* \oplus E_s^* \oplus E_0^*$$

$$\rho = \rho_u + \rho_s + \omega \mathcal{A}(m) \in T^*M, \quad \omega \in \mathbb{R} \text{ frequency.}$$

Then for $\rho(t) := \tilde{\phi}^t(\rho)$,

$$|\rho_u(t)| \geq \frac{1}{C} e^{\lambda t} |\rho_u(0)|, \quad |\rho_s(t)| \leq C e^{-\lambda t} |\rho_s(0)|, \qquad \omega(t) = \omega \text{ preserved}$$



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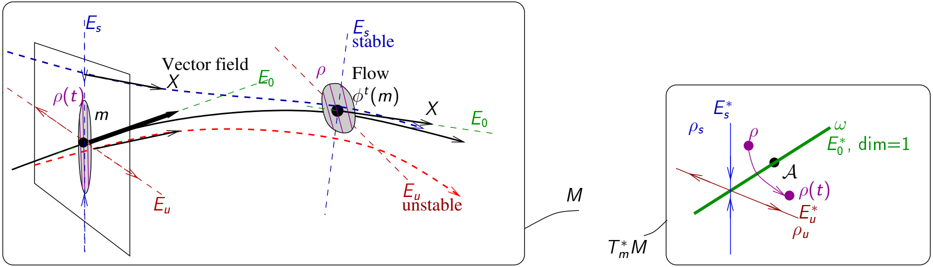

Weight (escape) function W in T^*M

Let $\alpha \geq \frac{1}{1+\beta_0}$ so that "the metric absorbs the β_0-Hölder fluctuations of E_0^*" (explained later..).

Notation: $\langle x \rangle := |x|$ if $|x| \geq 1$, otherwise $\langle x \rangle = 1$.

Let $R > 0$, $\rho \in T^*M$,

$$W(\rho) := \frac{\left\langle h_\gamma(\rho) \|\rho_s\|_g \right\rangle^R}{\left\langle h_\gamma(\rho) \|\rho_u\|_g \right\rangle^R}$$

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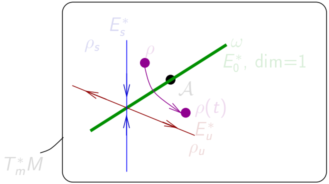
$$h_\gamma(\rho) = h_0 \left\langle \|\rho_u + \rho_s\|_g \right\rangle^{-\gamma}$$

$1 - \frac{\alpha \min(\beta_u, \beta_s)}{1-\alpha} \leq \gamma < 1$, $h_0 > 0$.



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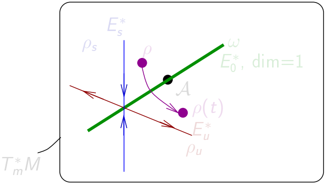

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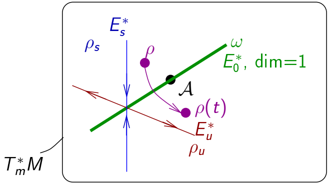

Properties of the escape function W in T^*M

Proposition

- *If* $R \geq 0$, $W(\rho) = \frac{\langle h_\gamma(\rho)\|\rho_s\|_g\rangle^R}{\langle h_\gamma(\rho)\|\rho_u\|_g\rangle^R}$ ***decays outside*** E_0^**:*

$$\forall t \geq 0, \quad \frac{1}{C} e^{-(\lambda_{\max} r)t} \leq \frac{W\left(\tilde{\phi}^t(\rho)\right)}{W(\rho)} \leq \begin{cases} C \\ Ce^{-(\lambda r)t} & \text{if } \operatorname{dist}_g(\rho, E_0^*) > C_t \end{cases} \tag{6}$$

with $r = R(1-\gamma)(1-\alpha)$ *(the order of* W*) i.e.* $C^{-1}\langle\rho\rangle^{-|r|} \leq W(\rho) \leq C\langle\rho\rangle^{|r|}$.

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Recall $X = -x\partial_x$ on $\mathbb{R}$. Lifted flow is $\tilde{\phi}^t(x,\xi) = (e^t x, e^{-t}\xi)$.

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$$\leq e^{-rt} \text{ if } |x| \geq h_0^{-1} \text{ or } |\xi| \geq h_0^{-1} e^t \qquad (7)$$

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Anisotropic Sobolev space $\mathcal{H}_W(M)$

Definition

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i.e. we have isometries

$$\mathcal{H}_W(M) \xrightarrow{\mathcal{T}} L^2(T^*M; W^2) \xrightarrow{W} L^2(T^*M)$$

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Good choice of symplectic boxes $\alpha \geq \frac{1}{1+\beta_0}$ for the phase space metric g

Recall, (for the trapped set $|\eta| \sim |\omega| \gg 1$):

$$g_\rho = \left(\frac{dx}{\langle\omega\rangle^{-\alpha}}\right)^2 + \left(\frac{dz}{\delta}\right)^2 + \left(\frac{d\xi}{\langle\omega\rangle^{\alpha}}\right)^2 + \left(\frac{d\omega}{\delta^{-1}}\right)^2$$

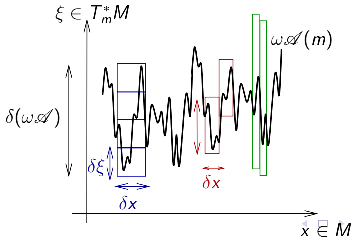
The trapped set is $E_0^* = \mathbb{R}_\omega\mathcal{A}$, with $\mathcal{A}$ that is β_0-Hölder. Hence variation $\delta x := \langle\omega\rangle^{-\alpha}$ gives $\delta(\omega\mathcal{A}) = \omega(\delta x)^{\beta_0} = \omega^{1-\alpha\beta_0}$.

For **temperate property** of W one requires

$$\delta(\omega\mathcal{A}) \leq \delta\xi := \langle\omega\rangle^{\alpha} \Leftrightarrow \omega^{1-\alpha\beta_0} \leq \omega^{\alpha}$$

$$\Leftrightarrow 1-\alpha\beta_0 \leq \alpha$$

$$\Leftrightarrow \alpha \geq \frac{1}{1+\beta_0}$$



Remark about Hörmander classes of symbols $S^m_{\rho,\delta}$

In transverse direction (x,ξ), after smoothing,

$$W \in S^m_{\rho,\delta} \quad \overset{\text{def}}{\Leftrightarrow} \quad \left|\partial_\xi^a \partial_x^b W\right| \leq C_{a,b} \langle\xi\rangle^{m-\rho|a|+\delta|b|}$$

with

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Ref: **Hörmander's PDE book Vol.3**, or **Lerner's book** p.68, with a conformal "phase's space metric"

$$\begin{aligned}
\tilde{g}_\rho &:= h_\gamma^2(\rho)\, g_\rho \\
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PART 3: Some results

Versions of the next Theorem about **discrete RP spectrum** have already been obtained

- For **Anosov diffeomorphisms**, by Blank, Keller, Liverani 2002, Gouëzel, Liverani 2005, Baladi, Tsujii 2005,2008, F-Roy-Sjöstrand 2008.
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Question: discrete spectrum of the generator X_F?

- **Imagine:** if $X_F = \sum_{j=1}^{N} z_j \Pi_j$, were a **matrix** with complex **eigenvalues** $z_j = a_j + i\omega_j$ and eigen-projectors (rank 1) Π_j then

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$$\underset{t\to+\infty}{\sim} e^{ta_0} e^{it\omega_0}\Pi_0 + .. \quad : \text{if } a_0 > a_{j\neq 0}, \quad : \text{"futur emerging behavior"}$$

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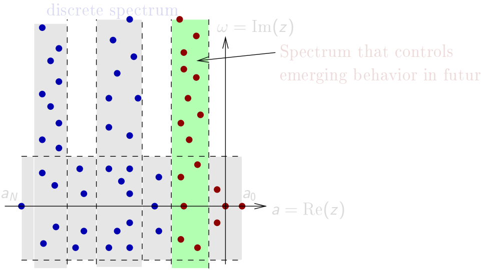

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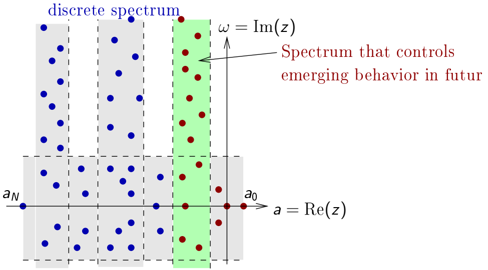

Meaning of this internal spectrum? Illustration

- In first approximation, the sea is quite, deep, flat, gentle $\equiv$ **Equilibrium state** z_0, z_N**, dominant.**

- But the behavior at the surface may be furious $\equiv$**internal spectrum** $(z_j)_j$

Notations

Consider **Anosov vector field** X on M and **potential** $V \in C^{\infty}(M)$,

$$X_F := X + V$$

(or more general $X_F : C^{\infty}(M; F) \circlearrowleft$).

$$C_{X,V} := \overline{\max}\left(\frac{1}{2}\mathrm{div}X + \mathrm{Re}(V)\right) \tag{9}$$

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Ruelle Pollicott discrete spectrum of resonances

Recall weight function $W(\rho) = \frac{\langle h_\gamma(\rho) \|\rho_s\|_g \rangle^R}{\langle h_\gamma(\rho) \|\rho_u\|_g \rangle^R}$ has order $r = R(1-\gamma)(1-\alpha)$, i.e. $C^{-1} \langle \rho \rangle^{-|r|} \leq W(\rho) \leq C \langle \rho \rangle^{|r|}$ and $h_\gamma(\rho) = h_0 \left\langle \|\rho_u + \rho_s\|_g \right\rangle^{-\gamma}$.

Theorem (F.-Tsujii 17)

$X_F := X + V$. The family of operators

$$e^{tX_F} : \mathcal{H}_W(M) \to \mathcal{H}_W(M), t \in \mathbb{R}$$

form a ***strongly continuous group****: for $t \geq 0$,*

$$\left\| e^{tX_F} \right\|_{\mathcal{H}_W} \leq C e^{t(C_{X,V} + \epsilon + \max(0, -2r\lambda))}$$

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X_F has ***"future" RP discrete spectrum*** *on $C_{X,V} - r\lambda + \epsilon \leq \mathrm{Re}(z) \leq C_{X,V}$, and* ***past RP discrete spectrum*** *on $C'_{X,V} \leq \mathrm{Re}(z) \leq C'_{X,V} - r\lambda_{\max} - \epsilon$.*

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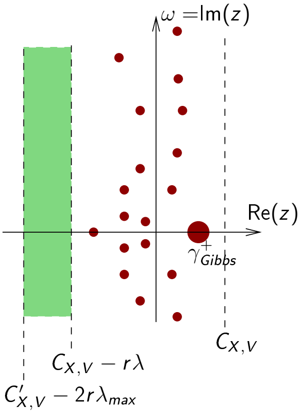

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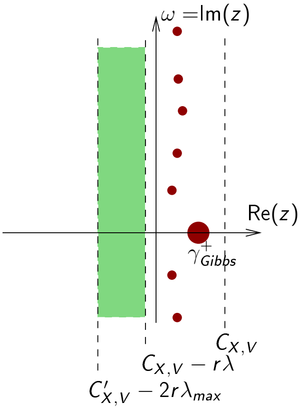
Ruelle-Pollicott spectrum $\sigma_+(X_F)$ of the future dynamics.



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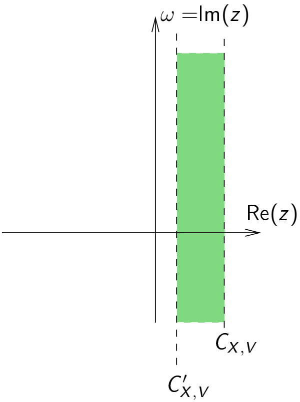
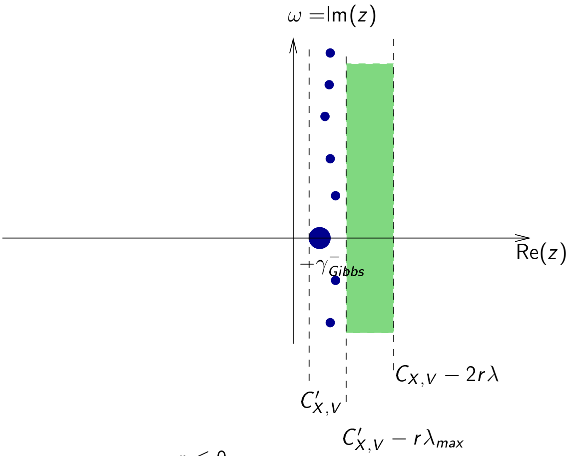


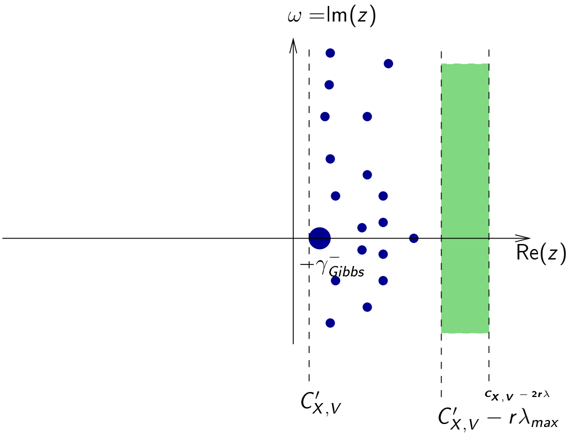
$r = 0$

Spectrum of X_F in $\mathcal{H}_W(M) = L^2(M)$



$r \leq 0$

Ruelle-Pollicott spectrum $\sigma_-(X_F)$
of the past dynamics.



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Remark on past/future spectrum

- If $V' + \overline{V} + \mathrm{div} X = 0$ then the **future** spectrum λ_j^+ of $A = -X + V$ is related to the **past** spectrum $\lambda_j'^-$ of $A' = -X + V'$ by

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Fractal Weyl law

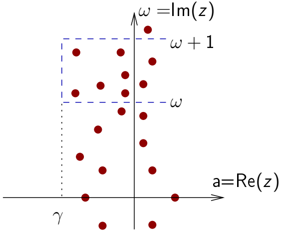
(after J. Sjöstrand 90 for quantum resonances)

Theorem (F.-Tsujii 17. "**Upper bound for the density of eigenvalues**")

$\forall \gamma \in \mathbb{R}, \exists C > 0, \forall \omega \geq 1,$

$$\sharp \left\{ z \in \sigma(X) ; \quad \mathrm{Re}(z) > \gamma, \ \mathrm{Im}(z) \in [\omega, \omega + 1] \right\} \leq C \left| \omega \right|^{\frac{\dim \boldsymbol{M} - 1}{1 + \beta_0}} .$$

with $\beta_0 \in]0, 1]$ *is Hölder exponent of* $E_u \oplus E_s$.



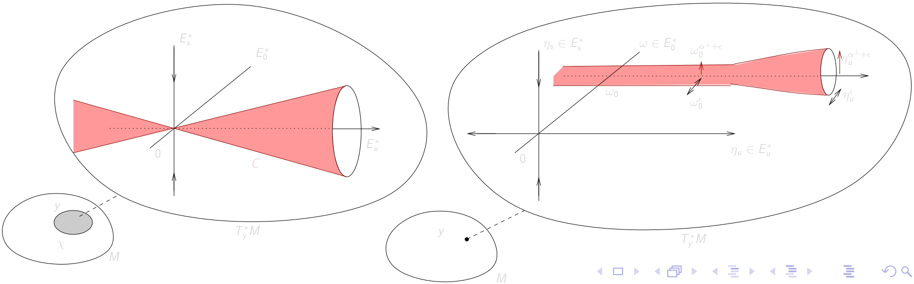
About the wave front set of resonances

Theorem (F.-Tsujii 17. "Parabolic wave front set")

$\forall C, N, \epsilon, \exists C_N$, for any (generalized) Ruelle Pollicott eigenfunction u with $\mathrm{Re}(z) > -C$ then $\forall (y, \eta) \in T^*M$,

$$|(\mathcal{T}u)(y,\eta)| \leq \frac{C_N}{\left\langle |\eta|^{-\epsilon} \mathrm{dist}_g \left(\rho, E_u^* + \underbrace{\mathrm{Im}(z)}_{\omega_0} \mathscr{A} \right) \right\rangle^N} \|u\|_{\mathcal{H}_W(M)}$$

We choose $\alpha = \frac{1}{1+\min(\beta_u, \beta_s)}$ (but expect $\alpha = \frac{1}{1+\beta_u}$) so that **uncertainty principle absorbs Hölder fluctuations**.



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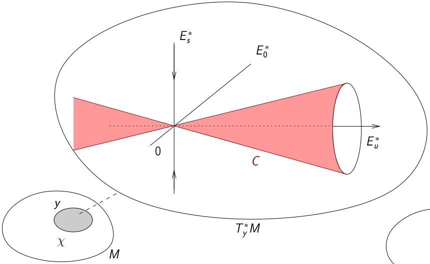

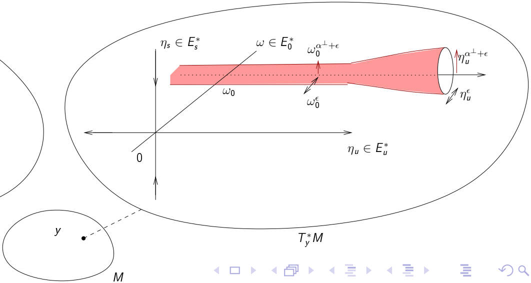

Sketch of proof for group property (1)

Recall the isometries:

$$\mathcal{H}_W(M) \overset{\mathcal{T}}{\to} L^2\left(T^*M; W^2\right) \overset{W}{\to} L^2\left(T^*M\right)$$

The strategy of microlocal analysis is to analyse the conjugated operator $W\mathcal{T}e^{tX}\mathcal{T}^*W^{-1}$ in $L^2\left(T^*M\right)$ instead of e^{tX}:

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Sketch of proof for "discrete RP spectrum" and Weyl law

Recall decay property (6): $\frac{W\left(\tilde{\phi}^{t}(\rho)\right)}{W(\rho)} \leq Ce^{-(\lambda r)t}$ if $\mathrm{dist}_g(\rho, E_0^*) \geq \sigma \gg 1$. Let $\chi_\sigma(\rho)$ characteristic function for $\mathrm{dist}_g(\rho, E_0^*) \geq \sigma$.

If we repeat the previous argument, the **norm decays very fast far from the trapped set**:

$$\left\| W\mathcal{T}e^{tX}\mathcal{T}^* W^{-1}\chi_\sigma \right\|_{L^2(T^*M)} \leq Ce^{(C_{X,V} - \lambda r + o(1))t}.$$

Near the trapped set E_0^*, let

$$A := \{\rho \in T^*M, \quad \mathrm{dist}_g(\rho, E_0^*) \leq \sigma, \quad \omega(\rho) \in [\omega, \omega+1]\}.$$

Using **trace formula** (and Jensen inequality..), the **density of RP spectrum** is then

$$\begin{aligned} \sharp\{z \in \sigma(X), \quad \mathrm{Im}(z) \in [\omega, \omega+1], \mathrm{Re}(z) \geq -\gamma\} &\lesssim \mathrm{Tr}(\mathcal{T}^*\chi_A\mathcal{T}) \\ &\asymp \mathrm{Vol}(A) \\ &\asymp (\sigma\omega^\alpha)^{\dim M - 1} \\ &\asymp \omega^{\frac{\dim M - 1}{1+\beta_0}} \end{aligned}$$

Sketch of proof for "discrete RP spectrum" and Weyl law

Recall decay property (6): $\frac{W(\tilde{\phi}^t(\rho))}{W(\rho)} \leq Ce^{-(\lambda r)t}$ if $\mathrm{dist}_g(\rho, E_0^*) \geq \sigma \gg 1$. Let $\chi_\sigma(\rho)$ characteristic function for $\mathrm{dist}_g(\rho, E_0^*) \geq \sigma$.

If we repeat the previous argument, the **norm decays very fast far from the trapped set**:

$$\left\| W \mathcal{T} e^{tX} \mathcal{T}^* W^{-1} \chi_\sigma \right\|_{L^2(T^*M)} \leq Ce^{(C_{X,V} - \lambda r + o(1))t}.$$

Near the trapped set E_0^*, let

$$A := \{\rho \in T^*M, \quad \mathrm{dist}_g(\rho, E_0^*) \leq \sigma, \quad \omega(\rho) \in [\omega, \omega + 1]\}.$$

Using **trace formula** (and Jensen inequality..), the **density of RP spectrum** is then

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Thank you for your attention