Math. Res. Lett.
Volume 25, Number 5, 1405–1427, 2018

Horocyclic invariance of Ruelle resonant states for contact Anosov flows in dimension 3

Colin Guillarmou and Frédéric Faure

We show that for contact Anosov flows in dimension 3 the resonant states associated to the first band of Ruelle resonances are distributions that are invariant by the unstable horocyclic flow.

1. Introduction

By the work of Liverani [Li], Butterley-Liverani [BuLi], Faure-Sjöstrand [FaSj] or Dyatlov-Zworski [DyZw2], one can define an intrinsic discrete spectrum for the vector field X generating a smooth Anosov flow on a compact manifold $\mathcal{M}$. More precisely, one view $P := -X$ as a first order differential operator and we can construct appropriate anisotropic Sobolev spaces $\mathcal{H}^N$ (depending on parameter $N > 0$) related to the stable/unstable splitting of the flow, on which the first order differential operator $P - \lambda$ is an analytic family of Fredholm operators of index 0 in the complex half-plane $\{\mathrm{Re}(\lambda) > C_0 - \mu N\}$ for some $C_0 \geq 0$ and $\mu > 0$ depending on X; here $N > 0$ can be taken as large as we like. The eigenvalues and the eigenstates of P are independent of N, they are called *resonances* and *resonant states*. The operator is not self-adjoint on $\mathcal{H}^N$ and there can be Jordan blocks. We say that $u \in \mathcal{H}^N$ is a *generalized resonant state* with resonance $\lambda_0 \in \{\mathrm{Re}(\lambda) > C_0 - \mu N\}$ if $(P - \lambda_0)^j u = 0$ for some $j \in \mathbb{N}$. An equivalent way to define resonances for P is through the resolvent: the resolvent $R_P(\lambda) := (P - \lambda)^{-1}$ is an analytic family of bounded operators on $L^2(\mathcal{M}, dm)$ (for some fixed Lebesgue type measure dm) in $\{\mathrm{Re}(\lambda) > C_0\}$ for some $C_0 \geq 0$, there exists a meromorphic continuation of $R_P(\lambda)$ to $\lambda \in \mathbb{C}$ as a map

$$R_P(\lambda) : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$$

and the polar part of the Laurent expansion of $R_P(\lambda)$ at a pole λ_0 is a finite rank operator. The resonances are the poles of $R_P(\lambda)$ and the generalized resonant states are the elements in the range of the residue

$$\Pi_{\lambda_0} := -\mathrm{Res}_{\lambda_0} R_P(\lambda)$$

which turns out to be a projector.

We will now assume that $\mathcal{M}$ is a closed oriented manifold with dimension 3 and that X generates a contact Anosov flow, i.e there is a smooth one-form α such that $d\alpha$ is symplectic on $\ker\alpha$, $\alpha(X) = 1$ and $i_X d\alpha = 0$. We fix a smooth metric G on $\mathcal{M}$ and we denote by E_s and E_u the stable and unstable bundles, the tangent bundle has a flow-invariant continuous splitting

$$T\mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u \tag{1.1}$$

such that there is $C > 1$ and $\nu > 0$ such that for all $z \in \mathcal{M}$

$$\begin{aligned} \forall \xi \in E_s(z), \forall t \geq 0, \quad |d\varphi_t(z).\xi|_G \leq Ce^{-\nu t}|\xi|_G, \\ \forall \xi \in E_u(z), \forall t \geq 0, \quad |d\varphi_{-t}(z).\xi|_G \leq Ce^{-\nu t}|\xi|_G. \end{aligned} \tag{1.2}$$

We define the mimimal/maximal expansion rates of the flow

$$\begin{aligned} \mu_{\max} &:= \lim_{t\to+\infty} \sup_{z\in\mathcal{M}} -\frac{1}{t}\log \left| d\varphi_t(z)|_{E_s(z)} \right|_G \\ &= \lim_{t\to+\infty} \sup_{z\in\mathcal{M}} -\frac{1}{t}\log \left| d\varphi_{-t}(z)|_{E_u(z)} \right|_G, \\ \mu_{\min} &:= \lim_{t\to+\infty} \inf_{z\in\mathcal{M}} -\frac{1}{t}\log \left| d\varphi_t(z)|_{E_s(z)} \right|_G \\ &= \lim_{t\to+\infty} \inf_{z\in\mathcal{M}} -\frac{1}{t}\log \left| d\varphi_{-t}(z)|_{E_u(z)} \right|_G \end{aligned} \tag{1.3}$$

that satisfy $0 < \mu_{\min} < \mu_{\max}$. The equality of the limits in (1.3) is due to the fact that we work with a contact flow. We assume that E_u is an orientable bundle and let U_- be a global non-vanishing section of E_u, called an *unstable horocyclic vector field*. Since the flow is contact, we know by Hurder-Katok [HuKa] that U_- is a vector field that can be chosen with regularity $C^{2-\epsilon}(\mathcal{M})$ for all $\epsilon > 0$. We will show in Lemma 2.2 that U_- satisfies a commutation relation

$$[X, U_-] = -r_- U_- \tag{1.4}$$

for some function $r_- \in C^{2-\epsilon}(\mathcal{M})$. There is a preserved smooth measure $dm := \alpha \wedge d\alpha$, thus P is skew-adjoint on $L^2(\mathcal{M}, dm)$ and $R_P(\lambda)$ is analytic in $\mathrm{Re}(\lambda) > 0$, the L^2-spectrum being the whole imaginary line. The operator U_- can be viewed as acting on the negative Sobolev space $H^{-s}(\mathcal{M})$ for $s < 1$ as follows: for $u \in H^{-s}(\mathcal{M})$, for all $f \in C^\infty(\mathcal{M})$,

$$\langle U_- u, f\rangle := \langle u, -U_- f - \mathrm{div}(U_-) f\rangle$$

where $\mathrm{div}(U_-)$ is the divergence of U_- with respect to dm.

Theorem 1. *Let $\mathcal{M}$ be a smooth 3-dimensional oriented compact manifold and let X be a smooth vector field generating a contact Anosov flow. Assume that the unstable bundle is orientable. For $P = -X$, if λ_0 is a resonance of P with $\mathrm{Re}(\lambda_0) > -\mu_{\min}$ and if u is a generalized resonant state of P with resonance λ_0, then $U_- u = 0$.*

We shall see in Corollary 1.1 that for flows with pinched Lyapunov exponents, there are infinitely many resonances in the region $\mathrm{Re}(\lambda_0) > -\mu_{\min}$, thus infinitely many resonant states are killed by U_-. We expect the result to hold more generally if $\mathrm{Re}(\lambda_0) > \mathrm{Pr}(-2r_-)$ if r_- is the function appearing in (1.4) and Pr denotes the topological pressure. The problem to reach that bound is that we do not know if a resonant state u with resonance $\lambda_0 \in \{\mathrm{Re}(\lambda) \le -\mu_{\min}\}$ is sufficiently regular to be able to define $U_- u$.

In view of the regularity of the stable/unstable foliation in our case, we have locally near each point $x_0 \in \mathcal{M}$ a decomposition of $\mathcal{M}$ as a product $W_u \times W_s \times (-\epsilon, \epsilon)_t$ using the stable/unstable foliation, where $W_{u/s}$ are diffeomorphic to $(-\epsilon, \epsilon)$. The flow is $X = \partial_t$ is those coordinates, and Theorem 1 says that a resonant state w with resonance λ_0 (if $\mathrm{Re}(\lambda_0) > -\mu_{\min}$) is of the form

$$w(u, s, t) = e^{-\lambda_0 t}\omega(s)$$

for some distribution ω on W_s. In fact, due to the wave-front set analysis of resonant states in [FaSj], a resonant state w can be restricted locally to each piece of local stable leaf (which is an embedded smooth submanifold), or alternatively the lift of w to the universal cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ can be restricted to the stable leaves in $\widetilde{\mathcal{M}}$.

The horocyclic invariance of the first band of resonances was shown for geodesic flows in constant negative curvature in any dimension by Dyatlov-Faure-Guillarmou [DFG]. For hyperbolic surfaces, this follows also from the work of Flaminio-Forni [FlFo]. We find quite stricking that this type of properties still holds for general contact Anosov flows in dimension 3. We also notice that for an Anosov diffeomorphism on $\mathbb{T}^2$, the first resonant state for a certain transfer operator associated to an Anosov diffeomorphism on

$\mathbb{T}^2$ has shown to be horocyclic invariant by Giuletti-Liverani [GiLi]. There are other related cases which appeared in the work of Dyatlov [Dy] for resonances of semi-classical operators with r-normally hyperbolic trapped set, but the resonant states are only microlocally killed by some smooth pseudo-differential operator playing the role of U_-.

In Theorem 2, we prove a more general result which applies to the operator $P := -X + V$ where V is a smooth potential, and where the unstable derivative U_- is replaced by $U_- + \alpha_V$ for some appropriate function α_V depending on V. After Corollary 3.6, we discuss some interesting particular cases of potentials, namely $V = cr_-$ with $c \in \mathbb{R}$, that do not technically fit our assumptions due to smoothness issues but could still be considered without problems using the works [BuLi, GoLi].

Using Faure-Tsujii [FaTs1][1], we deduce the following result about existence of an infinite dimensional space of horocyclic invariant distributions:

Corollary 1.1. *Let $\mathcal{M}$ be a smooth 3-dimensional oriented manifold and let X be a smooth vector field generating a contact Anosov flow. Assume that E_u is orientable and that $\mu_{\max} < 2\mu_{\min}$. Then, for each $\epsilon > 0$ small, there exist infinitely many resonant states in $\ker U_-$ with associated resonances contained in the band*

$$\left\{\mathrm{Re}(\lambda) \in \left[-\tfrac{1}{2}\mu_{\max} - \epsilon, -\tfrac{1}{2}\mu_{\min} + \epsilon\right]\right\}.$$

These resonant states belongs to the Sobolev space $H^{-\frac{1}{2}\frac{\mu_{\max}+2\epsilon}{\mu_{\min}}}(\mathcal{M})$.

The proof of Theorem 1 comes from the following commutation relation

$$U_-(-X-\lambda)^{-1} = (-X-\lambda-r_-)^{-1}U_-$$

between the resolvents of $-X$ and of $-X - r_-$ where r_- is the function obtained in(1.4). The main part of the argument is to show that such relation holds and makes sense as a map $C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ in a certain region of the complex λ-plane. This is not completely obvious since the resolvents $(-X - \lambda)^{-1}$ and $(-X - \lambda - r_-)^{-1}$ a priori map smooth functions to distributions and U_- is only a $C^{2-\epsilon}(\mathcal{M})$ vector field for all $\epsilon > 0$.

We notice that our proof would apply similarly in higher dimension under pinching conditions on the Lyapunov exponents, except that one needs to use a covariant derivative in the unstable direction instead of just a vector field

[1] Note that the result [FaTs1] is an announcement and has been proved only in the case $V = \frac{1}{2}r_-$ in [FaTs2]. The general case will appear in [FaTs3].

U_-. The horocyclic invariance of resonant states would typically apply only to finitely many resonant states, for there is only finitely many resonances in the complex region where our result would hold, by a result of Tsujii [Ts]. We have thus decided to focus only on the case of dimension 3, where in addition the regularity of E_u is known to be better.

We finally emphasize that an alternative proof of Theorem 1 using more advanced tools is given in the preprint version [FaGu] of this paper.

2. Stable/unstable bundles

2.1. Anosov flows and the regularity of stable/unstable bundles

Let $\mathcal{M}$ be a smooth compact 3-dimensional oriented manifold and let X be a smooth vector field with Anosov flow denoted by φ_t. We fix a smooth metric G on $\mathcal{M}$ and we denote by E_s and E_u the stable and unstable bundles so that one has the flow-invariant continuous splitting (1.1) with (1.2). Let α be the continuous flow-invariant 1-form on $\mathcal{M}$ defined by $\ker\alpha = E_u \oplus E_s$ and $\alpha(X) = 1$. By Hurder-Katok [HuKa, Theorem 2.3], if $\alpha \in C^1(\mathcal{M}; T^*\mathcal{M})$ then $\alpha \in C^\infty(\mathcal{M}; T^*\mathcal{M})$ and either $\alpha \wedge d\alpha = 0$ or it is a nowhere vanishing 3-form and φ_t is a *contact flow*, i.e $i_X d\alpha = 0$ and $d\alpha$ is symplectic on $\ker\alpha$. For simplicity, we shall assume in the whole paper that we are in case of a contact flow. Let us now define the dual Anosov decomposition

$$T^*\mathcal{M} = \mathbb{R}\alpha \oplus E_u^* \oplus E_s^*, \quad \text{with } E_s^*(E_s \oplus \mathbb{R}X) = 0, \quad E_u^*(E_u \oplus \mathbb{R}X) = 0.$$

In [HuKa], Hurder-Katok proved the following regularity statement on the unstable/stable bundles.

Lemma 2.1 (Hurder-Katok). *For a smooth contact flow in dimension* 3*, the regularity of the bundles E_u and E_s is as follows:*

$$\forall\epsilon > 0,\ E_u \in C^{2-\epsilon}, \quad \forall\epsilon > 0,\ E_s \in C^{2-\epsilon}. \tag{2.1}$$

By regularity C^r of a bundle, it is meant that the bundle is locally spanned by vector fields which have C^r coefficients in smooth charts on $\mathcal{M}$. For what follows, we will write $f \in C^{2-}(\mathcal{M})$ to mean that a function/vector field belongs to $\cap_{\epsilon>0} C^{2-\epsilon}(\mathcal{M})$. Anosov [An] proved that there exist local stable and unstable smooth submanifolds $W_s(z), W_u(z)$ of $\mathcal{M}$ at each point z such that $T_zW_u(z) = E_u(z)$ and $T_zW_s(z) = E_s(z)$. As in Lemma 2.1 and [HuKa], the dependence in z is in fact C^{2-}, but it is never C^2 except for geodesic flows in constant negative curvature.

The submanifolds W_u form a foliation near z and by [HPS, Theorem 6.1] (see also [DMM, Lemma 2.1]), there are continuous maps

$$\Lambda : V_1 \times V_2 \to \mathcal{M}, \quad V_1 \subset \mathbb{R}, V_2 \subset \mathbb{R}^2$$

such that $\Lambda_y : V_1 \to \mathcal{M}$ defined by $\Lambda_y(x) = \Lambda(x, y)$ is a C^∞ embedding with image an unstable local submanifold $W_u(z)$ for some z and the derivatives $\partial_x^\beta \Lambda$ are continuous on $V_1 \times V_2$ for all $\beta \in \mathbb{N}$. The same holds for the stable foliation.

Next, we want to make sense of unstable derivatives.

Lemma 2.2. *Assume that X generates a smooth contact flow on an orientable 3-dimensional manifold $\mathcal{M}$ and that E_u is an orientable bundle. There exists a non-vanishing vector field U_- on $\mathcal{M}$ with regularity $C^{2-}(\mathcal{M}; T\mathcal{M})$ such that $U_-(z) \in E_u(z)$ for all $z \in \mathcal{M}$, and there exists a function r_- with regularity $C^{2-}(\mathcal{M})$ such that*

$$[X, U_-] = -r_- U_-. \tag{2.2}$$

The function r_- satisfies for $t \geq 0$

$$d\varphi_{-t}(z).U_-(z) = e^{-\int_{-t}^0 r_-(\varphi_s(z))ds} U_-(\varphi_{-t}(z)).$$

If a_i are the coefficients of U_- in a smooth coordinates system, then $U_-^k(a_i)$ are continuous for all $k \in \mathbb{N}$. The same properties hold with U_+ replacing U_-, E_s replacing E_u, r_+ replacing r_-, with

$$d\varphi_t(z).U_+(z) = e^{-\int_0^t r_+(\varphi_s(z))ds} U_+(\varphi_t(z)),$$

and U_+ is a $C^{2-}(\mathcal{M}; T\mathcal{M})$ section of E_s with local coefficients b_i such that $U_+^k(b_i)$ are continuous for all $k \in \mathbb{N}$.

Proof. The orientability of E_u insures that there exists a non-vanishing vector field U which is a section of E_u, and we normalize it so that its G-norm is $|U|_G = 1$. It can be chosen to be globally $C^{2-}(\mathcal{M})$ by Lemma 2.1. By the remark following the Lemma (which describes the unstable foliation regularity), we also have that the coefficients a_i of U in local coordinates are such that $U^n(a_i)$ are continuous for all $n \in \mathbb{N}$. We approximate U by a smooth vector field U_ϵ in a way that $|U - U_\epsilon|_G \leq \epsilon$ for $\epsilon > 0$ small. Since $\mathcal{M}$ is oriented and 3-dimensional (thus parallelizable), we can find a smooth vector field S so that (X, U_ϵ, S) is a global smooth basis of $T\mathcal{M}$, and we write $U = a_\epsilon U_\epsilon + b_\epsilon X + c_\epsilon S$ with $|a_\epsilon - 1| = \mathcal{O}(\epsilon)$ and $a_\epsilon, b_\epsilon, c_\epsilon \in C^{2-}(\mathcal{M})$. Let us define $U_- := (1/a_\epsilon)U$ which is also a $C^{2-}(\mathcal{M})$ non-vanishing section of E_u for $\epsilon > 0$ fixed small enough. Since $d\varphi_t(z).E_u(z) = E_u(\varphi_t(z))$, we have $d\varphi_t(z).U_-(z) = f(t,z)U_-(\varphi_t(z))$ for some $f(t,z) \in C^{2-}(\mathbb{R}\times\mathcal{M})$ with $f(t,z) > 0$, and $\partial_t f(t,z) \in C^{1-}(\mathbb{R}\times\mathcal{M})$[2]. We also have $f(s+t,z) = f(s,z)f(t,\varphi_s(z))$. We differentiate at $t = 0$ and get (2.2) with $r_-(z) := \partial_t f(0,z)/f(0,z)$ and more generally $\partial_s f(s,z)/f(s,z) = r_-(\varphi_s(z))$. A priori $r_- \in C^{1-}(\mathcal{M})$ but a small computation using $[X, U_-] = -r_- U_-$ implies that

$$-r_- = h + \frac{c_\epsilon}{a_\epsilon} k$$

where $h, k \in C^\infty(\mathcal{M})$ are the U_ϵ components of $[X, U_\epsilon]$ and $[X, S]$ in the basis (X, U_ϵ, S). Thus $r_- \in C^{2-}(\mathcal{M})$. The regularity of the coefficients of U_- when differentiated twice in the direction U_- follows from the same property as for U. By definition of r_- we also have that

$$|d\varphi_{-t}(z)U_-(z)|_G = e^{-\int_{-t}^0 r_-(\varphi_s(z))ds}|U_-(\varphi_{-t}(z))|_G$$

and this completes the proof. □

Remark 1. *We notice that $U_\pm$ are not uniquely defined: one can always multiply $U_\pm$ by a positive smooth function f, and $fU_\pm$ would satisfy all the same properties as $U_\pm$ described in Lemma 2.2. On the other hand, the kernel of U_- is independent of the choice of non-vanishing section U_- of E_u.*

It is interesting to give the following interpretation to (2.2), which explains why the operator $P = -X - r_-$ appears naturally: the flow acts on the bundle E_s^*, and if ω is a non-vanishing section of E_s^* defined by $\omega|_{E_s \oplus \mathbb{R}X} = 0$ and $\omega(U_-) = 1$, we have $\mathcal{L}_X \omega = r_- \omega$; thus for each $f \in C^2(\mathcal{M})$,

$$\mathcal{L}_{-X}(f\omega) = (-Xf - r_- f)\omega.$$

The map $\pi : C^{2-}(\mathcal{M}; E_s^*) \to C^{2^-}(\mathcal{M})$ defined by $\pi(h) := h(U_-)$ is an isomorphism with inverse $e : C^{2^-}(\mathcal{M}) \to C^{2-}(\mathcal{M}; E_s^*)$ given by $e(f) = f\omega$, one has $\pi \mathcal{L}_{-X} e = -X - r_-$ and (2.2) can be reinterpreted as the identity: for

[2]It is probably known from experts that $\partial_t f(t,z) \in C^{2-}(\mathbb{R}\times\mathcal{M})$, from which $r_- \in C^{2-}(\mathcal{M})$ would follow, but we haven't found references for such a fact, which is the reason why we use the approximation argument involving U_ϵ.

each $f \in C^\infty(\mathcal{M})$

$$\mathcal{L}_{-X} d^u f = d^u \mathcal{L}_{-X} f$$

where $d^u : C^\infty(\mathcal{M}) \to C^{2-}(\mathcal{M}, E_s^*)$ is the operator defined by $d^u f := df|_{E_u}$. We refer to [FaTs2, Section 3.3.2] for a related discussion.

To conclude this section, we define the minimal and maximal expansion rates by

$$\begin{aligned} \mu_{\min} &:= \lim_{t\to+\infty} \inf_{z\in\mathcal{M}} \frac{1}{t}\int_0^t r_-(\varphi_s(z))ds, \\ \mu_{\max} &:= \lim_{t\to+\infty} \sup_{z\in\mathcal{M}} \frac{1}{t}\int_0^t r_-(\varphi_s(z))ds. \end{aligned} \tag{2.3}$$

First, we remark that the two limits exist as $t \to +\infty$ by Fekete's lemma since $F_1(t) := \sup_{z\in\mathcal{M}} \int_0^t r_-(\varphi_s(z))ds$ is easily seen to be a subadditive function and $F_2(t) := \inf_{z\in\mathcal{M}} \int_0^t r_-(\varphi_s(z))ds$ is superadditive. By Lemma (2.2), for each $\epsilon > 0$, there is C_ϵ such that for all $t \geq 0$ and all $z \in \mathcal{M}$

$$C_\epsilon^{-1} e^{-t(\mu_{\max}+\epsilon)} \leq \Big| d\varphi_{-t}(z)|_{E_u} \Big|_G \leq C_\epsilon e^{-t(\mu_{\min}-\epsilon)}. \tag{2.4}$$

2.2. The case of geodesic flow

To illustrate the discussion above, let us discuss the special case of the geodesic flow of negatively curved surfaces. Let (M, g) be a smooth oriented compact Riemannian surface with Gauss curvature $K(x) < 0$ and let SM be its unit tangent bundle with the projection $\pi_0 : SM \to M$. We define $\mathcal{M} = SM$ and the geodesic flow at time $t \in \mathbb{R}$ is denoted by $\varphi_t : SM \to SM$, its generating vector field is denoted by X as above. The generator of rotations $R_s(x, v) := (x, e^{is}v)$ in the fibers of SM is a smooth vertical vector field denoted by V. Let $X_\perp := [X, V]$, this is a horizontal vector field and $(X, X_\perp, V)$ is an orthonormal basis for the Sasaki metric G on SM. We have the commutator formulas (see for example [PSU])

$$[X, X_\perp] = -KV, \quad [V, X_\perp] = X. \tag{2.5}$$

The Jacobi equation along a geodesic $x(t) = \pi_0(\varphi_t(x, v))$ is

$$\ddot{y}(t) + K(x(t))y(t) = 0. \tag{2.6}$$

For $(x,v)\in SM$ and $a,b\in\mathbb{R}$, one has

$$d\varphi_t(x,v).(-aX_\perp+bV)=-y(t)X_\perp(\varphi_t(x,v))+\dot y(t)V(\varphi_t(x,v)) \tag{2.7}$$

if $y(t)$ solves the Jacobi equation with $y(0)=a,\dot y(0)=b$. Notice that the function $r(t)=\dot y(t)/y(t)$ solves the Riccati equation

$$\dot r(t)+r(t)^2+K(x(t))=0 \tag{2.8}$$

for the times so that $y(t)\neq 0$. For $T\in\mathbb{R}$, let $y_T(t,x,v)$ be the solution of the Jacobi equation (2.6) along the geodesic $x(t)=\pi_0(\varphi_t(x,v))$ with conditions

$$y_T(0,x,v)=1,\quad y_T(T,x,v)=0.$$

Since g has no conjugate points, $y_T(t,x,v)\neq 0$ when $t\neq T$. Let $r_T(t,x,v):=\dot y_T(t,x,v)/y_T(t,x,v)$ which solves (2.8), it is defined for $t<T$ and $r_T(t,x,v)\to-\infty$ as $t\to T$. By Hopf [Ho], the following limits exist for all t,x,v

$$r_+(t,x,v):=-\lim_{T\to+\infty}r_T(t,x,v),\quad r_-(x,v):=\lim_{T\to+\infty}r_{-T}(t,x,v).$$

We denote $r_\pm(x,v):=r_\pm(0,x,v)$ and we see that $r_\pm(t,x,v)=r_\pm(\varphi_t(x,v))$. We have $r_\pm>0$ and they solve the Riccati equation on SM

$$\mp Xr_\pm+r_\pm^2+K=0. \tag{2.9}$$

The functions $r_\pm(x,v)$ are smooth in the X direction and are globally Hölder, they are called the stable (for r_+) and unstable (for r_-) Riccati solutions. We define the vector fields

$$U_-:=X_\perp-r_-V,\quad U_+:=X_\perp+r_+V$$

Lemma 2.3. *The following commutation relations hold*

$$[X,U_-]=-r_-U_-,\quad [X,U_+]=r_+U_+,$$

the function $r_\pm$ are in $C^{2-}(\mathcal{M})$ and

$$d\varphi_t(x,v).U_-(x,v)=e^{\int_0^t r_-(\varphi_s(x,v))ds}U_-(\varphi_t(x,v)),$$
$$d\varphi_t(x,v).U_+(x,v)=e^{-\int_0^t r_+(\varphi_s(x,v))ds}U_+(\varphi_t(x,v)).$$

Proof. We just compute, using (2.5) and the fact that $r_\pm$ solves (2.9),

$$[X, X_\perp - r_- V] = -KV - X(r_-)V - r_- X_\perp = -r_-(X_\perp - r_- V) = -r_- U_-$$

and similarly for $[X, U_+]$. By (2.7), we have for each $(x, v) \in SM$

$$d\varphi_t(x, v).U_- = -y(t)X_\perp + \dot{y}(t)V$$

where $\ddot{y} + Ky = 0$ and $y(0) = -1$ and $\dot{y}(0) = -r_-(x, v)$. Clearly we have $w := \dot{y}/y$ which satisfies the Riccati equation (2.8) with $w(0) = r_-(x, v)$, thus $w(t) = r_-(\varphi_t(x, v))$. This implies

$$d\varphi_t(x, v).U_-(x, v) = -y(t)U_-(\varphi_t(x, v)).$$

and $y(t) = -e^{\int_0^t r_-(\varphi_s(x,v))ds}$, and it shows $U_\pm$ are sections of E_u and E_s. By Lemma 2.1, since $X_\perp, V$ is a smooth frame, we deduce that $r_\pm$ are in $C^{2-}(\mathcal{M})$. □

We remark that by Klingenberg [Kl], if the Gauss curvature satisfies $-k_0^2 \leq K(x) \leq -k_1^2$ for some $k_0 > k_1 > 0$, then there exists $C > 0$ (depending only on k_0/k_1) so that for each $z \in SM$

$$\forall \xi \in E_s(z), \forall t \geq 0, \ Ce^{-k_0 t}|\xi|_G \leq |d\varphi_t(z).\xi|_G \leq Ce^{-k_1 t}|\xi|_G.$$

In particular this implies the bounds

$$k_0 \geq \mu_{\max} \geq \mu_{\min} \geq k_1. \tag{2.10}$$

3. Resonant states and horocyclic invariance

3.1. Analytic preliminaries

We first recall basic facts about microlocal analysis. Let $dm := \alpha \wedge d\alpha$ be the contact measure on $\mathcal{M}$ associated to the contact form α, that is invariant by the flow. We use the notation $H^s(\mathcal{M})$ for the L^2-based Sobolev space of order $s \in \mathbb{R}$, the space $C^\gamma(\mathcal{M})$ denotes the Banach space of Hölder functions with order $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$; for $k \in \mathbb{N}_0$ we shall write $C^k(\mathcal{M})$ for the space of functions k-times differentiable and with continuous k-derivatives. We will write $(C^\gamma(\mathcal{M}))'$ for their dual spaces and $C^{\gamma-}(\mathcal{M}) = \cap_{\epsilon>0} C^{\gamma-\epsilon}(\mathcal{M})$. We recall the embedding (see [Hö, Chapter 7.9])

$$\text{(3.1)} \quad \text{if } \gamma \notin \mathbb{N},\ C^\gamma(\mathcal{M}) \subset H^s(\mathcal{M}) \text{ for } s<\gamma, \quad \text{if } k \in \mathbb{N}_0,\ C^k(\mathcal{M}) \subset H^k(\mathcal{M}).$$

We denote by $\Psi^s(\mathcal{M})$ the space of pseudo-differential operators of order $s \in \mathbb{R}$, which have Schwartz kernel that can be written in local coordinates as

$$K(x,x') = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i(x-x')\xi}\sigma(x,\xi)d\xi$$

where $\sigma(x,\xi)$ is smooth and satisfies the following symbolic estimates of order s

$$\forall \alpha,\beta \in \mathbb{N}^3, \exists C_{\alpha,\beta}>0, \quad |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}\langle \xi\rangle^{s-|\beta|}.$$

For $A \in \Psi^s(\mathcal{M})$, there is a homogeneous symbol σ_p on $T^*\mathcal{M}$ of order s, called principal symbol, so that in local coordinates $\sigma - \sigma_p$ is a symbol of order $s-1$ outside $\xi = 0$. We say that A is elliptic in a conic set $W \subset T^*\mathcal{M}$ if there is $C>0$ such that $|\sigma_p(x,\xi)| \ge C|\xi|^s$ in W for $|\xi|>1$. The wave-front set of a distribution $u \in \mathcal{D}'(\mathcal{M})$ is the closed conic subset $\mathrm{WF}(u) \subset T^*\mathcal{M} \setminus \{\xi = 0\}$ defined by: $(x_0,\xi_0) \notin \mathrm{WF}(u)$ if and only if there is $A \in \Psi^0(\mathcal{M})$ elliptic in a conic open set W containing (x_0,ξ_0) such that $Au \in C^\infty(\mathcal{M})$. The wave-front set $\mathrm{WF}(A)$ of $A \in \Psi^s(\mathcal{M})$ is the conic closed set in $T^*\mathcal{M}$ whose complement is the conic region where the symbol of A and its derivatives decay to infinite order as $|\xi| \to \infty$.

3.2. Discrete spectrum in Sobolev anisotropic spaces

We recall the results of Butterley-Liverani [BuLi] and Faure-Sjöstrand [FaSj] (see also Dyatlov-Zworski [DyZw1] for similar results).

Proposition 3.1 (Faure-Sjöstrand). *Let X be a smooth vector field generating an Anosov flow on a compact manifold $\mathcal{M}$, let $V \in C^\infty(\mathcal{M})$ and let $P = -X + V$ be the associated first-order differential operator.*

1) There exists $C_0 \ge 0$ such that the resolvent $R_P(\lambda) := (P-\lambda)^{-1} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ of P is defined for $\mathrm{Re}(\lambda) > C_0$ *and extends meromorphically to $\lambda \in \mathbb{C}$ as a family of bounded operators $R_P(\lambda) : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$. The poles are called Ruelle resonances, the operator* $\Pi_{\lambda_0} := -\mathrm{Res}_{\lambda_0} R_P(\lambda)$ *at a pole λ_0 is a finite rank projector and there exists $p \ge 1$ such that $(P-\lambda_0)^p\Pi_{\lambda_0} = 0$. The distributions in* $\mathrm{Ran}\,\Pi_{\lambda_0}$ *are called generalized resonant states and those in* $\mathrm{Ran}\,\Pi_{\lambda_0} \cap \ker(P-\lambda_0)$ *are called resonant states.*

2) There exists $c>0$ *such that for each* $N\in[0,\infty)$*, there exists a Hilbert space* $\mathcal{H}^N$ *so that* $C^\infty(\mathcal{M})\subset\mathcal{H}^N\subset H^{-N}(\mathcal{M})$ *and such that* $R_P(\lambda):\mathcal{H}^N\to\mathcal{H}^N$ *is a meromorphic family of bounded operators in* $\mathrm{Re}(\lambda)>C_0-cN$*, and* $(P-\lambda):\mathrm{Dom}(P)\cap\mathcal{H}^N\to\mathcal{H}^N$ *is an analytic family of Fredholm operators*[3] *in that region with inverse given by* $R_P(\lambda)$*.*

3) For each $N_0>0$ *large enough and each conic neighborhood* W *of* E_u^**,* $\mathcal{H}^N$ *can be chosen in such a way that* $H^{N_0}(\mathcal{M})\subset\mathcal{H}^N$*, and for each* $A\in\Psi^0(\mathcal{M})$ *microsupported outside* W*, one has* $Au\in H^{N_0}(\mathcal{M})$ *for all* $u\in\mathcal{H}^N$*. For a resonance* λ_0*, the wave-front set of each generalized resonant state* $u\in\mathrm{Ran}(\Pi_{\lambda_0})$ *is contained in* E_u^**.*

The space $\mathcal{H}^N$ is called an anisotropic Sobolev space. The statement in [FaSj] is only for the case with no potential (i.e $V=0$), but their proof applies as well to the case $P=-X+V$ as long as $V\in C^\infty(\mathcal{M})$. It also follows readily from the proof of [FaSj] that, if the flow of X preserves a smooth measure dm and $V=0$, then one can take $C_0=0$. Indeed one has $P^*=-P$ in $L^2(\mathcal{M},dm)$ in that case, thus $\lambda\mapsto(P-\lambda)^{-1}$ is analytic in $\mathrm{Re}(\lambda)>0$ as a family of bounded operators on $L^2(\mathcal{M},dm)$. Moreover, the proof of [FaSj, Lemma 3.4] can be done with the constant C of that Lemma to be 0 since the operator $\hat{P}_2$ appearing in [FaSj, Section 3.3.] has principal symbol given by $X(G_m)$ by using that $P^*=-P$ with respect to the $L^2(\mathcal{M},dm)$ product; here G_m is the escape function of [FaSj].

For a general potential and a flow preserving a smooth measure, we will give an estimate on C_0. Let us first define the quantity

$$V_{\max}:=\lim_{t\to-\infty}\sup_{z\in\mathcal{M}}\frac{1}{|t|}\int_t^0 V(\varphi_s(z))ds.$$

Lemma 3.2. *Let* $V\in C^\infty(\mathcal{M})$ *be real-valued and assume that* X *is a smooth vector field generating an Anosov flow preserving a smooth measure dm. The resolvent* $R_P(\lambda)$ *of Proposition 3.1 is analytic in* λ *as an* $L^2(\mathcal{M})$ *bounded operator in* $\mathrm{Re}(\lambda)>V_{\max}$*. For each* $N>0$*, there is* $N_0>0$ *such that* $R_P(\lambda):H^{N_0}(\mathcal{M})\to H^{-N}(\mathcal{M})$ *is a meromorphic family of bounded operators in the region* $\mathrm{Re}(\lambda)>V_{\max}-N\mu_{\min}$*.*

[3] Here $\mathrm{Dom}(P):=\{u\in\mathcal{H}^N;Pu\in\mathcal{H}^N\}$ is the domain of P equipped with the graph norm.

Proof. The resolvent of $P = -X + V$ for $\mathrm{Re}(\lambda) \gg 1$ large enough is given by the expression

$$R_P(\lambda)f = -\int_{-\infty}^{0} e^{\lambda t + \int_t^0 V\circ\varphi_s\, ds} f \circ \varphi_t \, dt.$$

We see that it converges in $L^2(\mathcal{M}, dm)$ in the region $\{\mathrm{Re}(\lambda) > V_{\max}\}$ by using first the estimate $||f \circ \varphi_t||_{L^2(dm)} = ||f||_{L^2(dm)}$ and the pointwise bounds (following from Cauchy-Schwarz)

$$|R_P(\lambda)f(z)|^2 \leq C_{\lambda,\epsilon} \int_{-\infty}^{0} e^{\mathrm{Re}(\lambda)t - t(V_{\max}+\epsilon)} |f(\varphi_t(z))|^2 \, dt$$

for some constant $C_{\lambda,\epsilon}$ depending on $\mathrm{Re}(\lambda)$ and $\epsilon > 0$, where $\epsilon > 0$ can be chosen as small as we want. We next prove the second statement. First, we know from Proposition 3.1 that there is $N_1 > 0$ such that $R_P(\lambda)f \in \mathcal{H}^{N_1} \subset H^{-N_1}(\mathcal{M})$ if $f \in \mathcal{H}^{N_1}$ and $\mathrm{Re}(\lambda) > V_{\max} - \mu_{\min} N$. By 3) of Proposition 3.1 and choosing appropriately the space $\mathcal{H}^{N_1}$, we also know that $BR_P(\lambda)f \in H^{N_0}(\mathcal{M})$ for some large $N_0 > 0$ if $B \in \Psi^0(\mathcal{M})$ is elliptic outside an arbitrarily small fixed conic neighborhood of E_u^*. The desired statement will be a consequence of the radial point estimates proved in Dyatlov-Zworski [DyZw2, Theorem E.56] (see also [DyZw1]). Let $\Phi_t(z,\xi) := (\varphi_t(z), (d\varphi_t(z)^{-1})^T \xi)$ be the symplectic lift of the flow φ_t to $T^*\mathcal{M}$ and let $\mathbf{X}$ be its vector field on $T^*\mathcal{M}$. In [DyZw2, Theorem E.56 and the following remark], it is shown that if for $T > 0$ large enough and for some $s \in \mathbb{R}$, one has

$$\int_0^T ((V - \mathrm{Re}(\lambda)) \circ \varphi_t + s\mathbf{X}(\log\langle\xi\rangle) \circ \Phi_t) dt < 0$$

on E_u^* for all $|\xi|$ large, then there is $A, B \in \Psi^0(\mathcal{M})$ with $\mathrm{WF}(B) \subset T^*\mathcal{M} \setminus E_u^*$ and A elliptic near E_u^* such that, if $u \in H^{-N_1}(\mathcal{M})$, $(P - \lambda)u \in H^s(\mathcal{M})$ and $Bu \in H^s(\mathcal{M})$, then $Au \in H^s(\mathcal{M})$. The quantity above can be rewritten as

$$\int_0^T V(\varphi_t(z)) \, dt - \mathrm{Re}(\lambda)T + s \log\Big(\frac{\langle\Phi_T(z,\xi)\rangle}{\langle\xi\rangle}\Big)$$

and, if $s < 0$, this is negative on E_u^* for large $|\xi|$ if $\mathrm{Re}(\lambda) > V_{\max} + s\mu_{\min}$ by using the bounds (2.4). Taking $s = -N$, $N_0 > 0$ large enough so that $H^{N_0}(\mathcal{M}) \subset \mathcal{H}^{N_1}$ and applying this with $u = R_P(\lambda)f$ and $f \in H^{N_0}(\mathcal{M})$, we obtain $R_P(\lambda)f \in H^{-N}(\mathcal{M})$. □

3.3. Horocyclic invariance of resonant states for contact flows

In this section, we shall assume that $\mathcal{M}$ is a 3-dimensional oriented compact manifold and X is a smooth vector field generating a contact Anosov flow, with oriented unstable bundle. Here dm will denote the contact measure and $V \in C^1(\mathcal{M})$ is a potential. Due to the C^{2-} regularity of U_-, for $u \in H^{-1+\epsilon}(\mathcal{M})$ we can define $\omega = U_- u$ as a distribution by the expression

$$\forall f \in C^\infty(\mathcal{M}), \quad \langle U_- u, f\rangle := \langle u, -(U_- f + \mathrm{div}(U_-)f)\rangle;$$

here div denotes the divergence with respect to dm and $-U_- - \mathrm{div}(U_-)$ is the adjoint to U_- with respect to dm. The quantity $\mathrm{div}(U_-)$ is in $C^{1-}(\mathcal{M})$, thus if u is a generalized resonant state of $-X$, $U_- u$ is well-defined as long as $\mathrm{Re}(\lambda) > -\mu_{\min}$ since $u \in H^{-1+\epsilon}(\mathcal{M})$ for some $\epsilon > 0$ in that case by Lemma 3.2.

We define the transfer operator

$$\mathcal{L}^t : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \quad (\mathcal{L}^t f)(x) := f(\varphi_t(x)).$$

It extends as a bounded operator on $L^2(\mathcal{M}, dm)$ with norm $||\mathcal{L}^t||_{L^2\to L^2} = 1$. If $V \in C^1(\mathcal{M})$ and $P = -X + V$, we also define the operator

$$e^{-tP} : C^1(\mathcal{M}) \to C^1(\mathcal{M}), \quad e^{-tP} f := e^{-\int_0^t \mathcal{L}^s V ds} \mathcal{L}^t f$$

satisfying $\partial_t(e^{-tP} f) = -Pe^{-tP} f$. Let us first prove an easy Lemma.

Lemma 3.3. *For each $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for each $s \in [-1, 1]$ and each $t \in \mathbb{R}$, the operator $\mathcal{L}^t$ is bounded on $C^1(\mathcal{M})$ with norm*

$$||\mathcal{L}^t||_{C^1\to C^1} \le C_\epsilon e^{(\mu_{\max}+\epsilon)|t|} \tag{3.2}$$

and on $H^s(\mathcal{M})$ with norm

$$||\mathcal{L}^t||_{H^s\to H^s} \le C_\epsilon e^{|s|(\mu_{\max}+\epsilon)|t|}. \tag{3.3}$$

Proof. The C^1 bound follows from the definition of $\mu_{\max}$. We have

$$||\mathcal{L}^t||_{L^2\to L^2} = 1$$

and for each $\epsilon > 0$, there is $C_\epsilon > 0$ such that for all $u \in C^\infty(\mathcal{M})$ and $x \in \mathcal{M}$

$$|d\mathcal{L}^t u|_{G_x} = |du_{\varphi_t(x)}.d\varphi_t(x)|_{G_x} \le C_\epsilon e^{(\mu_{\max}+\epsilon)|t|} |du(\varphi_t(x))|_{G_{\varphi_t(x)}}$$

thus by integrating the square of this inequality on $\mathcal{M}$ and using that φ_t preserves dm, we get $||d\mathcal{L}^t u||_{L^2} \leq C_\epsilon e^{(\mu_{\max}+\epsilon)|t|}||du||_{L^2}$ and $||\mathcal{L}^t||_{H^1\to H^1} \leq C_\epsilon e^{(\mu_{\max}+\epsilon)|t|}$. Interpolating between H^1 and L^2 we get the result for $s \geq 0$ and using that $(\mathcal{L}^t)^* = \mathcal{L}^{-t}$ we obtain the desired result for $s \leq 0$. □

As a direct corollary, we get

Corollary 3.4. *Let $V \in C^1(\mathcal{M})$ be real-valued. If $\mathrm{Re}(\lambda) > \mu_{\max} + V_{\max}$, then the resolvent $R_P(\lambda)$ of $P = -X + V$ is bounded as a map*

$$R_P(\lambda) : C^1(\mathcal{M}) \to C^1(\mathcal{M}).$$

Proof. The resolvent of $P = -X + V$ for $\mathrm{Re}(\lambda) > 0$ is given by the expression

$$R_P(\lambda) f = -\int_{-\infty}^{0} e^{\lambda t + \int_t^0 \mathcal{L}^s V ds} \mathcal{L}^t f \, dt$$

and (3.2) shows that the integral converges in C^1 norm if $\mathrm{Re}(\lambda) > \mu_{\max} + V_{\max}$. □

Next, define the potential $W := V - r_-$ and the quantities

$$W_{\max} := \lim_{t\to-\infty} \sup_{z\in\mathcal{M}} \frac{1}{|t|} \int_t^0 W(\varphi_s(z)) ds.$$

which in turn are bounded by $W_{\max} \leq V_{\max} - \mu_{\min}$. We obtain

Lemma 3.5. *Let r_- be the function of Lemma 2.2, $V \in C^1(\mathcal{M})$ and $W := V - r_-$. The operator $P' = -X + W$ has an analytic resolvent $R_{P'}(\lambda) : C^0(\mathcal{M}) \to C^0(\mathcal{M})$ in the region $\{\mathrm{Re}(\lambda) > W_{\max}\}$, given by the convergent expression*

$$R_{P'}(\lambda) f := -\int_{-\infty}^{0} e^{\lambda t + \int_t^0 \mathcal{L}^s W ds} (\mathcal{L}^t f) dt \tag{3.4}$$

and satisfying $(P' - \lambda) R_{P'}(\lambda) = \mathrm{Id}$ in the distribution sense. If $f \in C^1(\mathcal{M})$, then for $\mathrm{Re}(\lambda) > W_{\max} + s\mu_{\max}$ with $s \in (0, 1]$, we have

$$R_{P'}(\lambda) f \in C^{s-}(\mathcal{M}). \tag{3.5}$$

Finally, there is no $C^0(\mathcal{M})$ solution ω to $(P' - \lambda)\omega = 0$ in the region $\{\mathrm{Re}(\lambda) > W_{\max}\}$.

Proof. The proof of the first statement is straightforward using that for each $\epsilon > 0$ small, we have for $t < 0$ large enough and uniformly on $\mathcal{M}$

$$\int_t^0 \mathcal{L}^s W \, ds \leq (W_{\max} + \epsilon)|t|.$$

For the regularity (3.5), we observe that for $s = 1$ this follows directly from the expression (3.4) and the bound (3.2). To obtain the $s < 1$ case, it suffices to use interpolation (i.e Hadamard three line theorem) between the line $\mathrm{Re}(\lambda) = W_{\max} + 2\epsilon$ where we have C^0 bounds and the line $\mathrm{Re}(\lambda) = W_{\max} + \mu_{\max} + \epsilon$ where we have C^1 bounds.

To prove that $(P' - \lambda)$ is injective on $C^0(\mathcal{M})$, assume $(P' - \lambda)\omega = 0$ and let $\omega(t) = \mathcal{L}^t \omega \in C^0(\mathcal{M})$. We have in the weak sense

$$\partial_t \omega(t) = \mathcal{L}^t X\omega = -\mathcal{L}^t(r_- - V + \lambda)\omega = (\mathcal{L}^t(W) - \lambda)\omega(t)$$

and therefore $\omega(t) = \omega e^{-\lambda t - \int_t^0 \mathcal{L}^s W \, ds}$. Since $||\omega(t)||_{C^0} \leq ||\omega||_{C^0}$, we can let $t \to -\infty$ and we obtain a contradiction if $\omega \neq 0$. □

A first consequence of Lemma 3.5 is that for each $V \in C^{2-}(\mathcal{M})$ there exists a function $\alpha_V := R_{-X-r_-}(0)U_-(V)$ satisfying

$$\alpha_V \in C^{\frac{\mu_{\min}}{\mu_{\max}}-}(\mathcal{M}), \quad (-X - r_-)\alpha_V = U_-(V). \tag{3.6}$$

This will be useful for what follows. Note also that the operator U_- is not a priori skew-adjoint with respect to the measure dm: one has $U_-^* = -U_- - \mathrm{div}(U_-)$ where $\mathrm{div}(U_-) \in C^{1-}(\mathcal{M})$ is the divergence of U_- with respect to dm. We observe that

$$V = r_- \Longrightarrow \alpha_V = \mathrm{div}(U_-). \tag{3.7}$$

Indeed, taking the adjoint of (2.2), we have the identity of operators

$$(-X - r_-)U_-^* = U_-^*(-X + r_-) - r_- U_-^*$$

and therefore

$$\begin{aligned}(-X - r_-)(\mathrm{div}(U_-)) &= -(-X - r_-)U_-^*(1) \\ &= -U_-^*(r_-) + r_- U_-^*(1) = U_-(r_-).\end{aligned}$$

which shows (3.7). In particular we see that $\alpha_V \in C^{1-}(\mathcal{M})$ in that casen and more generally for $V = kr_-$ with $k \in \mathbb{R}$, the regularity of α_V is better than the regularity expected in (3.6). We can view the first order differential operator $U_- + \alpha_V$ as a connection along the unstable leaves associated to the potential V.

Now we can give a short proof of the following

Theorem 2. *Let $V \in C^\infty(\mathcal{M})$, $W := V - r_-$, $P := -X + V$ and $P' := -X + W$. Let α_V be the function of (3.6) and let $s \in [\frac{\mu_{\min}}{\mu_{\max}}, 1]$ be the largest number so that $\alpha_V \in C^{s-}(\mathcal{M})$. In the region $\{\mathrm{Re}(\lambda) > W_{\max}\}$, the operator $R_{P'}(\lambda)(U_- + \alpha_V) : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ is analytic and one has the identity*

$$(U_- + \alpha_V)R_P(\lambda) = R_{P'}(\lambda)(U_- + \alpha_V) \tag{3.8}$$

in $\{\mathrm{Re}(\lambda) > V_{\max} - s\mu_{\min}\}$. For each generalized resonant state u of P with resonance λ_0 contained in $\{\mathrm{Re}(\lambda) > V_{\max} - s\mu_{\min}\}$, we have $(U_- + \alpha_V)u = 0$.

Proof. It suffices to prove (3.8) for $\mathrm{Re}(\lambda)$ large enough and then use meromorphic continuation in λ. Let $u \in C^\infty(\mathcal{M})$ and assume that $\mathrm{Re}(\lambda) > \mu_{\max} + V_{\max}$. By Lemma 2.2, we have

$$\begin{aligned}[-X + V, U_- + \alpha_V] &= r_-(U_- + \alpha_V) - U_-(V) - (X + r_-)(\alpha_V)\\ &= r_-(U_- + \alpha_V)\end{aligned}$$

and thus

$$\begin{aligned}&(-X + V - r_- - \lambda)((U_- + \alpha_V)R_P(\lambda)u - R_{P'}(\lambda)(U_- + \alpha_V)u)\\ &= (U_- + \alpha_V)(-X + V - \lambda)R_P(\lambda)u - (U_- + \alpha_V)u = 0.\end{aligned} \tag{3.9}$$

To make sense of this identity, we use Corollary 3.4 showing that $(U_- + \alpha_V)R_P(\lambda)u \in C^0(\mathcal{M})$ and Lemma 3.5 that proves that $R_{P'}(\lambda)U_-u \in C^0(\mathcal{M})$. Thus

$$\omega := (U_- + \alpha_V)R_P(\lambda)u - R_{P'}(\lambda)(U_- + \alpha_V)u \in \ker(P' - \lambda).$$

By Lemma 3.5 again, we know that there is no C^0 solution to $(P' - \lambda)\omega = 0$ in $\{\mathrm{Re}(\lambda) > -\mu_{\min} + V_{\max}\}$ thus $\omega = 0$ and the proof of (3.8) in $\{\mathrm{Re}(\lambda) > \mu_{\max} + V_{\max}\}$ is complete. Among the terms in (3.8), all have meromorphic extension to $\{\mathrm{Re}(\lambda) > -\mu_{\min}s + V_{\max}\}$ as operators mapping $C^\infty(\mathcal{M})$ to $\mathcal{D}'(\mathcal{M})$: indeed, if $f \in C^\infty(\mathcal{M})$ then by Lemma 3.5 $R_{P'}(\lambda)(U_- + \alpha_V)f \in C^0(\mathcal{M})$ for all $\mathrm{Re}(\lambda) > W_{\max}$ thus for all $\mathrm{Re}(\lambda) > V_{\max} - s\mu_{\min}$, while by Lemma 3.2 we have $R_P(\lambda)f \in (C^s(\mathcal{M}))'$ if $\mathrm{Re}(\lambda) > V_{\max} - s\mu_{\min}$ thus $(U_- +$

$\alpha_V)R_P(\lambda)f$ is well defined. Taking the residue at a resonance $\lambda_0 \in \{\mathrm{Re}(\lambda) > V_{\max} - s\mu_{\min}\}$ in the identity (3.8), we obtain

$$(U_- + \alpha_V)\Pi_{\lambda_0} = 0$$

if $\Pi_{\lambda_0} := -\mathrm{Res}_{\lambda_0} R_P(\lambda)$, thus the range of Π_{λ_0} belongs to $\ker(U_- + \alpha_V)$, which means that generalized resonant states are in $\ker(U_- + \alpha_V)$. □

A particular case of interest is when $V = 0$.

Corollary 3.6. *The operator $R_{-X-r_-}(\lambda)U_- : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ is analytic in the region $\{\mathrm{Re}(\lambda) > -\mu_{\min}\}$ and one has in that region*

$$U_- R_{-X}(\lambda) = R_{-X-r_-}(\lambda)U_-. \tag{3.10}$$

Each generalized resonant state u of $-X$ with resonance λ_0 contained in the region $\{\mathrm{Re}(\lambda) > -\mu_{\min}\}$ satisfies $U_- u = 0$.

There are other natural cases of interest, namely when $V = kr_-$ for some $k \in \mathbb{R}$. This potential is not smooth, thus one would need a theory of resonances for operators with non-smooth coefficients. This has been developped by Butterley-Liverani [BuLi] for non-smooth flows. Even though it is not explicitely written in their paper, their technique should allow to deal with potentials $V \in C^{1+q}(\mathcal{M})$ for $q \in (0,1)$. In fact, the analysis with potentials has been done carefully by Gouëzel-Liverani [GoLi] for Anosov diffeomorphisms using the same technique. Combining the methods of [BuLi, Theorem 1] for flows with the arguments of [GoLi, Proposition 4.4. and Theorem 6.4.] (taking $p = 1, q < 1$ and $\iota = 0$ in their notations, since our flow is $C^\infty(\mathcal{M}) \subset C^{p+q+1}(\mathcal{M})$), one would in principle obtain the following result: for $V \in C^{1+q}(\mathcal{M})$ for some $0 < q < 1$ and X a smooth vector field generating an Anosov flow preserving a smooth measure dm in dimension 3, there exist a Banach space $\mathcal{B}_{1,q}$ satisfying that for each $q' > q$, one has $C^1(\mathcal{M}) \subset \mathcal{B}_{1,q} \subset (C^{q'}(\mathcal{M}))'$, the operator $P = -X + V$ has discrete spectrum in the region $\mathrm{Re}(\lambda) > \rho - q\mu_{\min}$, the resolvent $R_P(\lambda) = (P - \lambda)^{-1} : \mathcal{B}_{1,q} \to \mathcal{B}_{1,q}$ is meromorphic there and analytic in $\mathrm{Re}(\lambda) > \rho$. Here $\rho := \mathrm{Pr}(V - r_-)$ is the topological pressure of the potential $V - r_-$ and r_- is the function of Lemma 2.2. Since writing the proof of such result would be long, technical and not really in the scope of the paper, we just mention the expected results provided one accepts the combination of [BuLi] and [GoLi] works out: that result combined with the proof of Theorem 2 would give the following statement: for each $k \in \mathbb{R}$, the operator $R_{-X+(k-1)r_-}(\lambda)(U_- + k\,\mathrm{div}(U_-)) : C^\infty(\mathcal{M}) \to$

$\mathcal{D}'(\mathcal{M})$ is analytic in the region $\{\mathrm{Re}(\lambda) > \Pr((k-2)r_-)\}$ and the following identity holds

$$(U_- + k\,\mathrm{div}(U_-))R_{-X+kr_-}(\lambda) = R_{-X+(k-1)r_-}(\lambda)(U_- + k\,\mathrm{div}(U_-)) \tag{3.11}$$

in $\{\mathrm{Re}(s) > \Pr((k-1)r_-) - \mu_{\min}\}$. Each generalized resonant state of $-X + kr_-$ in that region is killed by $U_- + k\,\mathrm{div}(U_-)$.

An interesting particular case is when $k = 1$ and when $k = 1/2$. In the first case, this gives that $R_{-X}(\lambda)U_-^* : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ is analytic in the region $\{\mathrm{Re}(\lambda) > 0\}$ and one has in $\{\mathrm{Re}(s) > h_{\mathrm{top}} - \mu_{\min}\}$

$$U_-^* R_{-X+r_-}(\lambda) = R_{-X}(\lambda)U_-^*, \tag{3.12}$$

where $h_{\mathrm{top}} = \Pr(0)$ is the topological entropy of the flow of X. Each generalized resonant state u of $-X + r_-$ with resonance λ_0 contained in $\{\mathrm{Re}(\lambda) > h_{\mathrm{top}} - \mu_{\min}\}$ satisfies $U_-^* u = 0$. When $k = 1/2$, the operator

$$R_{-X-\frac{1}{2}r_-}(\lambda)(U_- + \tfrac{1}{2}\mathrm{div}(U_-)) : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$$

is analytic in the region $\{\mathrm{Re}(\lambda) > \Pr(-\frac{3}{2}r_-)\}$ and the following identity holds

$$(U_- + \tfrac{1}{2}\mathrm{div}(U_-))R_{-X+\frac{1}{2}r_-}(\lambda) = R_{-X-\frac{1}{2}r_-}(\lambda)(U_- + \tfrac{1}{2}\mathrm{div}(U_-)), \tag{3.13}$$

in $\{\mathrm{Re}(\lambda) > \Pr(-\frac{1}{2}r_-) - \mu_{\min}\}$. Each generalized resonant state u of $-X + \frac{1}{2}r_-$ with resonance λ_0 contained in $\{\mathrm{Re}(\lambda) > \Pr(-\frac{1}{2}r_-) - \mu_{\min}\}$ satisfies $(U_- + \frac{1}{2}\mathrm{div}(U_-))u = 0$. It can be noticed that the horocyclic derivative $\mathcal{U}_- := U_- + \frac{1}{2}\mathrm{div}(U_-)$ is skew-adjoint with respect to the contact measure dm. The study of the spectrum in the case $V = \frac{1}{2}r_-$ has been done in details by Faure-Tsujii [FaTs2] using the Grassmanian extension. It is particularly interesting since the first band of resonances concentrate near $\{\mathrm{Re}(\lambda) = 0\}$.

3.4. Invariant distributions for U_-

We recall the result of Faure-Tsujii [FaTs1] describing the localisation of Ruelle resonances. For a potential $V \in C^\infty(\mathcal{M})$ let us define the quantities for $k = 0, 1$

$$\gamma_k^+ := \lim_{t\to+\infty} \sup_{z\in\mathcal{M}} \frac{1}{t}\int_0^t (V - (\tfrac{1}{2}+k)r_-) \circ \varphi_s(z) ds,$$
$$\gamma_k^- := \lim_{t\to+\infty} \inf_{z\in\mathcal{M}} \frac{1}{t}\int_0^t (V - (\tfrac{1}{2}+k)r_-) \circ \varphi_s(z) ds.$$

In particular, when $V = 0$, this gives

$$\gamma_0^+ = -\frac{1}{2}\mu_{\min}, \quad \gamma_0^- = -\frac{1}{2}\mu_{\max}, \quad \gamma_1^+ = -\frac{3}{2}\mu_{\min}, \quad \gamma_1^- = -\frac{3}{2}\mu_{\max}.$$

Theorem 3 (Faure-Tsujii [FaTs1]). *Let $\mathcal{M}$ be a 3-dimensional oriented manifold and let X be a smooth vector field generating a contact Anosov flow and $V \in C^\infty(\mathcal{M})$. Then for each $\epsilon > 0$ small, there exists only finitely many resonances of $P = -X + V$ in the region*

$$\{\mathrm{Re}(\lambda) > \gamma_1^+ + \epsilon\} \setminus \{\mathrm{Re}(\lambda) \in [\gamma_0^- - \epsilon, \gamma_0^+ + \epsilon]\}.$$

If $\gamma_1^+ < \gamma_0^-$, then there are infinitely many resonances in $\{\mathrm{Re}(\lambda) \in [\gamma_0^- - \epsilon, \gamma_0^+ + \epsilon]\}$, with a Weyl type asymptotics. In the case $V = 0$, the condition $\gamma_1^+ < \gamma_0^-$ can be rewritten as the pinching condition $3\mu_{\min} > \mu_{\max}$.

By Lemma 3.2, the generalized resonant states in $\{\mathrm{Re}(\lambda) > -\frac{1}{2}\mu_{\max} - \epsilon\}$ have the regularity $H^{-\frac{1}{2}\frac{\mu_{\max}}{\mu_{\min}}-\epsilon'}(\mathcal{M})$ with $\epsilon' = \epsilon/\mu_{\min}$. The proof of Corollary 1.1 about the existence of infinitely many distributions in that Sobolev space that are horocyclic invariant (for all $\epsilon > 0$) is a direct consequence of Theorems 1 and 3, applied with $V = 0$. We notice that for general flows (not necessarily contact), existence of infinitely many resonances in some strip is proved by Jin-Zworski [JiZw] but we do not know if the associated resonant states would be regular enough to apply U_-.

Acknowledgements

C.G. is supported by ERC consolidator grant IPFLOW no 725967. F.F. and C.G. were supported by the grant ANR 13-BS01-0007-01. We thank T. Alazard, S. Crovisier, S. Dyatlov, S. Gouëzel, B. Hasselblatt, C. Liverani and T. De Poyferré for useful discussions and references. We also thank the anonymous referee for a careful reading of the manuscript and useful comments.

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Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS
Université Paris-Saclay, 91405 Orsay, France
E-mail address: colin.guillarmou@math.u-psud.fr

Institut Fourier, Université Grenoble Alpes, CNRS
38402 Saint Martin d'Hères, France
E-mail address: frederic.faure@univ-grenoble-alpes.fr

Received September 20, 2017