Microlocal analysis of Anosov geodesic flow

F. Faure (Grenoble) with M. Tsujii (Kyushu),

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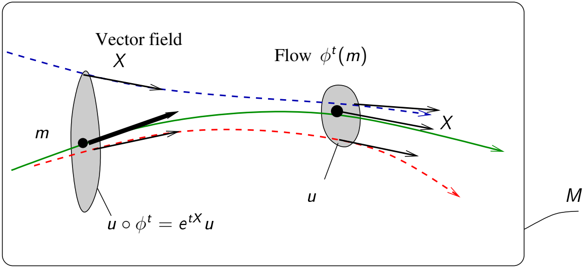
Deterministic dynamics: vector field X and flow ϕ^t



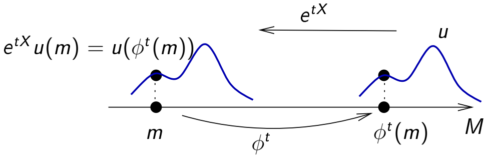
On a closed manifold M, let X be a C^∞ **vector field** that determines a **flow map:**

$$\phi^t : \begin{cases} M & \rightarrow M \\ m & \rightarrow \phi^t(m) \end{cases}, \qquad t \in \mathbb{R},$$

by

$$\left(\frac{d\phi^{t'}}{dt'}(m) \right)_{t'=t} = X\left(\phi^t(m)\right), \qquad \phi^{t=0}(m) = m.$$

Evolution of distributions on M by transfer operators e^{tX}



Pull-back action of the flow on functions $u \in C^\infty(M)$ is $u \circ \phi^t$.

Since $\frac{d(u \circ \phi^t)}{dt} = Xu$ with $X = \sum_{j=1}^{\dim M} X_j(m) \frac{\partial}{\partial m_j}$, we get

$$\boxed{e^{tX}u := u \circ \phi^t.}$$

- Remark: $e^{tX}\mathbf{1} = \mathbf{1}, X\mathbf{1} = 0$ hence the adjoint $\left(e^{tX}\right)^*$ called the **Perron Frobenius operator** is

$$\left(e^{tX}\right)^* u = \left|\det D\phi^{-t}\right| . u \circ \phi^{-t}$$

pushes forward probability distributions:

$$\int_M \left(e^{tX}\right)^* u d\mu = \langle 1 | \left(e^{tX}\right)^* u \rangle_{L^2} = \langle e^{tX} 1 | u \rangle_{L^2} = \langle 1 | u \rangle_{L^2} = \int_M u d\mu.$$

In particular and $e^{tX^*}\delta_m = \delta_{\phi^t(m)}$.

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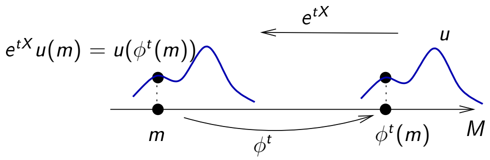

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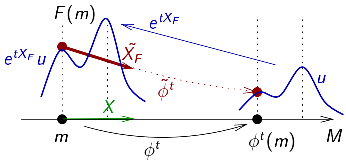
More general evolution of sections of $F \to M$



Let $F \to M$ a **vector bundle** and $\tilde{\phi}_F^t = e^{t\tilde{X}_F} : F \to F$ be smooth, linear, bundle map, extension of $\phi^t : M \to M$.

Definition

For a **section** $u \in C^\infty(M; F)$,

$$e^{tX_F} u := \tilde{\phi}_F^{-t}\left(u \circ \phi^t\right).$$

- Example: for tensor bundle $F = TM \otimes \ldots \otimes T^*M$, $\tilde{\phi}_F^t$ is determined by the differential $D\phi^t$ and X_F **is the Lie derivative**.

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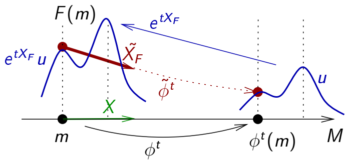

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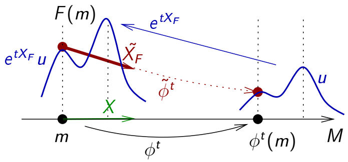

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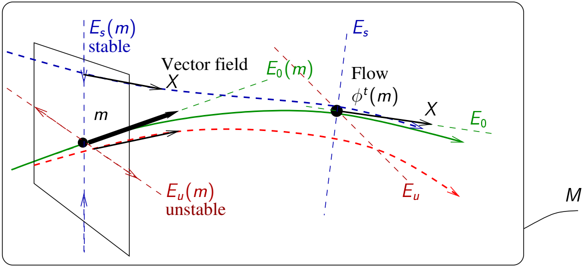
Anosov flow (or uniformly hyperbolic flow)



Definition

Vector field X is **Anosov** if there exists an invariant, **Hölder** continuous splitting, $\forall m \in M, \quad T_m M = E_u(m) \oplus E_s(m) \oplus \underbrace{E_0(m)}_{\mathbb{R}X}$, s.t.

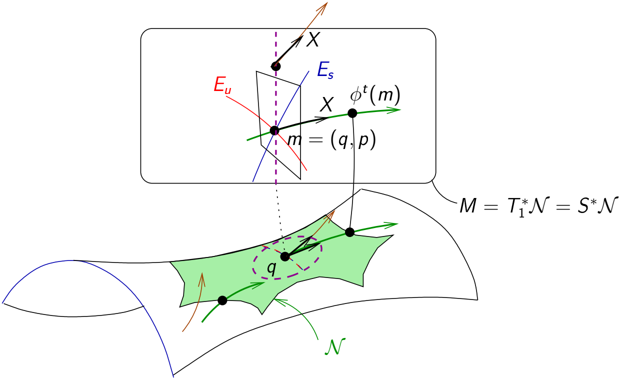
$\exists g, \exists C > 0, \lambda > 0, \forall t \geq 0, m \in M,$

$$\left\| D\phi^t_{/E_s(m)} \right\|_g \leq Ce^{-\lambda t}, \quad \left\| D\phi^{-t}_{/E_u(m)} \right\|_g \leq Ce^{-\lambda t},$$

this **"sensitivity to initial conditions"** generates "**chaos**" (confusion, unpredictability).

Special example: Anosov geodesic flow

Let $(\mathcal{N}, g)$ a closed Riemannian manifold, $\dim \mathcal{N} = d + 1$. Let $\mathcal{A} = \sum_j p_j dq_j$ be the canonical Liouville one form on phase space $T^*\mathcal{N}$.



- **Def:** The **Geodesic vector field** X on $M = (T^*\mathcal{N})_1$, $\dim M = 2d + 1$, is defined by Hamilton equation of motion of a free particle $(H(q;p) = \|p\|_{g(q)})$: $d\mathcal{A}(X, .) = 0$, $\mathcal{A}(X) = 1$.
- **Thm (Anosov 67):** if $(\mathcal{N}, g)$ **has negative curvature** then X **is Anosov** with $\dim E_u = \dim E_s = d$.

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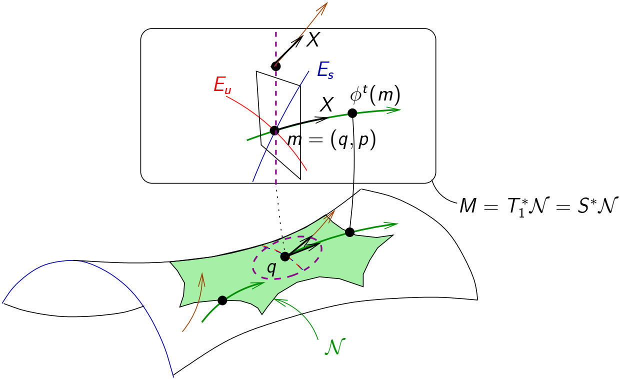

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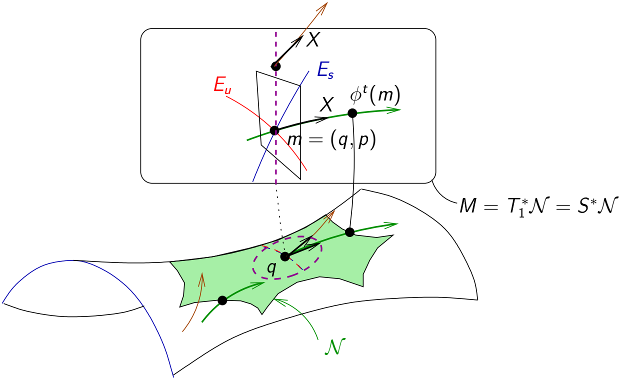

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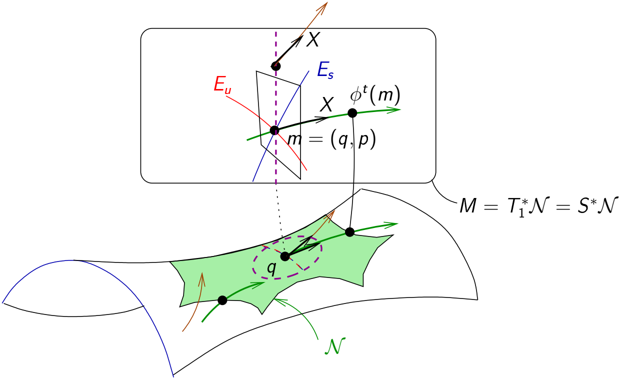

- "**special**" because $E_u \oplus E_s = \mathrm{Ker}\mathcal{A}$ is C^∞ and $d\mathcal{A}-$**symplectic** (maximally non integrable).
- Ex: surface of constant curvature $\mathcal{N} = \Gamma \setminus \underbrace{(\mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2)}_{\mathbb{D}^2}$ $\quad M = (T^*\mathcal{N})_1 = \Gamma\setminus\mathrm{SL}_2(\mathbb{R})$,

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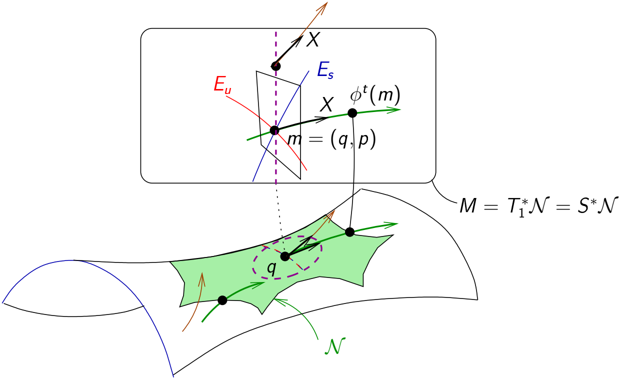

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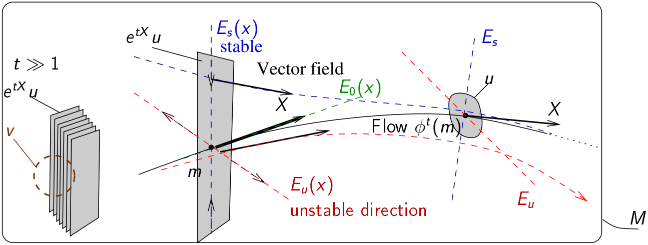
Dynamical correlation functions

Observe that for $t \to +\infty$, $e^{tX}u = u \circ \phi^t$ gets **high oscillations along** $E_u^* = (E_s \oplus E_0)^\perp$, i.e. informations goes towards microscopic scales.



- Rem: this is **reversible**: $e^{-tX}e^{tX}u = u$.
- **Objective:** describe $e^{tX}u$ in the "weak sense" i.e. for $u, v \in C^\infty(M)$ describe "**dynamical correlation functions**":

$$\langle v | e^{tX}u \rangle_{L^2} \underset{t \gg 1}{=} ?$$

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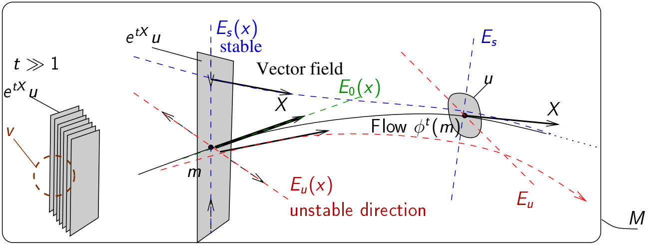

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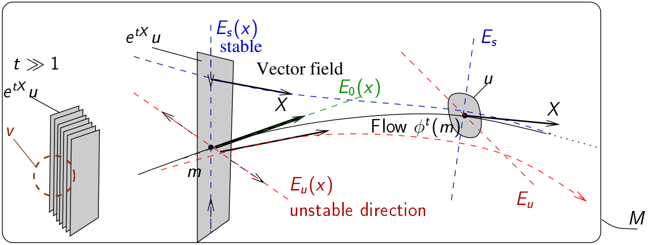

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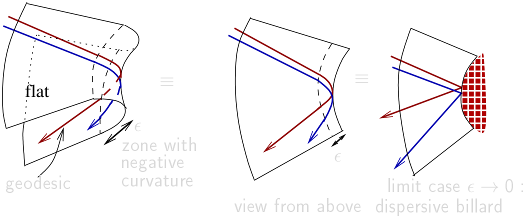

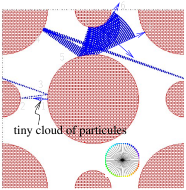

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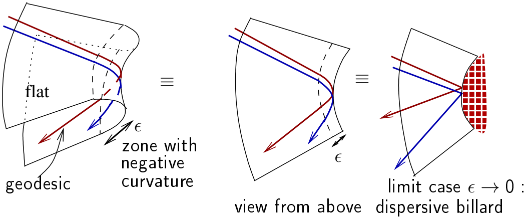

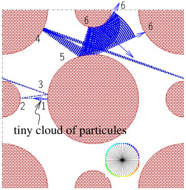

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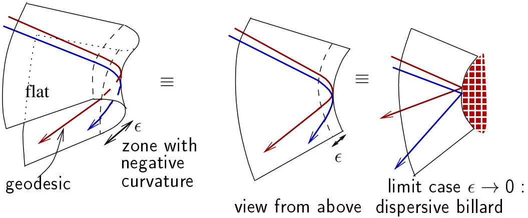

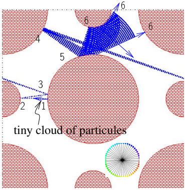

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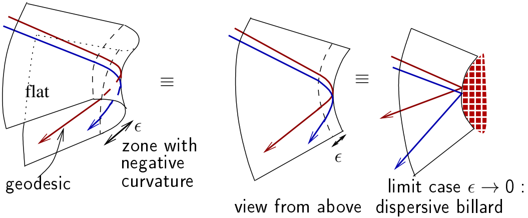

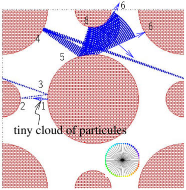

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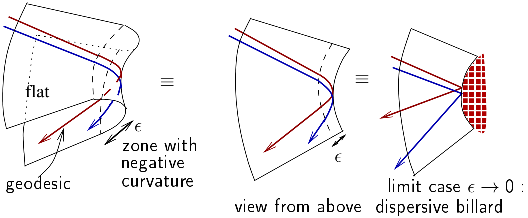

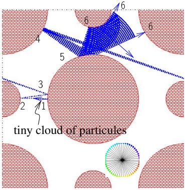

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Question: discrete spectrum of the generator X_F?

- **Imagine:** if $X_F = \sum_{j=1}^{N} z_j \Pi_j$, were a **matrix** with complex **eigenvalues** $z_j = a_j + i\omega_j$ and eigen-projectors (rank 1) Π_j then

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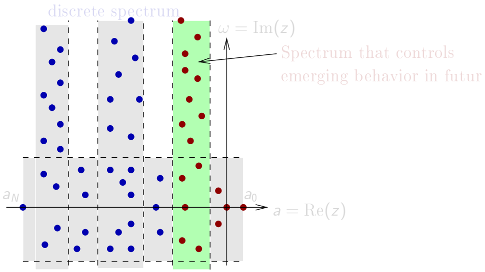

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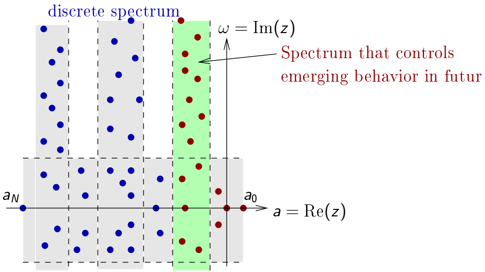

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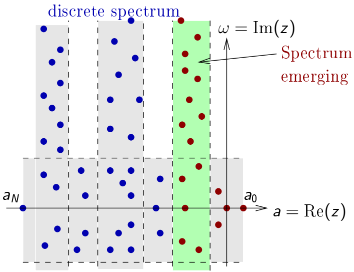

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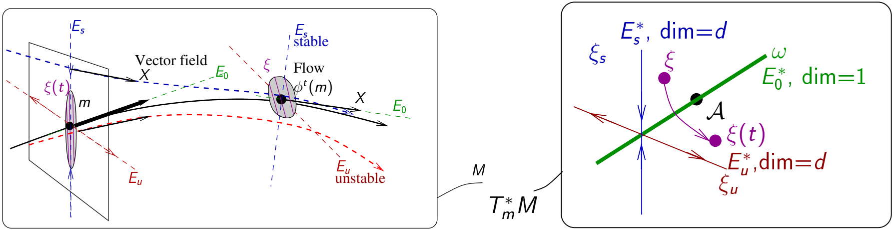
- **Series of work and interesting recent activity:** Ruelle, Bowen 70', Pollicott 86, Rugh 90', Blank, Keller, Liverani 2002, Gouëzel, Liverani 2005, Baladi, Tsujii 2005,2008, Butterley-Liverani 2007, Roy-Sjöstrand-F. 2008, Datchev-Dyatlov-Zworski, Dyatlov-Guillarmou 2014 for Axiom A flows. Dang-Rivière 2016 for Morse-Smale flows, Bonthonneau-Weich 2017 for cusps, Guillarmou-Hilgert-Weich 2018 for symmetric spaces.

Discrete spectrum of the generator X_F: "Pollicott-Ruelle resonances"

- We are using **micro-local analysis**, similarly to the approaches of Combes 70' (dilatation method), Helffer-Sjöstrand 86 (**escape functions, resonances as eigenvalues, quantum scattering on phase space**), Melrose 80',90'(scattering, radial estimates).
- **Series of work and interesting recent activity:** Ruelle, Bowen 70', Pollicott 86, Rugh 90', Blank, Keller, Liverani 2002, Gouëzel, Liverani 2005, Baladi, Tsujii 2005,2008, Butterley-Liverani 2007, Roy-Sjöstrand-F. 2008, Datchev-Dyatlov-Zworski, Dyatlov-Guillarmou 2014 for Axiom A flows. Dang-Rivière 2016 for Morse-Smale flows, Bonthonneau-Weich 2017 for cusps, Guillarmou-Hilgert-Weich 2018 for symmetric spaces.

Dual Anosov decomposition of T^*M.



We have $TM = E_u \oplus E_s \oplus E_0$.
Let $E_0^* := (E_u \oplus E_s)^\perp = \mathbb{R}\mathcal{A}$, $E_s^* := (E_u \oplus E_0)^\perp$, $E_u^* := (E_s \oplus E_0)^\perp$,

$$T^*M = E_u^* \oplus E_s^* \oplus E_0^*$$

$$\xi = \xi_u + \xi_s + (\omega\mathcal{A}) \in T^*M, \quad \omega \in \mathbb{R}.$$

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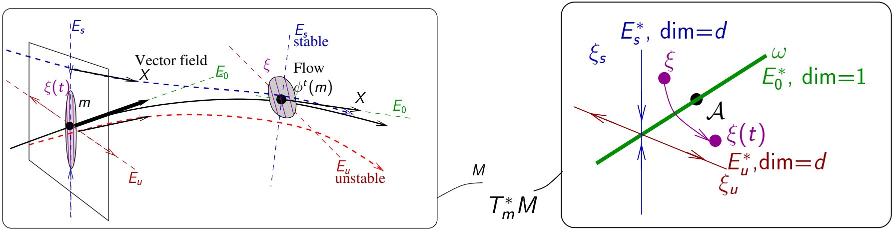

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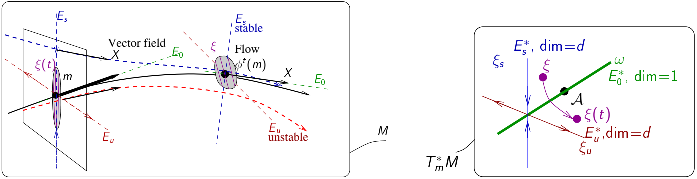
Weight W on T^*M. Anisotropic Sobolev space $\mathcal{H}_W(M)$



(Notation: $\langle x\rangle := |x|$ if $|x| \geq 1$, otherwise $\langle x\rangle = 1$.) Let $h_0 \ll 1$, $m \gg 1$,

$$W(\xi) := \frac{\langle h_0 |\xi_s|\rangle^m}{\langle h_0 |\xi_u|\rangle^m}, \qquad \Rightarrow \text{decay} : \frac{W(\xi(t))}{W(\xi)} \leq e^{-\lambda m t} \text{ outside } E_0^*.$$

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$$\boxed{\mathcal{H}_W(M) := \mathrm{Op}\left(W^{-1}\right)\left(L^2(M)\right)} : \textbf{anisotropic Sobolev space}$$

i.e. imposes smoothness along E_s^* and accepts irregularities along E_u^*.

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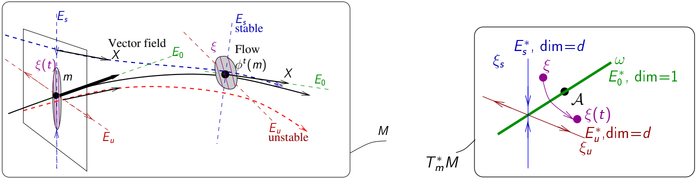

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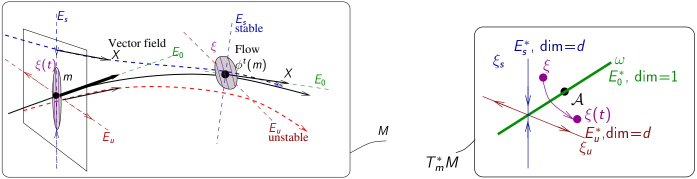

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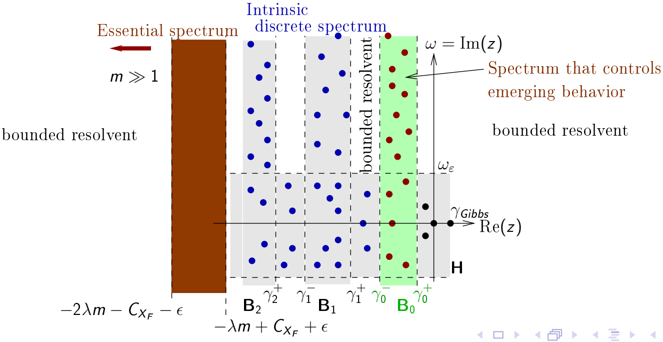
Theorem ("Discrete spectrum of X_F in vertical bands B_k" (F.-Tsujii 2006,12,13,16..))

The generator X_F generates a ***strongly cont. group*** *on $\mathcal{H}_W(M)$ and has* ***intrinsic discrete Ruelle-Pollicott spectrum*** *on $\mathrm{Re}(z) > -\lambda m + C_{X_F} + \epsilon$, $\forall \epsilon > 0$:*

$$\mathrm{Spec}(X_F) \subset \underbrace{([-\infty, z_0] \times [-i\omega_\epsilon, i\omega_\epsilon])}_{H} \cup \bigcup_{k\geq 0} \underbrace{([\gamma_k^- - \epsilon, \gamma_k^+ + \epsilon]) \times i\mathbb{R}}_{B_k}$$

$\gamma_k^{\pm}$ given below. ϵ depends on W. ***Weyl law*** *in each isolated band.* ***Bounded resolvent*** *in the gaps.*



Other models with spectral band structure

- S. Dyatlov (2015): similar **spectral band structure** for the **decay of waves around black holes**.
- F. (2006), band structure of the Ruelle-Pollicott spectrum of a $U(1)$-**contact extension** of a **linear hyperbolic "cat map"** $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on $\mathbb{T}^2$. (Simplified model of Anosov geodesic flow).
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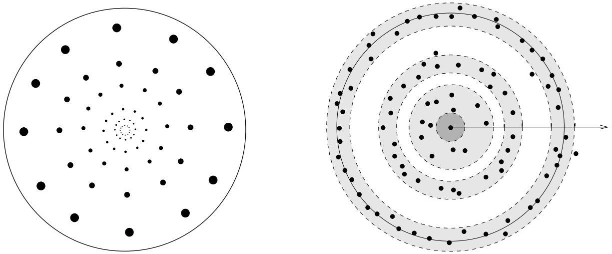

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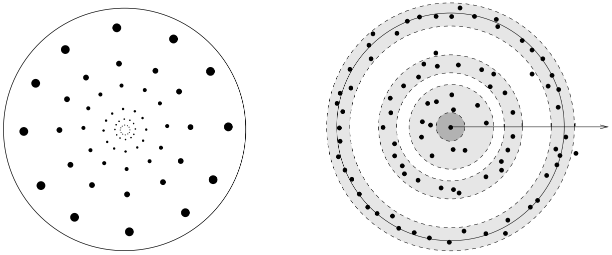
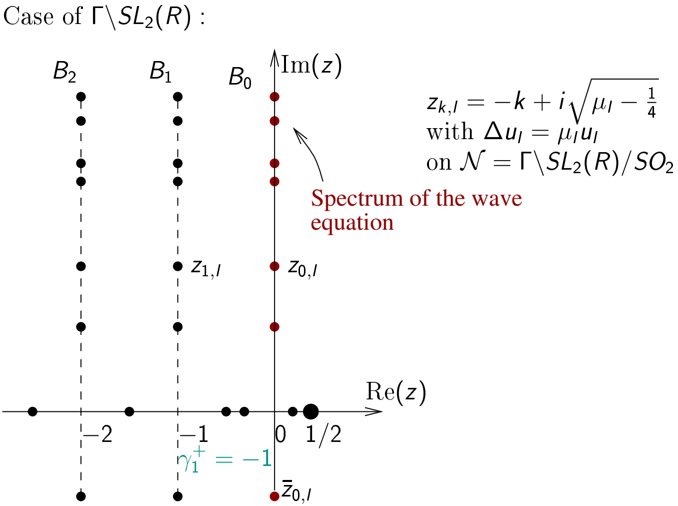


Ex: spectrum of $X_F = X + \frac{1}{2}$ in case of $\Gamma\backslash \mathrm{SL}_2\mathbb{R}$ (constant curvature), $F = |E_u|^{1/2}$.

Obtained by direct "equivariant" method (Ref: Dyatlov-F-Guillarmou 2015). Short explanation at the end.



Consequences for evolution of correlation functions

Let $\mathbf{\Pi}_{\mathrm{Band\,B_0}}$ be the spectral projector on band B_0. Then, $\forall u, v \in C^{\infty}(M)$,

$$\left\langle u | e^{tX_F} v \right\rangle_{L^2} = \langle u | \underbrace{\left(\Pi_{\mathrm{Band\,B_0}} e^{tX_F}\right)}_{\text{"quantum operator","wave op."}} v \rangle + O_{u,v}\left(e^{\left(\gamma_1^+ + \epsilon\right)t}\right)$$

Interpretations:

- **Emergence of an effective "quantum dynamics and uncertainty principle" (wave equation, discrete spectrum)** from classical correlation functions that describes the fluctuations.
- (F-Tsujii 2013) Operators $(\Pi_{\mathrm{Band\,B_0}}\mathcal{L}^t)$ and $(\Pi_{\mathrm{Band\,B_0}}A)$ are a **natural quantization** of the geodesic flow (exact trace formula, Egorov theorem etc..), that emerge from long time dynamics.

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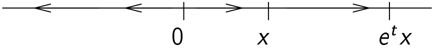
Consider the expanding vector field $\boxed{X = -x\partial_x \text{ on } \mathbb{R} \equiv E_s}$ that generates the flow $\phi^t(x) = e^{-t}x$, and the induced flow $\tilde{\phi}^t(x,\xi) = (e^t x, e^{-t}\xi)$ on $T^*\mathbb{R} \equiv T^*E_s$.



Rem: in $L^2(\mathbb{R})$, $X^* = -X + 1 \Leftrightarrow \left(X - \frac{1}{2}\right) = -\left(X - \frac{1}{2}\right)^*$, $\mathrm{Spec}(\mathrm{X}) = i\mathbb{R} + \frac{1}{2}$.

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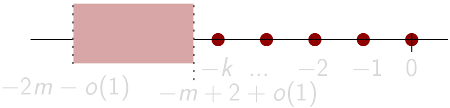

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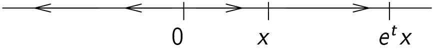

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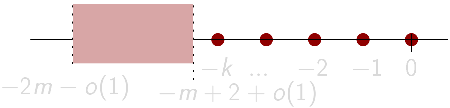

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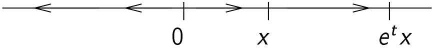

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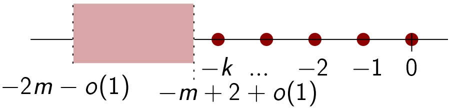

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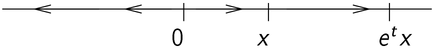

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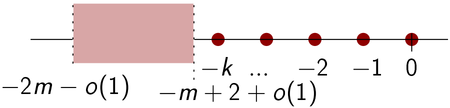

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$$\left(\mathrm{Op}^{\mathrm{Weyl}}(f)\, u\right)(x) := \frac{1}{(2\pi)^n} \int f\left(\frac{x+y}{2}, \xi\right) e^{i\xi(x-y)} u(y)\, dy d\xi$$

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$$\mathrm{Op}^{\mathrm{Geom.}}(f) := (-i)\, \mathcal{T}^* (X_f + if)\, \mathcal{T}$$

  with wave packets $\varphi_\rho, \rho \in T^*\mathbb{R}^n$ and the **wave packet transform** $\mathcal{T} : u \in L^2(\mathbb{R}^n) \to \langle \varphi_. | u \rangle_{L^2} \in L^2(T^*\mathbb{R})$.

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$$\mathrm{Op}^{\mathrm{Toeplitz}}(f) := \int_{\rho \in T^*\mathbb{R}^n} f(\rho)\, |\varphi_\rho\rangle\langle\varphi_\rho|\, d\rho = \mathcal{T}^* \mathcal{M}_f \mathcal{T}$$

Preliminary remark on "quantization"

Let $f : T^*\mathbb{R}^n \to \mathbb{R}$, a **Hamiltonian function,** or **"symbol"**, that generates the "**classical Hamiltonian vector field**" X_f by $\Omega(X_f, .) = df$.
Quantization: one associates an operator $\mathrm{Op}(f) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

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Spectrum of X_F: idea of proof and band estimates $\gamma_k^{\pm}$

- Consider the vicinity of the **symplectic trapped set** $E_0^* = \mathbb{R}\mathcal{A} \equiv M \times \mathbb{R}_\omega^* \subset T^*M$, with its **symplectic orthogonal** $\equiv T^*E_s$.

Thm (in progress): Using "**geometric quantization for vector bundle**", for $\omega \gg 1$, the operator X_F **is well approximated by** $\mathrm{Op}^{\mathrm{Geom}}(X_{\mathcal{F}}) := \mathcal{T}^*(X_{\mathcal{F}} + i\omega)\mathcal{T}$ with symbol $X_{\mathcal{F}}$ being the vector field X lifted on the bundle $\mathcal{F} \to E_0^*$:

$$\mathcal{F} := F \otimes \mathrm{Jet}(E_s) \otimes |E_s|^{1/2} = \bigoplus_{k\geq 0} \underbrace{F \otimes \mathrm{Sym}^k(E_s) \otimes |E_s|^{1/2}}_{\mathcal{F}_k}$$

($|E_s|^{1/2}$ is to get unitarity on E_0^*)

- In $L^2(E_0^*; \mathcal{F}_k)$ we have $\left(e^{t\mathcal{F}_k}\right)^* \left(e^{t\mathcal{F}_k}\right)(m) = \left(\tilde{\phi}_{\mathcal{F}_k}^{-t}\right)^* \tilde{\phi}_{\mathcal{F}_k}^{-t}(m)$: positive endomorphism in $\mathcal{F}_k(m)$. The spectrum of $X_{\mathcal{F}_k} : L^2 \to L^2$ is continuous and **contained in vertical band**

$$B_k = \left[\gamma_k^-, \gamma_k^+\right] \times i\mathbb{R}$$

with

$$\gamma_k^+ = \lim_{t\to+\infty} \frac{1}{t} \log \max_{m\in M} \left\|\tilde{\phi}_{\mathcal{F}_k}^{-t}(m)\right\|, \qquad \gamma_k^- = \lim_{t\to+\infty} \frac{1}{t} \log \max_{m\in M} \left\|\tilde{\phi}_{\mathcal{F}_k}^{-t}(m)^{-1}\right\|^{-1}$$

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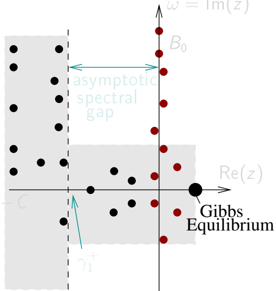
Special case: Let $F = |E_u|^{1/2} \equiv |E_s|^{-1/2}$ (not smooth! better to consider the smooth bundle $G_d(TM) \to M$ instead), then

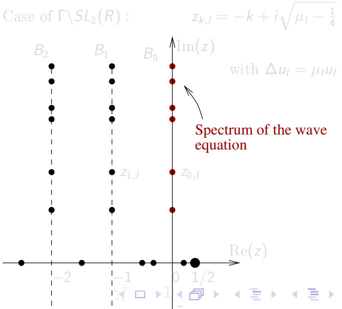
$$\mathcal{F}_{k=0} = F \otimes \mathrm{Sym}^0(E_s) \otimes |E_s|^{1/2} = \mathbb{C}, \quad : \text{trivial bundle.}$$

hence the symbol $X_{\mathcal{F}_0} \equiv X$: geodesic vector field $,\gamma_0^{\pm} = 0, \gamma_1^{+} = \lim_{t\to\infty} \sup_{x\in M} -\frac{1}{t} \log \left\| D\phi^t(x)_{/E_u} \right\|_{\min} < 0$. **We have** $X_F \underset{t\gg 1}{\sim} \mathrm{Op}(X)$!!

Theorem ((F.-Tsujii 2015))

*For $F = |E_u|^{1/2}$, the Ruelle-Pollicott spectrum of X_F in $\mathcal{H}_W(M)$ has eigenvalues $(z_{0,l})_l$ **that accumulate** on $\mathrm{Re}(z) = 0$, with density $\mathrm{Vol}(M)\frac{\omega^d}{(2\pi)^{d+1}}$, and an **asymptotic spectral gap** $\gamma_1^+ < \mathrm{Re}(z) < 0$.*





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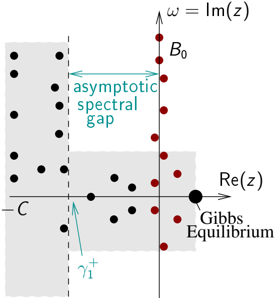

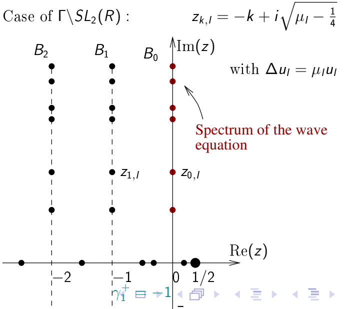

Relation with periodic orbits γ from Atiyah-Bott trace formula (65), Guillemin (79)

$$\mathrm{Tr}^{\flat}\left(e^{tX_F}\right) := \int_M \underbrace{K_t(m,m)}_{\text{Schwartz kernel of } e^{tX_F}} dm = \int_M \mathrm{Tr}\left(e^{tX_F}(m)\right)\delta\left(m - \phi^t(m)\right) dm$$

$$= \ldots = \sum_{\text{periodic orbits } \gamma} |\gamma| \sum_{n \geq 1} \frac{\mathrm{Tr}\left(\tilde{\phi}_F^t(m)\right).\delta\left(t - n|\gamma|\right)}{\left|\det\left(1 - D_{/E_u \oplus E_s}\phi^t(m)\right)\right|}, \quad m \in \gamma$$

This is a distribution in $\mathcal{D}'(\mathbb{R}_t)$.

Theorem (Giulietti-Pollicott-Liverani 12, Dyatlov-Zworsky 13)

*The **spectral determinant***

$$d(z) := \text{"}\det^{\flat}(z - X_F)\text{"} = \exp\left(-\sum_{\text{periodic orbits } \gamma} \sum_{n \geq 1} \frac{\mathrm{Tr}\left(\tilde{\phi}_F^t(m)\right) e^{-zn|\gamma|}}{n\left|\det\left(1 - D_{/E_u \oplus E_s}\phi^t(m)\right)\right|}\right)$$

*has a **holomorphic extension** on $\mathbb{C}$. Its **zeros are Ruelle eigenvalues** $\{z_j\}_j = \mathrm{Spect}(X_F)$.*

Relation with periodic orbits γ from Atiyah-Bott trace formula

Theorem (Tsujii-F. 12,13)

The ***semi-classical zeta function*** *(from "quantum chaos"), for* $F=|E_u|^{1/2}$*,:*

$$Z_{s.c.}(z) := \exp\left(-\sum_{\gamma}\sum_{n\geq 1}\frac{e^{-zn|\gamma|}}{n\left|\det\left(1-D_{/E_u\oplus E_s}\phi^t(m)\right)\right|^{1/2}}\right)$$

has a ***meromorphic extension*** *on* $\mathbb{C}$ *with finite number of poles on* $\mathrm{Re}(z)\geq\gamma_1^+ +\epsilon$*.* ***Zeros are Ruelle eigenvalues*** $(z_j)_j$*.*

In example $\Gamma\backslash\mathrm{SL}_2\mathbb{R}$, we have $D_{E_u\oplus E_s}\phi^{n|\gamma|}(\gamma)=\begin{pmatrix} e^{|\gamma|n} & 0\\ 0 & e^{-|\gamma|n}\end{pmatrix}$, giving

$$Z_{s.c.}(z)=\exp\left(-\sum_{\gamma}\sum_{n\geq 1}\sum_{m\geq 0}\frac{1}{n}e^{-n|\gamma|\left(z+\frac{1}{2}+m\right)}\right)=\prod_{\gamma}\prod_{m\geq 0}\left(1-e^{-\left(z+\frac{1}{2}+m\right)|\gamma|}\right)$$

$$=:\zeta_{\mathrm{Selberg}}\left(z+\frac{1}{2}\right)$$

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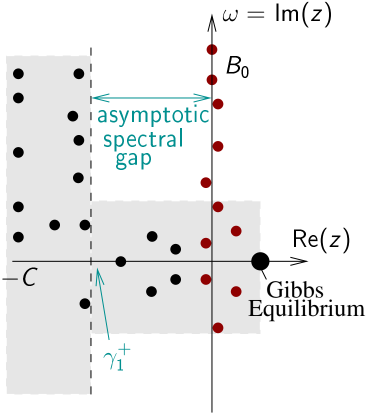

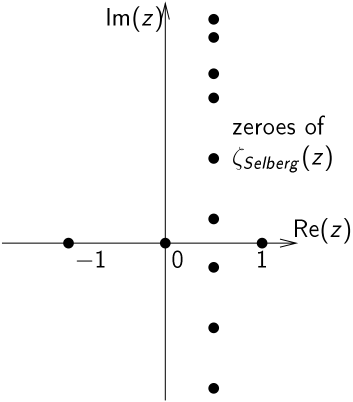

Thank you for your attention

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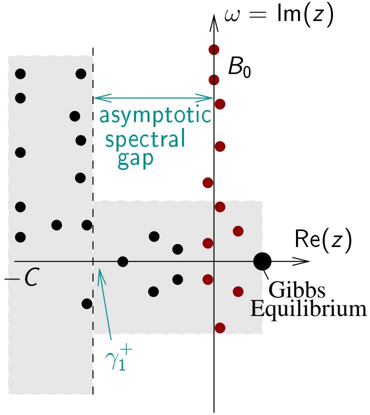

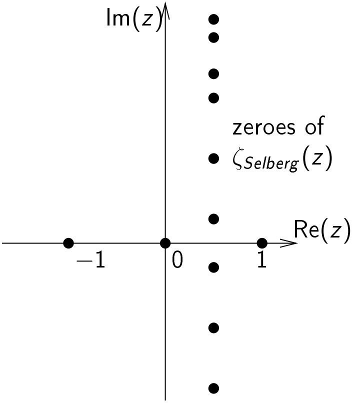

(*) Ex: Ruelle-Pollicott spectrum of geodesic flow on $\Gamma\backslash SL_2\mathbb{R}$

(Flaminio-Forni 02, Dyatlov-F-Guillarmou 14 is generalization to $\Gamma\backslash SO_{1,n}/SO_{n-1}$).
$\mathrm{sl}_2(\mathbb{R})$ algebra: $[U,X]=U,[S,X]=-S,[S,U]=2X$. X is the generator the geodesic flow on surface $\mathcal{N}=\Gamma\backslash SL_2\mathbb{R}/SO_2$.
Observations: if $Xu=zu$, then we get other resonances:

$$X(Uu)=(UX-U)u=(z-1)(Uu),$$
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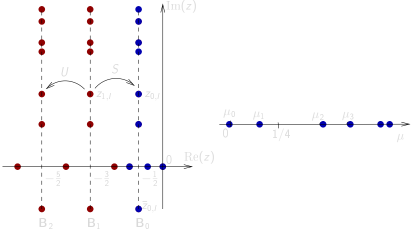
and $\exists k\geq 0$ s.t. $S^k u=0, S^{k-1}u\neq 0$. We say $u\in\mathbf{B}_k$ "band k".



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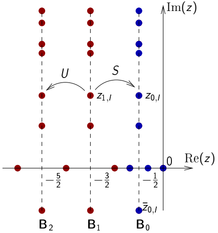

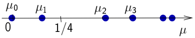

(*) Ex. spectrum on $\Gamma \backslash SL_2\mathbb{R}$ (2)

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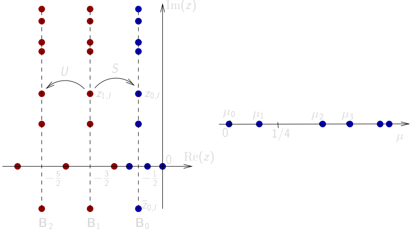
thus $\langle u \rangle_{SO_2} \in C^\infty(\mathcal{N})$ is an eigenfunction of $\Delta \equiv -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. Thus $\mu \in \mathbb{R}^+$ and

$$z = -\frac{1}{2} \pm i\sqrt{\mu - \frac{1}{4}}$$

$\rightarrow$ R-P spectrum has band structure (lines).

- Rem: $\mu_1 > 0$ gives the exponential rate for mixing.



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