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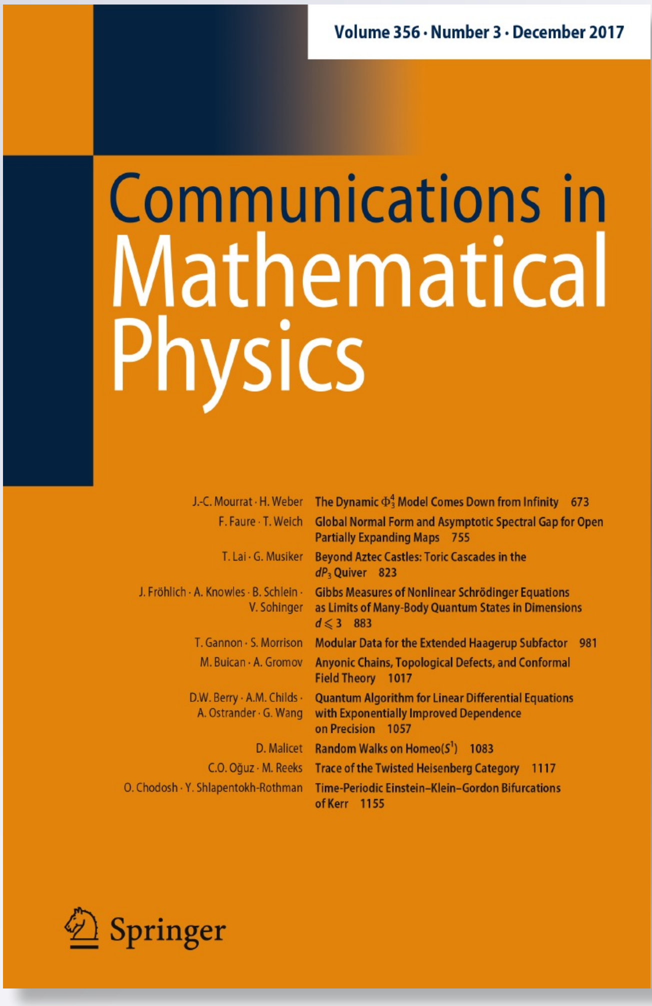

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Global Normal Form and Asymptotic Spectral Gap for Open Partially Expanding Maps

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Abstract: We consider a $\mathbb{R}$-extension of one dimensional uniformly expanding open dynamical systems and prove a new explicit estimate for the asymptotic spectral gap. To get these results, we use a new application of a "**global normal form**" for the dynamical system, a "**semiclassical expression beyond the Ehrenfest time**" that expresses the transfer operator at large time as a sum over rank one operators (each is associated to one orbit). In this paper we establish the validity of the so-called "**diagonal approximation**" **up to twice the local Ehrenfest time**.

Contents

1. Introduction

In this paper we consider a $\mathbb{R}$-extension of one dimensional uniformly expanding open dynamical systems, so called iterated function systems (IFS). The dynamical properties of these IFS are on the one hand interesting, because of relations to the spectral theory on Riemann surfaces and questions in number theory. On the other hand, the $\mathbb{R}$-extension adds a neutral direction to the dynamics and our model can also be considered as a toy model for more complicated dynamical systems such as Anosov or Axiom A flows [KH95]. The main object of study in this paper is the asymptotic spectral gap for the family of transfer operators associated to these specific open partially expanding maps. In Appendix C we propose a discussion for motivating the study of the "asymptotic spectral gap" $\gamma_{\text{asympt.}}$ from the general point of view of hyperbolic flows (in both classical and quantum mechanics). For related models we review known results on $\gamma_{\text{asympt.}}$ and we also discuss the conjecture for $\gamma_{\text{asympt.}}$ that generically $\gamma_{\text{asympt.}} = \gamma_{conj} := \frac{1}{2}\text{Pr}(2(V - J))$. This appendix may be consulted first by the reader who are interested in more detailed motivations. It is, however, not mandatory for understanding the main results.

In Sect. 2, we define the model under study. The transfer operator $\mathcal{L}_\nu$ acting on functions u is $\mathcal{L}_\nu u := e^{i\nu\tau+V} u\circ\phi^{-1}$ depending on a parameter $\nu\in\mathbb{R}$, smooth functions τ, V and ϕ^{-1} an expanding map on intervals.

In Sect. 3, we define the "asymptotic spectral gap" $\gamma_{\text{asympt.}} := \limsup_{\nu\to\infty} \log\left(r_s\left(\mathcal{L}_\nu\right)\right)$ (where $r_s\left(\mathcal{L}_\nu\right)$ stands for the spectral radius) and give the main results of this paper: in Theorem 3.3 we show that $\gamma_{\text{asympt.}} \leq \gamma_{\text{up}} := \frac{1}{2}\text{Pr}\left(2\left(V-J\right)\right)+\frac{1}{4}\left\langle J\right\rangle$ where $\text{Pr}\left(.\right)$ is the topological pressure, $J = \log\left|\left(\phi^{-1}\right)'\right| > 0$ is the expansion rate, $\langle J\rangle$ is an averaged expansion rate given in Eq. (27). In Theorem 3.6 we also get an upper bound for the norm of the resolvent of the transfer operator. We discuss their consequences in terms of decay of correlations. In Sect. 3.3 we discuss other interesting results obtained in this paper that may be extended to more general hyperbolic dynamics: a global normal form and an asymptotic expansion of the transfer operator. In Sect. 3.4 we provide a (very short) sketch of the proof of the main results.

From Sects. 4, 5, 6, 7 and 8 we provide the proof of the main Theorems and develop tools for this.

Under non local integrability (NLI) hypothesis it has been shown by D. Dolgopyat [Dol02] that $\exists\epsilon>0$, $\gamma_{\text{asympt.}}\leq\gamma_{\text{Gibbs}}-\epsilon$ with $\gamma_{\text{Gibbs}}=\text{Pr}\left(V-J\right)$. Using semiclassical analysis and some hypothesis it is also known [AFW13] that $\gamma_{\text{asympt.}}\leq\gamma_{\text{sc}}=\text{tsup}\left(V-\frac{1}{2}J\right)$ where tsup means supremum after time average (see (30)).

In Appendix A we consider examples based on linear maps and the Gauss map and compare our bounds with numerical results for the Ruelle spectrum. We also show that the new bound γ_{up} improves the previous bounds γ_{Gibbs} and γ_{sc} in some range of parameters. On the web site of the first author [Fau] we propose movies and additional multimedia contents that illustrate these models.

2. The Model

In this section we introduce the model which we study in this paper. This model has already been studied[1] in [AFW13] and we refer to this paper for more comments, examples or details.

2.1. Iterated function system. See Fig. 1 for an illustration.

Definition 2.1. "**An iterated function system (I.F.S.)**". Let $I_1,\dots I_N\subset\mathbb{R}$ be a finite collection of **disjoint bounded and closed** intervals with $N\geq 1$. Let $A\in\{0,1\}^{N\times N}$ called an adjacency matrix and assume that the matrix A is primitive, i.e. there is $T\geq 0$ such that $\forall i,j$, $\left(A^T\right)_{i,j}>0$. We will note $i\rightsquigarrow j$ if $A_{i,j}=1$. Assume that for each pair $i,j\in\{1,\dots,N\}$ such that $i\rightsquigarrow j$, we have a smooth invertible map $\phi_{i,j}:I_i\to\phi_{i,j}\left(I_i\right)\subset\text{Int}\left(I_j\right)$. Assume that the map $\phi_{i,j}$ is a **strict contraction**, i.e. there exists $0<\theta<1$ such that for every $x\in I_i$,

$$0<\phi'_{i,j}\left(x\right)\leq\theta. \tag{1}$$

We suppose that different images of the maps $\phi_{i,j}$ do not intersect (this is the "strong separation condition" in [Fal97, p.35]):

$$\left(i,j\right)\neq\left(k,l\right)\quad\Rightarrow\quad\phi_{i,j}\left(I_i\right)\cap\phi_{k,l}\left(I_k\right)=\emptyset. \tag{2}$$

[1] Compared to the previous paper [AFW13], we have changed the notation of the transfer operator from $\hat{F}$ to $\mathcal{L}$ and of its associated symplectic map from F to $\tilde{\phi}$. We have also replaced $\hbar$ by $\nu=1/\hbar$.

Remark 2.2. We have assumed for simplicity that the map $\phi_{i,j}$ preserves orientation, i.e. $0 < \phi'_{i,j}(x) \leq \theta$. The results of this paper also hold if we only suppose that $0 < \left|\phi'_{i,j}(x)\right| \leq \theta$. To treat this case, we can define $\sigma_{i,j} = \operatorname{sign}\left(\phi'_{i,j}\right) \in \{-1, 1\}$ and replace in every formula of this paper, the term $e^{J_{i,j}(x)}$ by $\sigma_{i,j}e^{J_{i,j}(x)}$. For example the truncated Gauss model presented in Sect. A.2 has negative derivatives $\phi'_{i,j}(x) < 0$.

2.2. The trapped set K. We define

$$I := \bigcup_{i=1}^{N} I_i. \tag{3}$$

The multivalued map:

$$\phi : I \to I, \quad \phi := \left(\phi_{i,j}\right)_{i,j}$$

can be iterated and generates a multivalued map $\phi^n : I \to I$ for $n \geq 1$. From Condition (2) the inverse map

$$\phi^{-1} : \phi(I) \to I$$

is uni-valued. Let

$$K_n := \phi^n(I) \tag{4}$$

and $K_0 = I$. We have $K_{n+1} \subset K_n$ so we can define the limit set

$$K := \bigcap_{n\in\mathbb{N}} K_n \tag{5}$$

called the **trapped set**. The map

$$\phi^{-1} : K \to K \tag{6}$$

is well defined and uni-valued.

2.3. The transfer operator $\mathcal{L}$.

Notations. We denote $C_0^\infty(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ with compact support. If $B \subset \mathbb{R}$ is a finite union of closed intervals, we denote by $C_0^\infty(B) \subset C_0^\infty(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ with support included in B. We denote by $C^\infty(B;\mathbb{R})$ and $C^\infty(B;\mathbb{C})$ the space of real (respect. complex) valued smooth functions on B.

Definition 2.3. Let $\tau \in C^\infty(\phi(I);\mathbb{R})$ and $V \in C^\infty(\phi(I);\mathbb{R})$ be smooth functions called respectively **roof function** and **potential** function. Let $\nu > 0$. We define the **transfer operator**:

$$\mathcal{L}_\nu : \begin{cases} C_0^\infty(I) & \to C_0^\infty(I) \\ \varphi = (\varphi_i)_i & \to \left(\sum_{i=1}^N \mathcal{L}_{i,j}\varphi_i\right)_j \end{cases} \tag{7}$$

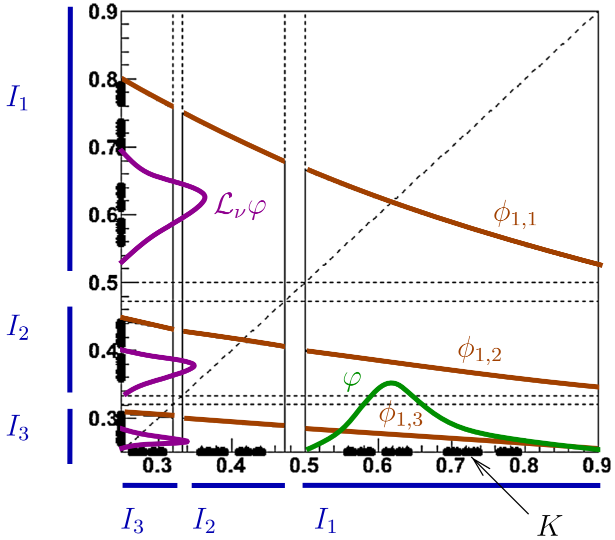

Fig. 1. Action of the transfer operator $\mathcal{L}_\nu$ on a function φ as defined in (7) for the dynamics of truncated Gauss map with $N = 3$ intervals $(I_j)_{j=1\ldots N}$, defined in Sect. A.2. In this picture φ is supported on I_1 and $\mathcal{L}_\nu\varphi$ is supported on the three intervals $I_1 \cup I_2 \cup I_3$. The maps ϕ: $\phi_{i,j} : I_i \to I_j, i, j = 1 \ldots N$ are contracting and given by $\phi_{i,j}(x) = \frac{1}{x+j}$. The trapped set K defined in (5) is a N-adic Cantor set. It is obtained as the limit of the sets $K_0 = (I_1 \cup I_2 \cdots \cup I_N) \supset K_1 = \phi(K_0) \supset K_2 = \phi(K_1) \supset \cdots \supset K$. In this example, $\dim_H(K) = 0.705\ldots$. In this schematic figure we have $\tau = 0$, $V = 0$. In general the factor $e^{V(x)}$ changes the amplitude of $\mathcal{L}_\nu\varphi$ and $e^{i\nu\tau(x)}$ creates some fast oscillations if $\nu \gg 1$

with

$$\mathcal{L}_{i,j} : \begin{cases} C_0^\infty(I_i) & \to C_0^\infty(I_j) \\ \varphi_i & \to (\mathcal{L}_{i,j}\varphi_i)(x) = \begin{cases} e^{i\nu\tau(x)+V(x)}\varphi_i\left(\phi_{i,j}^{-1}(x)\right) & \text{if } i \rightsquigarrow j \text{ and } x \in \phi_{i,j}(I_i) \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{8}$$

See Fig. 1.

Remark 2.4. 1. Eq. (7) is a family of transfer operators depending on the parameter $\nu \in \mathbb{R}$. We will be interested in the spectrum of these operators in the "semiclassical limit" $\nu \to +\infty$.

2. From Assumption (2), for any $x \in I$, the sum $\sum_{i=1}^N (\mathcal{L}_{i,j}\varphi_i)(x)$ which appears on the right hand side of (7) contains at most one non vanishing term.
3. For any $\varphi \in C_0^\infty(I)$, $n \geq 0$ we have

$$\text{supp}\left(\mathcal{L}_\nu^n\varphi\right) \subset K_n \tag{9}$$

with K_n defined in (4).
4. The family of operators $(\mathcal{L}_\nu)_{\nu\in\mathbb{R}}$ can naturally be obtained from a dynamical system (32) that is a $\mathbb{R}$-extension of the IFS and take the Fourier component with frequency ν in the neutral direction (see Sect. 3.2 or [AFW13, Sec.2.2] for a detailed explanation). The limit $\nu \to +\infty$ corresponds to the limit of high Fourier modes. In this sense

studying the spectral properties of the whole family of operators $(\mathcal{L}_\nu)_\nu$ corresponds to studying the spectral properties of this $\mathbb{R}$-extension of the IFS, i.e. a dynamical system with a neutral direction.

2.3.1. Extension of the transfer operator to distributions. In [AFW13, Sec.3.1] it is explained how the transfer operator $\mathcal{L}$, initially defined on smooth functions $C_0^\infty(I)$, can be extended to the space of distributions. For completeness we recall this construction. We first introduce a cut-off function $\chi \in C_0^\infty(K_a)$ such that $0 < \chi(x)$ for $x \in \mathrm{Int}(K_a)$ where $a \in \mathbb{N}$ and K_a is defined in (4) and $\chi(x) = 0$ for $x \in \partial K_a$. Let us remark that in the proof of Lemma 7.10, we will need to fix the value of a according to (59). We denote $\hat{\chi}$ the multiplication operator by the function χ. We define[2]

$$\mathcal{L}_{i,j,\chi} := \hat{\chi}^{-1}\mathcal{L}_{i,j}\hat{\chi} : C_0^\infty(\mathbb{R}) \to C_0^\infty(I_j), \quad \mathcal{L}_{\nu,\chi} := \hat{\chi}^{-1}\mathcal{L}_\nu\hat{\chi} \tag{10}$$

which is well defined since $\mathrm{supp}\left(\mathcal{L}_{i,j}\hat{\chi}\varphi\right) \subset \mathrm{Int}(K_a)$ where χ does not vanish, although $\hat{\chi}^{-1}$ is not defined by itself. The formal L^2-adjoint operator $\mathcal{L}^*_{i,j,\chi} : C_0^\infty(\mathbb{R}) \to C_0^\infty(I_i)$ is defined by

$$\langle \varphi_i, \mathcal{L}^*_{i,j,\chi}\psi_j\rangle_{L^2} = \langle \mathcal{L}_{i,j,\chi}\varphi_i, \psi_j\rangle_{L^2}, \quad \forall \varphi_i \in C_0^\infty(\mathbb{R}), \psi_j \in C_0^\infty(\mathbb{R}), \tag{11}$$

with the L^2-scalar product[3]

$$\langle u, v\rangle_{L^2} := \int \overline{u}(x)\, v(x)\, dx. \tag{12}$$

The L^2-adjoint operator $\mathcal{L}^*_{\nu,\chi} : C_0^\infty(\mathbb{R}) \to C_0^\infty(I)$ is defined by

$$\psi = (\psi_j)_j \to \left(\mathcal{L}^*_{\nu,\chi}\psi\right)_i(y) = \sum_{j \text{ s.t. } i \rightsquigarrow j} \left(\mathcal{L}^*_{i,j,\chi}\psi_j\right)(y)$$

whose components are given by [AFW13, Lemma 3.1]

$$\left(\mathcal{L}^*_{i,j,\chi}\psi_j\right)(y) = \frac{\chi(y)}{\chi\left(\phi_{i,j}(y)\right)}\left|\phi'_{i,j}(y)\right| e^{V(\phi_{i,j}(y))} e^{-i\nu\tau(\phi_{i,j}(y))}\psi_j\left(\phi_{i,j}(y)\right). \tag{13}$$

Proposition 2.5 [AFW13, Sec.3.2]. *By duality the transfer operators $\mathcal{L}_{\nu,\chi}$ and $\mathcal{L}^*_{\nu,\chi}$ extend to distributions:*

$$\begin{aligned} \mathcal{L}_{\nu,\chi} &: \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}) \\ \mathcal{L}^*_{\nu,\chi} &: \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}) \end{aligned} \tag{14}$$

[2] The conjugation by χ is necessary to extend the operators to distributions and thus, later in Sect. 2.4, to Sobolev spaces. This is due to the fact that the dynamics is "open". The spectral properties are, however, independent of the choice of χ (cf. Theorem 3.3).

[3] We will omit the index L^2 sometimes.

2.4. Escape function. In this section we want to introduce Hilbert spaces in which the transfer operator has discrete spectrum. We therefore consider the following "escape[4] function" on the cotangent space $T^*\mathbb{R} = \mathbb{R}^2$ with coordinates (x, ξ). Let $m > 0$ and let $A_m \in C^\infty\left(\mathbb{R}^2; \mathbb{R}\right)$ be the "symbol" given by[5]

$$A_m(x,\xi) := \langle \xi \rangle^{-m} \tag{15}$$

with $\langle \xi \rangle := \left(1+\xi^2\right)^{1/2}$. We will use the L^2-unitary $\nu-$Fourier transform $\mathcal{F}_\nu : L^2(\mathbb{R}_x) \to L^2\left(\mathbb{R}_\xi\right)$ and its inverse:

$$\begin{aligned}
(\mathcal{F}_\nu \varphi)(\xi) &:= \frac{1}{\sqrt{2\pi/\nu}} \int_{\mathbb{R}} e^{-i\nu\xi.x} \varphi(x)\, dx, \\
\left(\mathcal{F}_\nu^{-1} \psi\right)(x) &:= \frac{1}{\sqrt{2\pi/\nu}} \int_{\mathbb{R}} e^{i\nu\xi.x} \psi(\xi)\, d\xi.
\end{aligned} \tag{16}$$

Notice that the parameter ν is just a scaling in ξ. Let $\hat{A}_m := \mathrm{Op}_\nu(A_m) : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, be the linear operator defined as the semiclassical quantization of A_m [Zwo12]. In this case this is simple. For $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\left(\hat{A}_m \varphi\right)(x) &:= \frac{1}{2\pi/\nu} \int A_m(x,\xi)\, e^{i\nu(x-y)\xi} \varphi(y)\, dy d\xi \\
&= \mathcal{F}_\nu^{-1}\left(\langle \xi \rangle^{-m} (\mathcal{F}_\nu \varphi)\right)(x)
\end{aligned} \tag{17}$$

where in the last line $\langle \xi \rangle^{-m}$ denotes the multiplication operator. By duality $\hat{A}_m$ is extended to[6] $\hat{A}_m : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$. For $m \in \mathbb{R}$, the ν-Sobolev space of order m is defined by

$$H_\nu^{-m}(\mathbb{R}) := \hat{A}_m^{-1}\left(L^2(\mathbb{R})\right) \tag{18}$$

and the norm of $\varphi \in H_\nu^{-m}(\mathbb{R})$ is defined by

$$\|\varphi\|_{H_\nu^{-m}(\mathbb{R})} := \left\|\hat{A}_m \varphi\right\|_{L^2(\mathbb{R})}. \tag{19}$$

Let

$$\hat{Q}_{i,j} := \hat{A}_m \mathcal{L}_{i,j,\chi} \hat{A}_m^{-1} \quad : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

and

$$\hat{Q} := \hat{A}_m \hat{\chi}^{-1} \mathcal{L}_\nu \hat{\chi} \hat{A}_m^{-1} = \hat{A}_m \mathcal{L}_{\nu,\chi} \hat{A}_m^{-1}. \tag{20}$$

Equivalently we have the following commutative diagram

$$\begin{array}{ccc}
L^2(\mathbb{R}) & \xrightarrow{\hat{Q}} & L^2(\mathbb{R}) \\
\hat{A}_m^{-1}\big\downarrow & & \hat{A}_m^{-1}\big\downarrow \\
H_\nu^{-m}(\mathbb{R}) & \xrightarrow{\mathcal{L}_{\nu,\chi}} & H_\nu^{-m}(\mathbb{R})
\end{array}. \tag{21}$$

[4] The name "escape function" will be justified by Remark 4.2 which shows that A_m decays along the dynamics of $\tilde{\phi}$ for $|\xi| \geq C$.

[5] In fact $A_m(x,\xi)$ is independent on x.

[6] $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions, see [Tay96, p.204].

Theorem 2.6 [AFW13, th.2.6]. *"**Discrete spectrum**". For any $r > 0$, there is $m_0 > 0$ such that for all $m > m_0$ and for all $\nu \in \mathbb{R}$,*

$$\mathcal{L}_{\nu,\chi} : H_\nu^{-m}(\mathbb{R}) \to H_\nu^{-m}(\mathbb{R})$$

*has purely discrete spectrum $(\lambda_j(\nu))_{j\in\mathbb{N}}$ on the spectral domain $\{\lambda \in \mathbb{C}, |\lambda| > r\}$. The eigenvalues $(\lambda_j(\nu))_{j\in\mathbb{N}}$ in this domain are independent on m and χ and are called the **Ruelle–Pollicott resonances** of the transfer operator $\mathcal{L}_\nu$.*

Remark 2.7. From the commutative diagram (21), the spectral properties of $\hat{Q} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are equivalent to those of $\mathcal{L}_{\nu,\chi} : H_\nu^{-m}(\mathbb{R}) \to H_\nu^{-m}(\mathbb{R})$. In practice (in the proofs) we will work with $\hat{Q}$ on $L^2(\mathbb{R})$.

3. The Main Results

Let $r_s(\mathcal{L}_{\nu,\chi}) = \sup_{j\in\mathbb{N}}\{|\lambda_j(\nu)|\}$ be the spectral radius of the operator $\mathcal{L}_{\nu,\chi} : H_\nu^{-m}(\mathbb{R}) \to H_\nu^{-m}(\mathbb{R})$ with m large enough so that $r_s(\mathcal{L}_{\nu,\chi})$ does not depend on m nor on χ (for this we need that the Ruelle spectrum is non empty, otherwise we put $r_s(\mathcal{L}_{\nu,\chi}) := 0$). We are interested in the asymptotic value

$$\gamma_{\text{asympt.}} := \limsup_{\nu\to+\infty}\left(\log\left(r_s\left(\mathcal{L}_{\nu,\chi}\right)\right)\right). \tag{22}$$

To express the main results below we need to introduce the topological pressure. It can be defined from the periodic points as follows. A **periodic point** of period $n \geq 1$ is $x \in K$ such that $x = \phi^{-n}(x)$.

Definition 3.1 [Fal97, p.72]. The **topological pressure** of a Lipschitz function $\varphi \in U \to \mathbb{R}$ with U a neighborhood of the trapped set K, is

$$\Pr(\varphi) := \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{x=\phi^{-n}(x)} e^{\varphi_n(x)}\right) \tag{23}$$

where

$$\varphi_n(x) := \sum_{k=0}^{n-1}\varphi\left(\phi^{-k}(x)\right)$$

is the Birkhoff sum of φ along the periodic orbit.

We define the "**Jacobian function**"

$$J(x) := \log\frac{d\phi^{-1}}{dx}(x) > 0. \tag{24}$$

and

$$J_{max} := \operatorname{tsup}(J) := \lim_{n\to\infty}\sup_{x\in K}\left(\frac{1}{n}\sum_{k=0}^{n-1}J\left(\phi^{-k}(x)\right)\right), \tag{25}$$

$$J_{min} := \operatorname{tinf}(J) := \lim_{n\to\infty}\inf_{x\in K}\left(\frac{1}{n}\sum_{k=0}^{n-1}J\left(\phi^{-k}(x)\right)\right). \tag{26}$$

Remark 3.2. The limits on the right hand sides of (25), (26) exist because the sequences

$$a_n := \inf_{x\in K}\left(\sum_{k=0}^{n-1} J\left(\phi^{-k}(x)\right)\right), \quad b_n := \sup_{x\in K}\left(\sum_{k=0}^{n-1} J\left(\phi^{-k}(x)\right)\right)$$

are superadditive (i.e. $a_n+a_m \leq a_{n+m}$) and subadditive (i.e. $b_n+b_m \geq b_{n+m}$) respectively and Fekete's Lemma guaranties existence of the limits $J_{min} = \lim_{n\to\infty} a_n/n$ and $J_{max} = \lim_{n\to\infty} b_n/n$.

3.1. Theorems.

Theorem 3.3. ***"Bound of the spectral radius"****. Let $\beta > 0$ be defined by*

$$\Pr(2(V-J)+\beta J) = 2\Pr(V-J)$$

and

$$\langle J\rangle := \frac{2\Pr(V-J)-\Pr(2(V-J))}{\beta} \in [J_{min}, J_{max}]. \tag{27}$$

Under the Assumption 4.5 of minimal captivity defined below, if $\beta \geq \frac{1}{2}$ and $\langle J\rangle < 2J_{min}$ then

$$\gamma_{\text{asympt.}} \leq \gamma_{\text{up}} := \frac{1}{2}\Pr(2(V-J)) + \frac{1}{4}\langle J\rangle \tag{28}$$

otherwise

$$\gamma_{\text{asympt.}} \leq \gamma_{\text{Gibbs}} := \Pr(V-J). \tag{29}$$

Remark 3.4. The Assumption 4.5 of minimal captivity will be explained later but can be summarized as follows. If $\tilde{\phi}$ is the symplectic map on $T^*\mathbb{R}$ associated to the transfer operator $\mathcal{L}_\nu$ and $\mathcal{K} \subset T^*\mathbb{R}$ is its trapped set (it is a Cantor set) then the "minimal captivity assumption" is that $\tilde{\phi}$ is univalued on a small neighborhood of $\mathcal{K}$.

Remark 3.5. It is remarkable that the bound γ_{up} in (28) does not depend on the roof function τ, however beware that τ will appear in the expression of $\tilde{\phi}$ (see (40)) and therefore the Assumption 4.5 needed to get (28) depends on τ.

Previous known results about $\gamma_{\text{asympt.}}$:

- the bound (29) is already well known and holds without any assumption [Rue89].
- D. Dolgopyat [Dol02] has shown under a generic condition that $\exists\epsilon > 0$,

$$\gamma_{\text{asympt.}} \leq \gamma_{\text{Gibbs}} - \epsilon, \quad \gamma_{\text{Gibbs}} := \Pr(V-J).$$

- in [AFW13] it has been shown under the assumption of "minimal captivity" that we have

$$\gamma_{\text{asympt.}} \leq \gamma_{\text{sc}} := \text{tsup}(D) := \lim_{n\to\infty}\sup_{x\in K}\left(\frac{1}{n}\sum_{k=0}^{n-1} D\left(\phi^{-k}(x)\right)\right), \quad D := V - \frac{1}{2}J, \tag{30}$$

(here γ_{sc} stands for "$\gamma_{semi\text{-}classical}$" since we used semiclassical analysis to obtain it and $D = V - \frac{1}{2}J$ is called the "effective damping function" from [FT15]).

The next Theorem gives an upper bound for the norm of the resolvent of the transfer operator in $H_\nu^{-m}(\mathbb{R})$ outside the radius $e^{\gamma_{\mathrm{up}}}$. This is useful to control the asymptotic decay of correlation functions for the corresponding dynamical system (see Corollary 3.9). For an operator $O : \mathcal{H} \to \mathcal{H}$ we will use the notation $\|O\|_{\mathcal{H}} := \|O\|_{\mathcal{H}\to\mathcal{H}}$.

Theorem 3.6. *"bound of the resolvent" With γ_{up} and $\langle J \rangle$ given in Theorem 3.3, and $\gamma_{\mathrm{sc}} := \mathrm{tsup}\,(D)$ defined in (30), let us suppose that $\gamma_{\mathrm{up}} < \gamma_{\mathrm{sc}}$. Then for any $\epsilon > 0$, there exists $\nu_\epsilon > 0$, $C_\epsilon > 0$, such that for any $\nu > \nu_\epsilon$ we have for any $|z| > e^{(\gamma_{\mathrm{up}}+\epsilon)}$,*

$$\left\| \left(z - \mathcal{L}_{\nu,\chi}\right)^{-1} \right\|_{H_\nu^{-m}(\mathbb{R})} \leq C_\epsilon \nu^{\frac{2}{\langle J\rangle+\epsilon}(\gamma_{\mathrm{sc}}-\gamma_{\mathrm{up}})}, \tag{31}$$

Remark 3.7. 1. If $|z| > e^{(\gamma_{\mathrm{sc}}+\epsilon)}$ then a bound of the resolvent norm independent on ν has been obtained in [AFW13, Thm 2.9].

2. The positive power of ν in (31) (that diverges for $\nu \to \infty$), is related to our choice of the escape function that defines the norm in Sobolev space: the norm $\left\| \mathcal{L}_{\nu,\chi}^n \right\|$ is controlled by γ_{up} only for long time n of order $\frac{2}{\langle J\rangle} \log \nu$. This time is the required time for a wave packet starting on the trapped set to reach the region where there is an effective damping by the escape function. Using cutoff functions in some exotic symbol classes that allows a sharper cutoff in ξ, one should be able to improve this term.

3.2. Expansion of correlation functions for partially expanding maps. Theorem 3.6 has a direct application for decay of correlation functions for a related partially expanding dynamical system (see Remark 2.4(3)). One obtains exactly the same result as in [AFW13, Theorem 2.9] with the only change that we take any $\rho > e^{\gamma_{\mathrm{up}}}$ and initial functions $u \in H^{-m}(I) \otimes H^\sigma\left(S^1\right)$, $v \in H^m(I) \otimes H^\sigma\left(S^1\right)$ should have regularity of positive order $\sigma = \frac{2}{\langle J\rangle+\epsilon}\left(\gamma_{\mathrm{sc}} - \gamma_{\mathrm{up}}\right)$ in the neutral direction. Here is the precise statement.

Let ϕ be an iterated function system as defined in Definition 2.1. Recall that the map $\phi^{-1} : \phi(I) \to I$ is univalued and expanding. Let $\tau \in C^\infty(\phi(I); \mathbb{R})$ as in Definition 2.3. We define the map

$$f : \begin{cases} \phi(I) \times S^1 & \to I \times S^1 \\ (x, y) & \to \left(\phi^{-1}(x), y + \tau(x)\right) \end{cases} \tag{32}$$

with $S^1 := \mathbb{R}/(2\pi\mathbb{Z})$. Notice that the map f is expanding in the x variable whereas it is neutral in the y variable in the sense that $\frac{\partial f}{\partial y} = (0, 1)$. This is called a partially expanding map and may serve as a very simple model for the general study of partially open hyperbolic dynamics [Pes04] such as Axiom A flows. Let $V \in C^\infty(\phi(I); \mathbb{R})$.

Definition 3.8. The transfer operator of the map f with potential V is

$$\tilde{\mathcal{L}} : \begin{cases} C_0^\infty\left(I \times S^1\right) & \to C_0^\infty\left(\phi(I) \times S^1\right) \\ \psi(x, y) & \mapsto e^{V(x)} \psi(f(x, y)) \end{cases}. \tag{33}$$

A function $\psi \in C_0^\infty\left(I \times S^1\right)$ can be decomposed in Fourier modes in the y variable:

$$\psi(x, y) = \frac{1}{\sqrt{2\pi}} \sum_{\nu \in \mathbb{Z}} e^{i\nu y} \varphi_\nu(x) \tag{34}$$

then

$$\left(\tilde{\mathcal{L}}\psi\right)(x, y) = \frac{1}{\sqrt{2\pi}} \sum_{\nu \in \mathbb{Z}} e^{i\nu y} \left(\mathcal{L}_\nu \varphi_\nu\right)(x)$$

with $\mathcal{L}_\nu$ given by (7).

We introduce some notation: for a given $\nu \in \mathbb{Z}$, we have seen in Theorem 2.6 that the transfer operator $\mathcal{L}_\nu$ has a discrete spectrum of resonances. Let $\rho > 0$ such that there is no eigenvalue on the circle $|z| = \rho$ for any $\nu \in \mathbb{Z}$ and denote by $\Pi_{\rho,\nu}$ the spectral projector of the operator $\mathcal{L}_\nu$ on the domain $\{z \in \mathbb{C}, |z| > \rho\}$. These projection operators have obviously finite rank and each commutes with $\mathcal{L}_\nu$. Theorem 3.6 has the following corollary.

Corollary 3.9. ***"Expansion of correlations"***. *For m large enough (such that $r < e^{\gamma_{\mathrm{up}}}$ in Th. 2.6), for any $\epsilon > 0$, there exists $\nu_\epsilon \in \mathbb{N}$ and $C_\epsilon > 0$ such that if we put $\sigma_\epsilon = \frac{2}{\langle J \rangle + \epsilon}\left(\gamma_{\mathrm{sc}} - \gamma_{\mathrm{up}}\right)$, $\rho_\epsilon = e^{\gamma_{\mathrm{up}}+\epsilon}$, we have for any $u \in H^{-m}(I) \otimes H^{\sigma_\epsilon}\left(S^1\right)$, $v \in H^m(I) \otimes H^{\sigma_\epsilon}\left(S^1\right)$, any $n \in \mathbb{N}$,*

$$\left| \langle v | \tilde{\mathcal{L}}^n u \rangle - \sum_{|\nu| \le \nu_\epsilon} \langle v_\nu | \left(\mathcal{L}_{\nu,\chi} \Pi_{\rho_\epsilon,\nu}\right)^n u_\nu \rangle \right| \le C_\epsilon \rho_\epsilon^n \|u\|^2_{H^{-m}(I) \otimes H^{\sigma_\epsilon}(S^1)} \|v\|^2_{H^m(I) \otimes H^{\sigma_\epsilon}(S^1)}. \tag{35}$$

Here $u_\nu \in H^{-m}(I)$, $v_\nu \in H^m(I)$ stand for the Fourier components in S^1 direction of u, v defined as in (34) and $\langle v | u \rangle := \int \overline{v}(x) u(x) dx$ (extended to distributions).

Remark 3.10. The second term in Eq. (35) is a finite sum and each operator $\mathcal{L}_{\nu,\chi} \Pi_{\rho,\nu}$ has finite rank hence $\left(\mathcal{L}_{\nu,\chi} \Pi_{\rho_\epsilon,\nu}\right)^n$ can be expended over individual eigenvalues. Using the spectral decomposition of $\mathcal{L}_{\nu,\chi}$ we get an expansion of the correlation function $\langle v | \tilde{\mathcal{L}}^n u \rangle$ with a finite number of terms which involve the leading Ruelle resonances (i.e. those with modulus greater than ρ) and an error term that is $O(\rho^n)$.

Remark 3.11. In (32) we could consider $(x, y) \in \phi(I) \times \mathbb{R}$ instead which would give a Fourier decomposition $\psi(x, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\nu y} \varphi_\nu(x)\, d\nu$ with $\nu \in \mathbb{R}$. Then expansion of correlations would manifest some "diffusive behavior" governed by the range $|\nu| \le \nu_0$.

Proof. The proof of Corollary 3.9 is similar to the proof of [AFW13, Theorem 2.9].

3.3. Other interesting results: global normal form and asymptotic expansion. To get the result (28) we establish a "**global normal form**" for the transfer operator. The term "global" means here that the normal form is not specific to an individual fixed point or a periodic orbit as it is usually done [Arn88] but concerns the global dynamics in its whole. "Global normal forms" have already been considered for hyperbolic dynamics [DeL92, DeL95, Fau07] under the name "non stationary normal form". In this paper the use of global normal form shows that the transfer operator is conjugated to a simple dilation operator in a vicinity of any point and that the conjugation is Hölder continuous with respect to the point considered. This is particularly useful because dilation operators can be easily composed and this helps to study the dynamics for "long times" n: this is Theorem 5.2 that can be considered as an interesting result of this paper by itself. Then we use an expansion for large time[7] $n \gg \log \nu$, and obtain in Theorem 6.7 an asymptotic expression for the transfer operator $\mathcal{L}_\nu^n$ as a sum of rank one operators Π_w that can be written[8]:

$$\mathcal{L}_\nu^n \sim \sum_w e^{i\nu\tau_w + V_w - J_w} \Pi_w, \tag{36}$$

where the sum is over symbolic words w of length n that represent orbits (explained in Sect. 4.2), V_w is the Birkhoff sum of the function V along the orbit w (similarly for τ_w and J_w) and Π_w is a rank one operator of the form

$$\Pi_w = |\mathcal{U}_w\rangle\langle\mathcal{S}_w| \quad : \begin{cases} H_\nu^{-m}(\mathbb{R}) & \to H_\nu^{-m}(\mathbb{R}) \\ u & \to \mathcal{U}_w \langle \mathcal{S}_w | u \rangle_{H_\nu^{-m}} \end{cases} \tag{37}$$

where $\mathcal{U}_w, \mathcal{S}_w \in H_\nu^{-m}(\mathbb{R})$ are distributions associated respectively to the unstable/stable manifolds of the orbit w as shown on Fig. 4 (more precisely $\mathcal{U}_w, \mathcal{S}_w$ are Lagrangian WBK states, they will be precisely defined in Theorem 6.7).

3.4. Sketch of the proof. Let us shortly outline the principal mechanism in the proof of the new asymptotic gap bound (28) without discussing the technical difficulties. If the operator $\mathcal{L}_\nu$ would be trace class in $H_\nu^{-m}(\mathbb{R})$ then $r_s(\mathcal{L}_\nu) \leq \left|\mathrm{Tr}\left(\left(\mathcal{L}_\nu^n\right)^\dagger \mathcal{L}_\nu^n\right)\right|^{1/(2n)}$ for any $n \geq 1$, where $\dagger$ stands for adjointness in the specific Hilbert space $H_\nu^{-m}(\mathbb{R})$. In order to obtain good bounds on the trace norm we develop in a first step a global normal form (Sect. 5) as well as an asymptotic expansion (Sect. 6). This leads to the expansion (36) for $\mathcal{L}_\nu^n$ and similarly for its adjoint $\left(\mathcal{L}_\nu^n\right)^\dagger \sim \sum_{w'} e^{-i\nu\tau_{w'} + V_{w'} - J_{w'}} \Pi_{w'}^\dagger$. Using the definition of the asymptotic spectral gap (22) we get for $\nu \to \infty$,

$$\gamma_{\text{asympt.}} \leq \log\left(r_s\left(\mathcal{L}_\nu\right)\right) \leq \frac{1}{2n} \log \sum_{w,w'} e^{(V-J)_w + (V-J)_{w'} + i\nu\left(\tau_w - \tau_{w'}\right)} \mathrm{Tr}\left(\Pi_{w'}^\dagger \Pi_w\right). \tag{38}$$

We have $\mathrm{Tr}\left(\Pi_{w'}^\dagger \Pi_w\right) \underset{(37)}{=} \langle\mathcal{S}_w|\mathcal{S}_{w'}\rangle\langle\mathcal{U}_{w'}|\mathcal{U}_w\rangle$. In Proposition 7.8 we will show that for time n less than twice the Ehrenfest time i.e. such that $n < 2\frac{\log \nu}{\langle J\rangle}$ (or $e^{n\langle J\rangle} < \nu^2$) then the unstable/stable manifolds $\mathcal{S}_w, \mathcal{U}_w$ are well separated[9] in the sense that $w \neq w' \Rightarrow \mathrm{Tr}\left(\Pi_{w'}^\dagger \Pi_w\right) \simeq 0$. Applying this separation to the double sum (38) means that the non

[7] The symbol $n \gg \log \nu$ means here the right hand side of (94) get relatively small.

[8] We don't give here a precise statement of the result. We just write the main terms and ignore the remainders. We refer to Theorem 6.7 for a precise statement.

[9] This is up to some few pairs (w, w') that give negligible contributions. To control these terms we use large deviation techniques explained in Appendix B.

diagonal terms can be neglected and we call this the "diagonal approximation". For the diagonal terms that remain we compute that $\mathrm{Tr}\left(\Pi_w^\dagger \Pi_w\right) \leq \nu C$. This gives

$$\begin{aligned}\gamma_{\text{asympt.}} &\underset{(38)}{\leq} \frac{1}{2n}\log\left(\sum_w e^{2(V-J)_w} C\nu\right)\\ &\underset{(159)}{=} \frac{1}{2}\mathrm{Pr}\left(2\left(V-J\right)\right) + \frac{1}{2n}\log C + \frac{1}{2n}\log\nu + o\left(1\right) \end{aligned} \tag{39}$$

Then we take the time $n = 2\frac{\log\nu}{\langle J\rangle}(1-\epsilon)$ with any $\epsilon > 0$ and $\nu \to +\infty$ and get the result (28) that $\gamma_{\text{asympt.}} \leq \gamma_{\text{up}} := \frac{1}{2}\mathrm{Pr}\left(2\left(V-J\right)\right) + \frac{1}{4}\langle J\rangle$.

4. The Canonical Map $\tilde{\phi}$ and Its Trapped Set $\mathcal{K}$ in Phase Space

According to [AFW13, Lemma 4.2], the transfer operator $\mathcal{L}_{i,j}$ in (8) is a ν-semiclassical Fourier integral operator (FIO).[10] It is a general fact in semiclassical analysis that various properties of Fourier integral operators are obtained from the properties of their associated symplectic map (or canonical map) which are maps on the cotangent space. Here we use coordinates $x \in I_i$ and $\xi \in T_x^* I_i$. The canonical map associated to $\mathcal{L}_{i,j}$ is defined by [AFW13, Lemma 4.2]

$$\tilde{\phi}_{i,j} : \begin{cases} T^* I_i & \to T^* I_j \\ (x,\xi) & \to \begin{cases} x' & = \phi_{i,j}(x) \\ \xi' & = \frac{1}{\phi'_{i,j}(x)}\xi + \tau'\left(x'\right). \end{cases} \end{cases} \tag{40}$$

This gives a multi-valued canonical map $\tilde{\phi} : T^*I \to T^*I$ on the phase space $T^*I \cong I \times \mathbb{R}$, given by:

$$\tilde{\phi} : \begin{cases} T^* I & \to T^* I \\ (x,\xi) & \to \left\{\tilde{\phi}_{i,j}(x,\xi) \quad \text{ with } i,j \text{ s.t. } x \in I_i,\ i \rightsquigarrow j\right\}. \end{cases} \tag{41}$$

We have the following property of "escape at infinity outside a compact" for the dynamics defined by $\tilde{\phi} : T^*I \to T^*I$:

Lemma 4.1 [AFW13, Lemma 4.4]. *For any* $1 < \kappa < e^{J_{\min}}$, *there exists* $C \geq 0$ *such that* $\forall x \in I_i$, $\forall \xi \in \mathbb{R}$, $\forall j$ *s.t.* $i \rightsquigarrow j$,

$$\left(x',\xi'\right) = \tilde{\phi}_{i,j}(x,\xi) \text{ and } |\xi| > C \Rightarrow \left|\xi'\right| > \kappa|\xi|. \tag{42}$$

Remark 4.2. At this stage we observe from (42) that the function $A_m(x,\xi) = \langle\xi\rangle^{-m}$ defined in (15) satisfies $|\xi| > C \Rightarrow \frac{A_m\left(\tilde{\phi}_{i,j}(x,\xi)\right)}{A_m(x,\xi)} \leq c^m$ with $c = \sqrt{\frac{C^2+1}{(C\kappa)^2+1}} < 1$, i.e. A_m decreases strictly with the dynamics. This explains why we call A_m an "escape function".

[10] The reader does not need to be familiar with the theory of global Fourier integral operators for the rest of the article. For a discussion of FIOs in the context of IFS-transfer operators we refer to [AFW13, Section 4]. For a more general introduction we refer to [Zwo12, Chap.10].

4.1. The trapped set $\mathcal{K}$. We define the trapped set $\mathcal{K}$ for the dynamics of the canonical map $\tilde{\phi}$ in (41), as the points which do not escape "totally" neither in the past nor the future:

Definition 4.3. The **trapped set** in phase space $T^*I \equiv I \times \mathbb{R}$ is defined as

$$\mathcal{K} = \{(x, \xi) \in I \times \mathbb{R}, \quad \exists \mathcal{D} \Subset I \times \mathbb{R} \text{ compact}, \\ s.t.\ \forall n \in \mathbb{Z} \quad \tilde{\phi}^n(x, \xi) \cap \mathcal{D} \neq \emptyset\Big\}$$

Remark 4.4. Since the map $\tilde{\phi}$ is a lift of the map $\phi : I \to I$ we conclude that $\mathcal{K} \subset (K \times \mathbb{R})$ with K the trapped set of ϕ defined in (5). Using any value of C given from Lemma 4.1 one can make this precise and obtain that

$$\mathcal{K} \subset (K \times \{\xi \in \mathbb{R}, |\xi| \leq C\}).$$

For $\varepsilon > 0$, let $\mathcal{K}_\varepsilon$ denote a closed $\varepsilon-$neighborhood of the trapped set $\mathcal{K}$, namely

$$\mathcal{K}_\varepsilon := \left\{(x, \xi) \in T^*I, \quad \exists (x_0, \xi_0) \in \mathcal{K}, \quad \max(|x - x_0|, |\xi - \xi_0|) \leq \varepsilon\right\}.$$

From now on we will make the following hypothesis on the multi-valued map $\tilde{\phi}$.

Assumption 4.5. We assume the following property called "**minimal captivity**":

$$\exists \varepsilon > 0, \quad \forall (x, \xi) \in \mathcal{K}_\varepsilon, \quad \sharp\left\{\tilde{\phi}(x, \xi) \bigcap \mathcal{K}_\varepsilon\right\} \leq 1. \tag{43}$$

This means that the dynamics of $\tilde{\phi}$ is univalued on a neighborhood of the trapped set $\mathcal{K}$.

Remark 4.6. 1. The minimal captivity assumption has been introduced in [AFW13] and it allows an easy description of the trapped set $\mathcal{K}$. We refer to [AFW13, Prop. 4.1] for further discussions and alternative equivalent formulations.
2. The minimal captivity assumption is a stronger assumption than the Dolgopyat non-local integrability condition, [Dol98], [Nau05, Definition 2.1]. We refer to [AFW13, Section 4.3] for a comparison.
3. In [AFW13, Prop. 7.3] an explicit procedure to verify the minimal captivity assumption is discussed and minimal captivity assumption is proven for the examples of Bowen Series map and the truncated Gauss maps.
4. Given the minimal captivity Assumption 4.5, let $U \subset \mathbb{R}^2$ open such that $\mathcal{K} \subset U \subset (\mathcal{K}_\varepsilon \cap \tilde{\phi}^{-1}(\mathcal{K}_\varepsilon))$. We can extend the univalued map $\tilde{\phi}$ on $\mathcal{K}$ to an embedding $\tilde{\phi} : U \to \mathbb{R}^2$. With this point of view $\mathcal{K}$ is simply the maximal invariant hyperbolic set of the diffeomorphism $\tilde{\phi}$ (cf. [Has02, Section 1.b, Section 2.h]). This observation will be useful in the proof of Lemma 7.11 to use regularity estimates on the stable foliation of $\tilde{\phi}$.

4.2. Symbolic dynamics on the trapped set $K \subset I$. In the following two sub-sections we introduce the symbolic dynamics on the trapped set $K \subset I$ and the trapped set $\mathcal{K} \subset T^*I$ in phase space. Note that the symbolic dynamics is not necessary for the definition of Ruelle–Pollicott resonances neither for the statement of the results. It is a useful tool to keep track of orbits and is natural in the context of I.F.S. dynamics that is defined from a set of intervals $(I_j)_{j=1\ldots N}$.

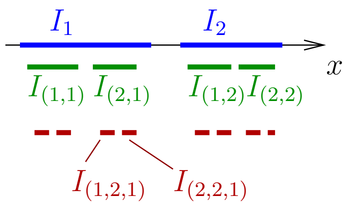

Fig. 2. This picture illustrates the definition of intervals $I_{w_{-n,0}}$ defined in (47) from a word $w_{-n,0} = (w_{-n}, \dots w_{-1}, w_0)$ and the contracting maps $\phi_{i,j} : I_i \to I_j$. For example here $I_{(2,2,1)} := \phi_{(2,2,1)}(I_2) := \phi_{2,1} \circ \phi_{2,2}(I_2) = \phi_{2,1}(I_{2,2}) \subset I_1$

We first consider the dynamics of $\phi : I \to I$ on the "base space" I. Let

$$\mathcal{W}_- := \left\{ (\dots, w_{-2}, w_{-1}, w_0) \in \{1, \dots, N\}^{-\mathbb{N}}, w_{l-1} \rightsquigarrow w_l, \forall l \le 0 \right\} \tag{44}$$

be the set of admissible left semi-infinite sequences. In other words, $\mathcal{W}_-$ is a subshift of finite type [BS02, p.56]. For $w_- \in \mathcal{W}_-$ and $i < j \le 0$ we write

$$w_{i,j} := \left(w_i, w_{i+1}, \dots w_j \right) \tag{45}$$

for an extracted sequence. For simplicity we will use the following notation for the composition of maps:

$$\phi_{w_{i,j}} := \phi_{w_{j-1}, w_j} \circ \phi_{w_{j-2}, w_{j-1}} \circ \dots \circ \phi_{w_i, w_{i+1}} : I_{w_i} \to I_{w_j}. \tag{46}$$

For $n \ge 0$, let

$$I_{w_{-n,0}} := \phi_{w_{-n,0}} \left(I_{w_{-n}} \right) \subset I_{w_0}. \tag{47}$$

See Fig. 2.

For any $0 < m < n$ we have the strict inclusions $I_{w_{-n,0}} \subset I_{w_{-m,0}} \subset I_{w_0}$ and from (1), the size of $I_{w_{-n,0}}$ is bounded by $\left| I_{w_{-n,0}} \right| \le \theta^n \left| I_{w_0} \right|$, hence the sequence of sets $\left(I_{w_{-n,0}} \right)_{n \ge 1}$ is a sequence of non empty and decreasing closed intervals and $\bigcap_{n=1}^{\infty} I_{w_{-n,0}}$ is a point in the trapped set K, Eq. (5). So we can define

Definition 4.7. The "**symbolic coding map** of K" is

$$S : \begin{cases} \mathcal{W}_- & \to K \\ w_- & \mapsto S(w_-) := \bigcap_{n=1}^{\infty} I_{w_{-n,0}} \end{cases} \tag{48}$$

Let us introduce the **left shift** L, a multivalued map, defined by

$$L : \begin{cases} \mathcal{W}_- & \to \mathcal{W}_- \\ (\dots, w_{-2}, w_{-1}, w_0) & \to (\dots, w_{-2}, w_{-1}, w_0, w_1) \end{cases} \tag{49}$$

with $w_1 \in \{1, \dots, N\}$ such that $w_0 \rightsquigarrow w_1$ and let the **right shift** R be the univalued map defined by

$$R : \begin{cases} \mathcal{W}_- & \to \mathcal{W}_- \\ (\dots, w_{-2}, w_{-1}, w_0) & \to (\dots, w_{-2}, w_{-1}) \end{cases}. \tag{50}$$

Proposition 4.8 [AFW13, Prop. 4.12]. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{W}_- & \xrightarrow{S} & K \\ R \uparrow\downarrow L & & \phi^{-1} \uparrow\downarrow \phi \\ \mathcal{W}_- & \xrightarrow{S} & K \end{array} \tag{51}$$

and the map $S : \mathcal{W}_- \to K$ is one to one.

4.3. Symbolic dynamics on the trapped set $\mathcal{K} \subset T^*I$. We consider now the dynamics of the canonical map $\tilde{\phi} : I \times \mathbb{R} \to I \times \mathbb{R}$ on the phase space. Let

$$\mathcal{W}_+ := \left\{ (w_0, w_1, w_2 \ldots) \in \{1, \ldots, N\}^{\mathbb{N}}, \quad w_l \rightsquigarrow w_{l+1}, \forall l \geq 0 \right\}$$

be the set of admissible right semi-infinite sequences. For any $n \geq 0$ let

$$\tilde{I}_{w_{0,n}} := \tilde{\phi}^{-n} \left(I_{w_{0,n}} \times [-C, C] \right) \tag{52}$$

be the image of the rectangle under the univalued map $\tilde{\phi}^{-n}$, where $I_{w_{0,n}}$ is given in (47) and C is given by Lemma 4.1. See Fig. 3a. Notice that $\pi \left(\tilde{I}_{w_{0,n}} \right) = I_{w_0}$ where $\pi (x, \xi) = x$ is the canonical projection map. Since the map $\tilde{\phi}^{-1}$ contracts strictly in the variable ξ by the factor $\theta < 1$ the sequence $\left(\tilde{I}_{w_{0,n}} \right)_{n \in \mathbb{N}}$ is strictly decreasing: $\tilde{I}_{w_{0,n+1}} \subset \tilde{I}_{w_{0,n}}$ and we can define the limit

$$\tilde{S} : \begin{cases} \mathcal{W}_+ & \to I \times \mathbb{R} \\ w_+ & \mapsto \tilde{S}(w_+) := \bigcap_{n \geq 0} \tilde{I}_{w_{0,n}} \end{cases}. \tag{53}$$

Proposition 4.9 [AFW13, Prop. 4.13]. *For every $w_+ \in \mathcal{W}_+$, the set $\tilde{S}(w_+) \subset T^* I_{w_0}$ is a smooth curve given by*

$$\tilde{S}(w_+) = \left\{ \left(x, \zeta_{w_+}(x) \right), x \in I_{w_0} \right\} \tag{54}$$

with

$$\zeta_{w_+}(x) := -\sum_{k \geq 1} e^{-J_{w_{0,k}}(x)} \tau' \left(\phi_{w_{0,k}}(x) \right), \tag{55}$$

and

$$J_{w_{0,n}}(x) := \sum_{k=1}^{n} J_{w_k, w_{k+1}} \left(\phi_{w_{0,k}}(x) \right) \tag{56}$$

is the Birkhoff sum of the "Jacobian function" defined in (24)

$$J_{i,j}(x) = -\log \left(\phi'_{i,j}(x) \right) > 0. \tag{57}$$

We have an estimate of regularity, uniform in w: $\forall \alpha \in \mathbb{N}$, $\exists C_\alpha > 0$, $\forall w_+ \in \mathcal{W}_+$, $\forall x \in I_{w_0}$,

$$\left| \left(\partial_x^\alpha \zeta_{w_+} \right)(x) \right| \leq C_\alpha. \tag{58}$$

Moreover, with the hypothesis 4.5 of minimal captivity with a neighborhood $\mathcal{K}_\varepsilon$ of $\mathcal{K}$, there exists $a \geq 1$ and K_a defined in (4) *such that*

$$\forall x \in K_a, \forall w_+ \in \mathcal{W}_+, \quad \left(x, \zeta_{w_+}(x) \right) \in \mathcal{K}_\varepsilon. \tag{59}$$

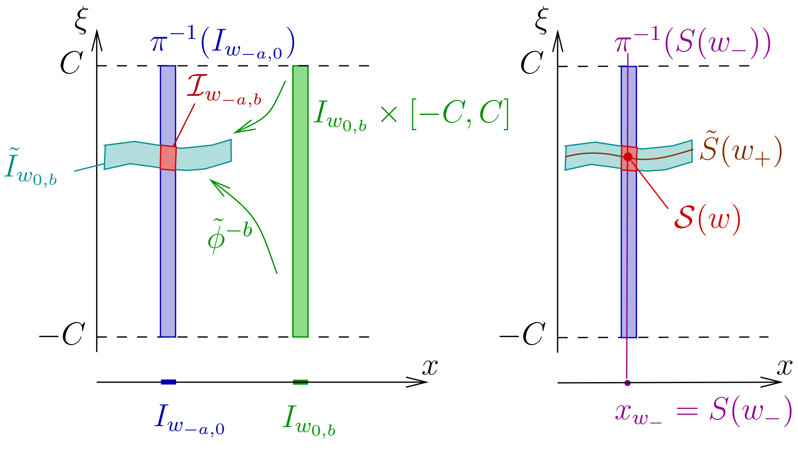

Fig. 3. **a** Illustrates the construction of $\tilde{I}_{w_{0,b}}$ given in (52), the construction of $\mathcal{I}_{w_{-a,b}}$ given in (60). **b** Illustrates the limit of these sets for semi-infinite words, i.e. $a, b \to \infty$. This gives the smooth curve $\tilde{S}(w_+)$ given in (54), the point $x_{w_-} = S(w_-) \in K$ given in (48), the vertical line $\pi^{-1}(S(w_-))$ and finally the intersection $\mathcal{S}(w) := \left(\pi^{-1}(S(w_-)) \cap \tilde{S}(w_+)\right) \in \mathcal{K}$ given in (61), that depends on a bi-infinite word $w \equiv (w_-, w_+) \in \mathcal{W}$

Proof. For (55) and (58) see [AFW13, Prop. 4.13]. For (59) see [AFW13, Prop. 4.10 (1)].

Let

$$\mathcal{W} := \left\{(\ldots w_{-2}, w_{-1}, w_0, w_1, \ldots) \in \{1, \ldots, N\}^{\mathbb{Z}}, \quad w_l \rightsquigarrow w_{l+1}, \forall l \in \mathbb{Z}\right\}$$

be the set of bi-infinite admissible sequences. For a given $w \in \mathcal{W}$ and $a, b \in \mathbb{N}$, let

$$\mathcal{I}_{w_{-a,b}} := \left(\pi^{-1}\left(I_{w_{-a,0}}\right) \cap \tilde{I}_{w_{0,b}}\right). \tag{60}$$

See Fig. 3a.

Definition 4.10. The **symbolic coding map** of $\mathcal{K}$ is

$$\mathcal{S} : \begin{cases} \mathcal{W} & \to \mathcal{K} \\ w & \mapsto \mathcal{S}(w) := \bigcap_{n=1}^{\infty} \mathcal{I}_{w_{-n,n}} = \left(\pi^{-1}(S(w_-)) \cap \tilde{S}(w_+)\right) \end{cases} \tag{61}$$

with $w_- = (\ldots w_{-1}, w_0) \in \mathcal{W}_-$, $w_+ = (w_0, w_1, \ldots) \in \mathcal{W}_+$ (with the same extreme letter w_0).

See Fig. 3b. More precisely we can express the point $\mathcal{S}(w) \in \mathcal{K}$ as

$$\mathcal{S}(w) = (x_w, \xi_w), \quad x_w = S(w_-) \in K, \quad \xi_w = \zeta_{w_+}(x_w) \in T^*_{x_w} I, \tag{62}$$

with $S(w_-)$ given in (48) and ζ_{w_+} given in (55).

Let L, R denote the full left/right shift on $\mathcal{W}$ defined similarly to (49) and (50) by $(Lw)_j = w_{j+1}$ and $(Rw)_j = w_{j-1}$.

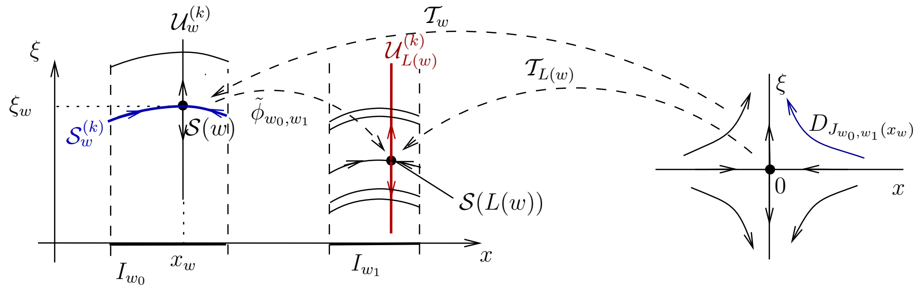

Fig. 4. According to (63) and (62), a word $w \in \mathcal{W}$ is associated to a point $\mathcal{S}(w) = (x_w, \xi_w) \in \mathcal{K}$. By the canonical map $\tilde{\phi}$, this point is sent to $\tilde{\phi}(\mathcal{S}(w)) = \mathcal{S}(L(w))$. In fact, in a vicinity of $\mathcal{S}(w)$ the map $\tilde{\phi}$ is conjugated to the dilation map: $\tilde{\phi}_{w_0,w_1} = \mathcal{T}_{L(w)} \circ D_{J_{w_0,w_1}(x_w)} \circ \mathcal{T}_w^{-1}$. Equation (65) shows that this conjugation is also true for the component $\mathcal{L}_{w_0,w_1}$ of the transfer operator. The stable (respectively unstable) manifold of the point $\mathcal{S}(w)$ (in blue, respect. in red) supports a Lagrangian state $\mathcal{S}_w^{(k)}$ (respect. $\mathcal{U}_w^{(k)}$) that are defined in Theorem 6.7 and used to express the transfer operator $\mathcal{L}_\nu^n$ for large time, as an asymptotic expansion (color figure online)

Proposition 4.11 [AFW13, Prop. 4.15]. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\mathcal{S}} & \mathcal{K} \\ R \uparrow\downarrow L & & \tilde{\phi}^{-1} \uparrow\downarrow \tilde{\phi} \\ \mathcal{W} & \xrightarrow{\mathcal{S}} & \mathcal{K} \end{array} \tag{63}$$

If Assumption 4.5 of minimal captivity holds true then the map $\mathcal{S} : \mathcal{W} \to \mathcal{K}$ is one to one. This means that the univalued dynamics of points on the trapped set $\mathcal{K}$ under the maps $\tilde{\phi}^{-1}, \tilde{\phi}$ is equivalent to the symbolic dynamics of the full shift maps R, L on the set of words $\mathcal{W}$.

Remark 4.12. Considering the trapped set $\mathcal{K}$ as the hyperbolic set with a local product structure of $\tilde{\phi}$ (cf. Remark 4.6(4)), the curve $\tilde{S}_{w_+} = (x, \zeta_w(x))$ is precisely the stable manifold [Has02, Section 2.e] through the point $(x_w, \zeta_{w^+}(x_w)) \in \mathcal{K}$. The unstable manifold is the vertical line $\pi^{-1}(S(w_-)) = \{(x_w, \xi), \xi \in \mathbb{R}\}$ (cf. Figs. 4 or 3b).

5. Global Normal Form

Normal forms are usually constructed for individual fixed points or individual periodic orbits [Arn88]. In few papers, normal forms have already been considered globally for a hyperbolic dynamics [DeL92,DeL95,Fau07]. We present here the global normal form for the transfer operator $\mathcal{L}_\nu$ considered in this paper. This is Theorem 5.2 below. We will need the following elementary (Fourier integral) operators on $C_0^\infty(\mathbb{R})$ and their associated symplectic (or canonical) maps on $T^*\mathbb{R} \equiv \mathbb{R}^2_{x,\xi}$ [Zwo12, chap.10]. Let $\varphi \in C_0^\infty(\mathbb{R})$.

- For $\lambda \in \mathbb{R}$, the **dilation operator** is

$$\left(\hat{D}_\lambda \varphi\right)(y) := \varphi\left(e^\lambda y\right) \tag{64}$$

whose canonical map is $D_\lambda : (x, \xi) \to \left(x' = e^{-\lambda} x, \xi' = e^\lambda \xi\right)$.

- For $x \in \mathbb{R}$, the **translation operator** is
$$\left(\hat{T}_x\varphi\right)(y) := \varphi(y-x)$$
whose canonical map is $T_x : (y,\xi) \to \left(y' = y + x, \xi' = \xi\right)$.
- For a smooth diffeomorphism $f : \mathbb{R} \to \mathbb{R}$, the **composition operator** is
$$\left(\hat{\mathcal{L}}_f\varphi\right)(y) := \varphi\left(f^{-1}(y)\right)$$
whose canonical map is $\mathcal{L}_f : (x,\xi) \to \left(x' = f(x), \xi' = \left(f'(x)\right)^{-1} \cdot \xi\right)$.
- For two smooth functions $\Upsilon^{(0)}, \Upsilon^{(1)} \in C^\infty(\mathbb{R};\mathbb{R})$ and $\nu > 0$, let $\Upsilon = \Upsilon^{(0)} + \frac{i}{\nu}\Upsilon^{(1)}$ and
$$\left(\hat{\Theta}_\Upsilon\varphi\right)(y) := e^{i\nu\Upsilon(y)}\varphi(y)$$
whose canonical map is $\Theta_\Upsilon : (x,\xi) \to \left(x' = x, \xi' = \xi + \frac{d\Upsilon^{(0)}}{dx}(x)\right)$.

Remark 5.1. The dilation operator is a special case of a composition operator: $\hat{D}_\lambda = \hat{\mathcal{L}}_f$ with $f(y) = e^{-\lambda}y$. Similarly for the translation operator: $\hat{T}_x = \hat{\mathcal{L}}_f$ with $f(y) = y - x$. For any two diffeomorphism f, g one has $\hat{\mathcal{L}}_f \circ \hat{\mathcal{L}}_g = \hat{\mathcal{L}}_{f\circ g}$.

The next theorem shows that using a combination of these previous simple operators, the transfer operator $\mathcal{L}_\nu : C_0^\infty(I) \to C_0^\infty(I)$ defined in (7) is "globally conjugated" to a simple dilation operator. This is illustrated in Fig. 4.

Theorem 5.2. ***"Global normal form"***. *For any word $w = (\ldots w_{-1}, w_0, w_1, \ldots) \in \mathcal{W}$ there exist functions $\Upsilon_w^{(0)}, \Upsilon_w^{(1)} \in C^\infty(I_{w_0};\mathbb{R})$, $\Upsilon_w = \Upsilon_w^{(0)} + \frac{i}{\nu}\Upsilon_w^{(1)} \in C^\infty\left(I_{w_0};\mathbb{C}\right)$ as well as a map $H_w : \mathcal{J} \to \mathbb{R}$ defined on an neighborhood $\mathcal{J} \subset \mathbb{R}$ of 0, which is independent of the word $w \in \mathcal{W}$. H_w is a C^∞ diffeomorphism onto its image and the following points hold.*

1. *There exists a neighborhood U of x_w such that the transfer operator in (8) acting on $C_0^\infty(U)$ can be expressed as*
$$\mathcal{L}_{w_0,w_1} = e^{i\nu\tau\left(x_{L(w)}\right)+V\left(x_{L(w)}\right)} \cdot \widehat{\mathcal{T}}_{L(w)} \circ \hat{D}_{J_{w_0,w_1}(x_w)} \circ \widehat{\mathcal{T}}_w^{-1} \tag{65}$$
with
$$\widehat{\mathcal{T}}_w := \hat{\Theta}_{\Upsilon_w} \circ \hat{T}_{x_w} \circ \hat{\mathcal{L}}_{H_w}, \tag{66}$$
$x_w \in I_{w_0}$ defined in (62) and J_{w_0,w_1} defined in (57). Equation (65) means that the components of the transfer operator (8) are conjugated to some dilation operator multiplied by a constant. The operator $\widehat{\mathcal{T}}_w$ is a FIO whose canonical map $\mathcal{T}_w$ sends $(0,0)$ to the point $\mathcal{S}(w) = (x_w, \xi_w) \in \mathcal{K}$.
2. *$H_w(0) = 0$, $H_w'(0) = 1$, $\Upsilon_w(x_w) = 0$ and*
$$\frac{d\Upsilon_w^{(0)}}{dy}(y) = \zeta_w(y)$$
with $\zeta_w(y) := \zeta_{w_+}(y)$ given in (55).

3. For any $\alpha \in \mathbb{N}$ there exists C_α such that for any $w \in \mathcal{W}$,

$$\left|\partial^\alpha H_w\right|_{L^\infty} < C_\alpha, \left|\partial^\alpha H_w^{-1}\right|_{L^\infty} < C_\alpha, \quad \left|\partial^\alpha \Upsilon_w\right|_{L^\infty} < C_\alpha. \tag{67}$$

that express some regularity of the functions H_w, Υ_w.

Remark 5.3. Note that for a given operator $\mathcal{L}_{w_0,w_1}$, the word $w \in \mathcal{W}$ appearing in the conjugation $\widehat{\mathcal{T}}_w$ can be an arbitrary extension of (w_0, w_1). This freedom for the choice of extension will be used from time to time in the sequel. When necessary, we will check that the choice does give bounded or negligible corrections, see e.g. Lemma B.1. The right hand side in (65) is acting on functions with support in a neighborhood of x_w that contains $K \cap I_{w_0}$. This is enough for us.

Remark 5.4. A consequence of (67) is that for any $\chi \in C_0^\infty(I)$, $m \in \mathbb{R}$,

$$\begin{aligned} \widehat{\mathcal{T}}_w \hat{\chi} &: H_\nu^{-m}(\mathbb{R}) \to H_\nu^{-m}(\mathbb{R}), \\ \widehat{\mathcal{T}}_w^{-1} \hat{\chi} &\quad H_\nu^{-m}(\mathbb{R}) \to H_\nu^{-m}(\mathbb{R}), \end{aligned} \tag{68}$$

are bounded uniformly with respect to $w \in \mathcal{W}$ and $\nu \in \mathbb{R}$.

The next Corollary uses Theorem 5.2 and iterations of it to express long time evolution.

Corollary 5.5. *For any $n \geq 1$,*

$$\mathcal{L}_\nu^n = \sum_{w_{0,n}} \mathcal{L}_{w_{0,n}}. \tag{69}$$

For each term $\mathcal{L}_{w_{0,n}}$, let $w \in \mathcal{W}$ be an arbitrary extension of $w_{0,n}$. We can write

$$\mathcal{L}_{w_{0,n}} = e^{i\nu\tau_{w_{0,n}}(x_w) + V_{w_{0,n}}(x_w)} \widehat{\mathcal{T}}_{L^n(w)} \hat{D}_{J_{w_{0,n}}(x_w)} \widehat{\mathcal{T}}_w^{-1}, \quad w \in \mathcal{W} \tag{70}$$

with $V_{w_{0,n}}$, $\tau_{w_{0,n}}$ and $J_{w_{0,n}}$ being Birkhoff sums defined as in (56).

Proof. of Theorem 5.2. For simplicity we define

$$\mathcal{V}(x) := \tau(x) - \frac{i}{\nu} V(x). \tag{71}$$

Let us denote $\hat{y} : \varphi(y) \to y\varphi(y)$ the multiplication operator by y. The operator $\mathcal{L}_{w_0,w_1}$ in (8) can be written:

$$\mathcal{L}_{w_0,w_1} = e^{i\nu\mathcal{V}(\hat{y})} \mathcal{L}_{\phi_{w_0,w_1}} \tag{72}$$

with $\mathcal{L}_{\phi_{w_0,w_1}} \varphi := \varphi \circ \phi_{w_0,w_1}^{-1}$. The aim is to transform progressively (72) into the expression (65). For the first step we write

$$\begin{aligned} \mathcal{L}_{w_0,w_1} &= e^{i\nu\mathcal{V}(x_{L(w)})} e^{i\nu(\mathcal{V}(\hat{y}) - \mathcal{V}(x_{L(w)}))} \mathcal{L}_{\phi_{w_0,w_1}} \\ &= e^{i\nu\mathcal{V}(x_{L(w)})} \mathcal{L}_{\phi_{w_0,w_1}} e^{i\nu(\mathcal{V}(\phi_{w_0,w_1}(\hat{y})) - \mathcal{V}(x_{L(w)}))} \end{aligned}$$

For any $y \in I_{w_0}$ and $k \geq 1$ we have $\left|\phi_{w_{0,k}}(y) - x_{L^k(w)}\right| \leq C.e^{-kJ_{\min}}$ with some $C > 0$ independent on w and y. Hence

$$\Upsilon_w(y) := -\sum_{k \geq 0} \left(\mathcal{V}\left(\phi_{w_{0,k+1}}(y)\right) - \mathcal{V}\left(x_{L^{k+1}(w)}\right)\right) \tag{73}$$

defines a smooth complex valued function with regularity estimate given in (67). We have also $\Upsilon_w(x_w) \underset{(73)}{=} 0$ and

$$\frac{d}{dy}\Upsilon_w(y) \underset{(73)}{=} -\sum_{k\geq 0}\mathcal{V}'\left(\phi_{w_{0,k+1}}(y)\right)\cdot\phi'_{w_{0,k+1}}(y)$$

so $\frac{d}{dy}\Upsilon_w^{(0)}(y) \underset{(55)}{=} \zeta_{w_+}(y)$. The family of functions Υ_w solves the homological equation

$$\begin{aligned}\Upsilon_{L(w)}\left(\phi_{w_0,w_1}(y)\right) &= -\sum_{k\geq 0}\left(\mathcal{V}\left(\phi_{w_{0,k+2}}(y)\right)-\mathcal{V}\left(x_{L^{k+2}(w)}\right)\right)\\ &= -\sum_{k\geq 1}\left(\mathcal{V}\left(\phi_{w_{0,k+1}}(y)\right)-\mathcal{V}\left(x_{L^{k+1}(w)}\right)\right)\\ &= \Upsilon_w(y)+\left(\mathcal{V}\left(\phi_{w_0,w_1}(y)\right)-\mathcal{V}\left(x_{L(w)}\right)\right).\end{aligned}$$

Therefore we get

$$\begin{aligned}\mathcal{L}_{w_0,w_1} &= e^{i\nu\mathcal{V}\left(x_{L(w)}\right)}\mathcal{L}_{\phi_{w_0,w_1}}e^{i\nu\left(\mathcal{V}\left(\phi_{w_0,w_1}(\hat{y})\right)-\mathcal{V}\left(x_{L(w)}\right)\right)}\\ &= e^{i\nu\mathcal{V}\left(x_{L(w)}\right)}\mathcal{L}_{\phi_{w_0,w_1}}e^{i\nu\left(\Upsilon_{L(w)}\left(\phi_{w_0,w_1}(\hat{y})\right)-\Upsilon_w(\hat{y})\right)}\\ &= e^{i\nu\mathcal{V}\left(x_{L(w)}\right)}e^{+i\nu\Upsilon_{L(w)}(\hat{y})}\mathcal{L}_{\phi_{w_0,w_1}}e^{-i\nu\Upsilon_w(\hat{y})}.\end{aligned}$$

For the second step we write

$$\mathcal{L}_{w_0,w_1} = e^{i\nu\mathcal{V}\left(x_{L(w)}\right)}e^{+i\nu\Upsilon_{L(w)}(\hat{y})}\hat{T}_{x_{L(w)}}\mathcal{L}_{f_{0,1}}\hat{T}_{-x_w}e^{-i\nu\Upsilon_w(\hat{y})} \tag{74}$$

with[11]

$$f_{0,1}(z) := \phi_{w_0,w_1}(z+x_w)-x_{L(w)},$$

satisfying $f_{0,1}(0)=0$ and $f'_{0,1}(0)=\phi'_{w_0,w_1}(x_w)=e^{-J_{w_0,w_1}(x_w)}$ with $J_{w_0,w_1}(x_w) := -\log\phi'_{w_0,w_1}(x_w)$. For the third step, as shown in [Nel69, th7, p.45], there exists a family of smooth functions $H_w : \mathcal{J}\to\mathbb{R}$ defined on $\mathcal{J}\subset\mathbb{R}$ a sufficiently small neighborhood of the origin[12] and satisfying $H_w(0)=0$, $H'_w(0)=1$ and

$$\forall z\in\mathcal{J},\quad H_{L(w)}\left(e^{-J_{w_0,w_1}(x_w)}z\right) = f_{0,1}\left(H_w(z)\right) \tag{75}$$

which gives

$$\mathcal{L}_{f_{0,1}} = \mathcal{L}_{H_{L(w)}}\circ\hat{D}_{J_{w_0,w_1}(x_w)}\circ\mathcal{L}_{H_w}^{-1}. \tag{76}$$

In other words, the contracting map $f_{0,1}$ is "globally conjugated" to the linear contracting map $z\to e^{-J_{w_0,w_1}(x_w)}z$. The functions H_w can be constructed by the "scattering

[11] Beware that $f_{0,1}$ depends on the full word w.

[12] This is possible because the points x_w are bounded away from the boundary of I, uniformly with respect to the words $w\in\mathcal{W}$.

process"[13] as follows. For $n \geq 1$ let $f_{0,n} := f_{n-1,n} \circ f_{n-2,n-1} \ldots \circ f_{0,1}$. For $z \in \mathbb{R}$ (closed enough to 0) let

$$H_w^{(n)}(z) := f_{0,n}^{-1}\left(e^{-J_{w_{0,n}}(x_w)} z\right).$$

As $n \to \infty$, the uniform convergence of $H_w^{(n)}$ and its derivatives can be obtained using bounded distortion estimates [Fal97, prop 4.2]. This gives the existence of the limit

$$H_w(z) := \lim_{n \to \infty} H_w^{(n)}(z).$$

We also get (67) from bounded distortion estimates. Then

$$\begin{aligned} H_{L(w)}^{(n-1)}\left(e^{-J_{w_0,w_1}(x_w)}.z\right) &= f_{1,n}^{-1}\left(e^{-J_{w_{0,n}}(x_w)} z\right) \\ &= f_{0,1}\left(f_{0,n}^{-1}\left(e^{-J_{w_{0,n}}(x_w)} z\right)\right) \\ &= f_{0,1}\left(H_w^{(n)}(z)\right). \end{aligned}$$

Taking the limit $n \to \infty$ and noting $L(w) = (w_1, w_2, \ldots)$, we get (75). With (74) and (76) we have obtained that:

$$\begin{aligned} \mathcal{L}_{w_0,w_1} &= e^{i\nu\mathcal{V}(x_{L(w)})} e^{+i\nu\Upsilon_{L(w)}(\hat{y})} \hat{T}_{x_{L(w)}} \mathcal{L}_{H_{L(w)}} \circ \hat{D}_{J_{w_0,w_1}(x_w)} \circ \mathcal{L}_{H_w}^{-1} \hat{T}_{-x_w} e^{-i\nu\Upsilon_w(\hat{y})} \\ &= e^{i\nu\mathcal{V}(x_{L(w)})} \widehat{\mathcal{T}}_{L(w)} \hat{D}_{J_{w_0,w_1}(x_w)} \widehat{\mathcal{T}}_w^{-1} \end{aligned}$$

This is (65).

6. Asymptotic Expansion

In this section we first give a simple but useful expansion for the dilation operator defined in (64) in terms of rank one operators in Theorem 6.3. Then we use this expansion and the global normal form (70) to deduce an expansion for the transfer operator $\mathcal{L}_\nu^n$ for large time n in Theorem 6.7.

6.1. Asymptotic expansion for the dilation operator. Fix $\lambda_0 > 0$ and let $\lambda \geq \lambda_0$, let $y_0 > 0$, $\varphi \in C_0^\infty(]-y_0, y_0[)$. Recall from (64) that we defined $\left(\hat{D}_\lambda \varphi\right)(y) := \varphi\left(e^\lambda y\right)$. Let $\chi_0 \in C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}\chi_0 \subset [-y_0, y_0]$ and $\chi_0(x) = 1$ for $x \in \left[-e^{-\lambda_0} y_0, e^{-\lambda_0} y_0\right]$ so that $\chi_0 \equiv 1$ on $\mathrm{supp}\left(\hat{D}_\lambda \chi_0\right)$. Hence we have $\hat{\chi}_0^{-1} \circ \hat{D}_\lambda \circ \hat{\chi}_0 = \hat{D}_\lambda \circ \hat{\chi}_0$ where $\hat{\chi}_0$ is the multiplication operator by χ_0.

For $k \geq 0$ let us denote $\delta^{(k)}$ for the k-th derivative of the Dirac distribution. Let $\langle x^k, \hat{\chi}_0 \psi \rangle := \int_{\mathbb{R}} x^k \chi_0(x) \psi(x)\, dx$. We introduce the rank one operator[14]

$$\Pi_k : \psi \in \mathcal{S}(\mathbb{R}) \to \frac{1}{k!} \langle x^k, \hat{\chi}_0 \psi \rangle \delta^{(k)} \in \mathcal{S}'(\mathbb{R}) \tag{77}$$

[13] The term "scattering process" comes from [Nel69]. Here in the "non interacting region" is $z \to 0$ whereas in the usual theory of scattering of waves, the non interacting region is the infinity, far from the action of the potential.

[14] $\mathcal{S}(\mathbb{R})$ is the Schwartz space [Tay74, p.197].

We will use the Dirac notations of physics and write

$$\langle x^k | : \psi \in C_0^\infty(\mathbb{R};\mathbb{C}) \to \langle x^k, \psi \rangle = \int_{\mathbb{R}} x^k \psi(x)\, dx \in \mathbb{C}$$

$$\Pi_k = |\frac{1}{k!}\delta^{(k)}\rangle \langle x^k | \hat{\chi}_0 \tag{78}$$

Lemma 6.1. *If $k < m - \frac{1}{2}$ then $\Pi_k : H_\nu^{-m}(\mathbb{R}) \to H_\nu^{-m}(\mathbb{R})$ is a bounded operator and*

$$\exists C > 0, \forall \nu > 0, \quad \|\Pi_k\|_{H_\nu^{-m}} \leq C\nu^{k+1/2}. \tag{79}$$

Furthermore

$$\exists C > 0, \forall m, \nu, \quad \left\|\hat{\chi}_0\right\|_{H_\nu^{-m}} \leq C. \tag{80}$$

Proof. In order to prove (80) we use ν-semiclassical calculus: we have $\hat{\chi}_0 = \mathrm{Op}_\nu(\chi_0)$ and using composition of PDO as well as L^2-continuity [Zwo12],

$$\begin{aligned}\left\|\hat{\chi}_0\right\|_{H_\nu^{-m}} &= \left\|\mathrm{Op}_\nu(A_m)\,\mathrm{Op}_\nu(\chi_0)\,\mathrm{Op}_\nu\left(A_m^{-1}\right)\right\|_{L^2(\mathbb{R})} \\ &= \left\|\mathrm{Op}_\nu(\chi_0)\right\|_{L^2(\mathbb{R})} + O\left(\nu^{-1}\right) \leq C.\end{aligned}$$

We will use (80) later in (92).

One has for $2(k-m) < -1$,

$$\begin{aligned}\left\|\delta^{(k)}\right\|_{H_\nu^{-m}} &= \left\|\langle\xi\rangle^{-m}\mathcal{F}_\nu[\delta^{(k)}]\right\|_{L^2} = \left\|\langle\xi\rangle^{-m}(2\pi/\nu)^{-1/2}(i\nu\xi)^k\right\|_{L^2} \\ &\leq C\nu^{k+1/2}\end{aligned} \tag{81}$$

and

$$\begin{aligned}\left\|x^k\chi_0(x)\right\|_{H_\nu^m} &= \nu^{1/2}\left\|\langle\xi\rangle^m\mathcal{F}_1[x^k\chi_0(x)](\nu\xi)\right\|_{L^2} \\ &= \| \underbrace{\langle\xi/\nu\rangle^m}_{\leq\langle\xi\rangle^m}\mathcal{F}_1[x^k\chi_0(x)](\xi)\|_{L^2} \leq C\end{aligned}$$

This gives

$$\|\Pi_k\|_{H_\nu^{-m}} = \left\||\frac{1}{k!}\delta^{(k)}\rangle\langle\chi_0(x)x^k|\right\|_{H_\nu^{-m}} \leq C\nu^{k+1/2} \tag{82}$$

Remark 6.2. We have

$$\Pi_k \circ \Pi_{k'} \underset{(78)}{=} \delta_{k,k'}\Pi_k, \quad \left[\hat{D}_\lambda \circ \hat{\chi}_0, \Pi_k\right] = 0, \quad \hat{D}_\lambda \circ \hat{\chi}_0 \circ \Pi_k = e^{-(k+1)\lambda}\Pi_k,$$

i.e. the operators $(\Pi_k)_k$ form a set of commuting spectral projection operators for the operator $\hat{D}_\lambda \circ \hat{\chi}_0$. The next Theorem shows that they are complete in the sense that they give a spectral decomposition of $\hat{D}_\lambda \circ \hat{\chi}_0$ up to some remainder with small norm.

Theorem 6.3. *Fix $\lambda_0 > 0$ and $\nu_0 > 0$. Let $m > \frac{1}{2}$, $\nu > \nu_0$, $\lambda \geq \lambda_0$ and χ_0 as above. The operator $\hat{D}_\lambda \circ \hat{\chi}_0 : H_\nu^{-m} \to H_\nu^{-m}$ is bounded and for any $d < m - \frac{3}{2}$, and we have*

$$\left\| \hat{D}_\lambda \circ \hat{\chi}_0 - \sum_{k=0}^{d} e^{-(k+1)\lambda} \Pi_k \right\|_{H_\nu^{-m}(\mathbb{R})} \leq C e^{-\lambda/2} \left(e^{-\lambda}\nu\right)^{(d+1)+\frac{1}{2}} \tag{83}$$

where the constant C in the remainder does not depend on λ, ν but depends on d, χ_0 and m.

Remark 6.4. In higher dimensions, there is a similar result in [FT15, Prop. 4.19]. Formula (83) can be considered as a Taylor expansion of the dilation operator. From (79) we have $e^{-(k+1)\lambda} \|\Pi_k\|_{H_\nu^{-m}(\mathbb{R})} = O\left(e^{-\lambda/2}\left(e^{-\lambda}\nu\right)^{k+1/2}\right)$, therefore if $e^\lambda > \nu$, (83) can be interpreted as an asymptotic expansion in powers of $e^{-\lambda}\nu$.

Proof. We directly see that

$$\hat{D}_\lambda \left(\hat{\chi}_0 - \sum_{k=0}^{d} \Pi_k\right) = \left(\hat{D}_\lambda \hat{\chi}_0 - \sum_{k=0}^{d} e^{-(k+1)\lambda} \Pi_k\right) \tag{84}$$

and first prove the following Lemma.

Lemma 6.5. *Let $\phi \in C_0^\infty$ then*

$$\mathcal{F}_\nu \left[\left(\hat{\chi}_0 - \sum_{k=0}^{d} \Pi_k\right)\phi\right](\xi) = \sqrt{\nu}\left(\tilde{\Psi}(\nu\xi) - \sum_{k=0}^{d} \frac{1}{k!}(\nu\xi)^k \tilde{\Psi}^{(k)}(0)\right) \tag{85}$$

where $\tilde{\Psi} := \mathcal{F}_1[\chi_0\phi]$.

Proof. First we consider

$$\tilde{\Psi}^{(k)}(\xi) = \left(\frac{d}{d\xi}\right)^k \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} \chi_0(x)\phi(x)dx = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi}(-ix)^k \chi_0(x)\phi(x)dx$$

and conclude that

$$\tilde{\Psi}^{(k)}(0) = \frac{(-i)^k}{\sqrt{2\pi}} \langle x^k, \hat{\chi}_0\phi\rangle. \tag{86}$$

Furthermore we calculate

$$\mathcal{F}_\nu\left[\delta^{(k)}\right](\xi) = (2\pi/\nu)^{-1/2}(-i\nu\xi)^k \tag{87}$$

and

$$\mathcal{F}_\nu\left[\hat{\chi}_0\phi\right](\xi) = \nu^{1/2}\tilde{\Psi}(\nu\xi). \tag{88}$$

Finally (78) with (86), (87) and (88) give (85) which finishes the proof of Lemma 6.5.

We can now return to the proof of Theorem 6.3. We have $\mathcal{F}_\nu[\hat{D}_\lambda \Psi](\xi) = e^{-\lambda}\mathcal{F}_\nu[\Psi]\left(e^{-\lambda}\xi\right)$ which follows directly from variable substitution. Using Taylor's Theorem we have

$$
\begin{aligned}
&\mathcal{F}_\nu\left[\left(\hat{D}_\lambda \hat{\chi}_0 - \sum_{k=0}^{d} e^{-(k+1)\lambda}\Pi_k\right)\phi\right](\xi)\\
&\underset{(84)}{=} e^{-\lambda}\mathcal{F}_\nu\left[\left(\hat{\chi}_0 - \sum_{k=0}^{d}\Pi_k\right)\phi\right]\left(e^{-\lambda}\xi\right)\\
&\underset{(85)}{=} \sqrt{\nu}e^{-\lambda}\left(\tilde{\Psi}\left(\nu e^{-\lambda}\xi\right) - \sum_{k=0}^{d}\frac{1}{k!}\left(\nu e^{-\lambda}\xi\right)^k \tilde{\Psi}^{(k)}(0)\right)\\
&= \sqrt{\nu}e^{-\lambda}\frac{1}{(d+1)!}\left(\nu e^{-\lambda}\xi\right)^{d+1}\tilde{\Psi}^{(d+1)}\left(\theta_{\nu e^{-\lambda}\xi}\nu e^{-\lambda}\xi\right) \qquad (89)
\end{aligned}
$$

with $|\theta_{\nu e^{-\lambda}\xi}| \le 1$. By definition of H_ν^{-m} in (19) we have

$$
\begin{aligned}
(*) :&= \left\|\left(D_\lambda \hat{\chi}_0 - \sum_{k=0}^{d} e^{-(k+1)\lambda}\Pi_k\right)\phi\right\|_{H_\nu^{-m}}\\
&= \left\|\langle\xi\rangle^{-m}\mathcal{F}_\nu\left[\left(D_\lambda \hat{\chi}_0 - \sum_{k=0}^{d} e^{-(k+1)\lambda}\Pi_k\right)\phi\right](\xi)\right\|_{L^2}\\
&\underset{(89)}{=} \left\|\langle\xi\rangle^{-m}\sqrt{\nu}e^{-\lambda}\frac{1}{(d+1)!}\left(\nu e^{-\lambda}\xi\right)^{d+1}\tilde{\Psi}^{(d+1)}\left(\theta_{\nu e^{-\lambda}\xi}\nu e^{-\lambda}\xi\right)\right\|_{L^2}\\
&= e^{-\lambda/2}\left(\nu e^{-\lambda}\right)^{d+3/2}\frac{1}{(d+1)!}\sqrt{\int_{\mathbb{R}}\langle\xi\rangle^{-2m}\xi^{2(d+1)}\left|\tilde{\Psi}^{(d+1)}\left(\theta_{\nu e^{-\lambda}\xi}\nu e^{-\lambda}\xi\right)\right|^2 d\xi}
\end{aligned}
$$

To finish the proof of Theorem 6.3 we have to show that

$$
(*) \le C\|\phi\|_{H_\nu^{-m}}e^{-\lambda/2}\left(\nu e^{-\lambda}\right)^{d+3/2}
$$

with C independent of ϕ, λ, ν. We decompose the integral over $\mathbb{R}$ under the square root above into

$$
\int_{\mathbb{R}} \dots d\xi = \underbrace{\int_{[-e^\lambda, e^\lambda]} \dots d\xi}_{(A)} + \underbrace{\int_{\mathbb{R}\setminus[-e^\lambda, e^\lambda]} \dots d\xi}_{(B)}
$$

and treat them separately. We have:

$$
\begin{aligned}
(A) &\le \left(\int_{\mathbb{R}}\langle\xi\rangle^{-2m}\xi^{2(d+1)}d\xi\right)\max_{\xi\in[-e^\lambda, e^\lambda]}\left|\tilde{\Psi}^{(d+1)}\left(\theta_{\nu e^{-\lambda}\xi}\nu e^{-\lambda}\xi\right)\right|^2\\
&\le \left(\int_{\mathbb{R}}\langle\xi\rangle^{-2m}\xi^{2(d+1)}d\xi\right)\max_{\xi\in[-1,1]}\left|\tilde{\Psi}^{(d+1)}\left(\theta_{\nu\xi}\nu\xi\right)\right|^2\\
&\underset{(93)}{\le} C\|\phi\|^2_{H_\nu^{-m}}.
\end{aligned}
$$

Recall that we have assumed that

$$d < m - \frac{3}{2} \tag{90}$$

for the convergence of the integral. For the second term (B) we first observe that

$$\exists C > 0, \forall \xi \in \mathbb{R} \setminus [-1, 1], \frac{\left(e^{-\lambda}\left\langle \frac{\xi}{e^{-\lambda}} \right\rangle\right)^{-2m}}{\langle \xi \rangle^{-2m}} \leq C,$$

with C that depends on m but is independent of $\lambda > 0$. Also taking $\lambda \to 0$ in (89) we have

$$\left| \mathcal{F}_\nu \left[\left(\hat{\chi}_0 - \sum_{k=0}^{d} \Pi_k \right) \phi \right] (\xi) \right| = \frac{\sqrt{\nu}}{(d+1)!} (\nu\xi)^{d+1} \left| \tilde{\Psi}^{(d+1)} \left(\theta_{\nu\xi} \nu\xi\right) \right|. \tag{91}$$

If $d < m - 1/2$ then

$$\exists C > 0, \forall \nu > 0, \quad \left\| \left(\hat{\chi}_0 - \sum_{k=0}^{d} \Pi_k \right) \right\|_{H_\nu^{-m}} \underset{(79),(80)}{\leq} C\nu^{d+1/2}. \tag{92}$$

Hence

$$\begin{aligned} (B) &= e^{\lambda} \int_{\mathbb{R}\setminus[-1,1]} \langle e^{\lambda}\xi \rangle^{-2m} (e^{\lambda}\xi)^{2(d+1)} \left| \tilde{\Psi}^{(d+1)} \left(\theta_{\nu\xi} \nu\xi\right) \right|^2 d\xi \\ &= e^{-2(m-d-3/2)\lambda} \nu^{-2(d+1)} \\ &\quad \int_{\mathbb{R}\setminus[-1,1]} \frac{\left(\langle e^{\lambda}\xi \rangle e^{-\lambda}\right)^{-2m}}{\langle \xi \rangle^{-2m}} \langle \xi \rangle^{-2m} (\nu\xi)^{2(d+1)} \left| \tilde{\Psi}^{(d+1)} \left(\theta_{\nu\xi} \nu\xi\right) \right|^2 d\xi \\ &\underset{(91)}{\leq} Ce^{-2(m-d-3/2)\lambda} \nu^{-2(d+1)} \left(\nu^{-1/2} (d+1)!\right)^2 \left\| \left(\hat{\chi}_0 - \sum_{k=0}^{d} \Pi_k \right) \phi \right\|_{H_\nu^{-m}}^2 \\ &\underset{(90),(92)}{\leq} C\nu^{-2} \|\phi\|_{H_\nu^{-m}}^2 \end{aligned}$$

Thus taking (A) and (B) together we obtain

$$(*) \leq C \|\phi\|_{H_\nu^{-m}} e^{-\lambda/2} \left(\nu e^{-\lambda}\right)^{d+3/2}$$

which finishes the proof of Theorem 6.3.

The next Lemma has be used in the previous proof.

Lemma 6.6. *For every $\chi_0 \in C_0^\infty(\mathbb{R})$, $d, m \geq 0$ there exists $C > 0$ such that for any $\phi \in C_0^\infty(\mathbb{R})$,*

$$\max_{\xi \in [-1,1]} \left| \tilde{\Psi}^{(d+1)} (\nu\xi) \right|^2 \leq C \|\phi\|_{H_\nu^{-m}}^2 \tag{93}$$

with $\tilde{\Psi} := \mathcal{F}_1 [\chi_0 \phi]$ and where C depends only on χ_0, d and m.

Proof. By definition of $\tilde{\Psi}$ we have

$$\tilde{\Psi}^{(d+1)}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} (-ix)^{d+1} \chi_0(x) \phi(x) dx$$

thus

$$\tilde{\Psi}^{(d+1)}(\nu\xi) = \frac{1}{\sqrt{2\pi}} \left\langle e^{ix\nu\xi} (ix)^{d+1} \chi_0(x), \phi \right\rangle_{L^2}$$

$$\left| \tilde{\Psi}^{(d+1)}(\nu\xi) \right| \leq \frac{1}{\sqrt{2\pi}} \left\| e^{i\nu x\xi} (ix)^{d+1} \chi_0(x) \right\|_{H_\nu^m} \|\phi\|_{H_\nu^{-m}}$$

Thus we have to find for any $\xi_0 \in [-1, 1]$ an estimate of

$$\begin{aligned}
\left\| e^{i\nu x\xi_0} (ix)^{d+1} \chi_0(x) \right\|_{H_\nu^m} &= \left\| \langle \xi \rangle^m \mathcal{F}_\nu[(ix)^{d+1} \chi_0(x)](\xi - \xi_0) \right\|_{L^2} \\
&\leq \nu^{1/2} \left\| \langle \xi \rangle^m \mathcal{F}_1[(ix)^{d+1} \chi_0(x)] (\nu (\xi - \xi_0)) \right\|_{L^2} \\
&\leq \left\| \langle \xi/\nu + \xi_0 \rangle^m \mathcal{F}_1[(ix)^{d+1} \chi_0(x)] (\xi) \right\|_{L^2} \\
&\leq C_m \left\| \langle \xi \rangle^m \mathcal{F}_1[(ix)^{d+1} \chi_0(x)] (\xi) \right\|_{L^2} \leq C
\end{aligned}$$

where C depends on m, d and χ_0.

6.2. Asymptotic expansion for the transfer operator. In Corollary 5.5 we have shown that $\mathcal{L}_\nu^n$ is a sum of operators $\mathcal{L}_{w_{0,n}}$ and each of these operators is conjugated to a dilation operator. For the next Theorem we additionally use the asymptotic expansion in Theorem 6.3 for the dilation operator to deduce an asymptotic expansion for $\mathcal{L}_\nu^n$. In order to simplify the notation we will write $J_{w_{0,n}} = J_{w_{0,n}}(x_w)$ and $\tau_{w_{0,n}} = \tau_{w_{0,n}}(x_w)$, $V_{w_{0,n}} = V_{w_{0,n}}(x_w)$ for the Birkhoff sums defined in (56) and where w is an arbitrary extension of $w_{0,n}$ as explained in Corollary 5.5. In the limit of large n the bounded distortion principle implies that the impact of the arbitrary extension becomes small, anyway, see Lemma B.1.

Theorem 6.7. *For any $0 \leq d < m - \frac{3}{2}$, there exists $C > 0$ such that for any $n \geq 1$, any $\nu > 0$,*

$$\begin{aligned}
&\left\| \mathcal{L}_{\nu,\chi}^n - \sum_{w_{0,n}} e^{i\nu\tau_{w_{0,n}} + V_{w_{0,n}}} \sum_{k=0}^{d} e^{-(k+1)J_{w_{0,n}}} \Pi_{k,w,n} \right\|_{H_\nu^{-m}} \\
&\quad \leq C\nu^{\left(d+\frac{3}{2}\right)} e^{n(\Pr(V-(d+2)J)+R(n))}, \qquad (94)
\end{aligned}$$

with some function $R(n) \underset{n\to\infty}{\longrightarrow} 0$ and with the rank one operators

$$\Pi_{k,w,n} := |\mathcal{U}_{L^n(w)}^{(k)}\rangle\langle\mathcal{S}_w^{(k)}| \quad : H_\nu^{-m}(\mathbb{R}) \to H_\nu^{-m}(\mathbb{R}), \qquad (95)$$

where w is an arbitrary extension of $w_{0,n}$ and where we used the Dirac notation of Sect. 6.1 for the following distributions (cf. Fig. 4)

$$\begin{aligned}
|\mathcal{U}_{L^n(w)}^{(k)}\rangle &:= \widehat{\mathcal{T}}_{L^n(w)} | \frac{1}{k!} \delta^{(k)} \rangle \in H_\nu^{-m}(\mathbb{R}), \\
\langle \mathcal{S}_w^{(k)} | &:= \langle x^k | \widehat{\mathcal{T}}_w^{-1} \hat{\chi} \in H_\nu^{+m}(\mathbb{R}).
\end{aligned}$$

Remark 6.8. $\mathcal{U}_w^{(k)}$ and $\mathcal{S}_w^{(k)}$ are WKB Lagrangian states [BW97]. This is a geometric but non necessary remark.

Remark 6.9. For $k < m - 1/2$ we conclude from Remark 5.4 and Eq. (82) that $\Pi_{k,w,n}$ is bounded by

$$\left\| \Pi_{k,w,n} \right\|_{H_\nu^{-m}} \leq C \nu^{k+\frac{1}{2}}. \tag{96}$$

with C independent of ν, w and n.

Proof. Recall from Sect. 2.4 that

$$\hat{Q} := \hat{A}_m \hat{\chi}^{-1} \mathcal{L}_\nu \hat{\chi} \hat{A}_m^{-1} \quad : L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{97}$$

is bounded. For further use, let

$$\langle \tilde{\mathcal{S}}_w^{(k)} | := \langle \mathcal{S}_w^{(k)} | \hat{A}_m^{-1} = \langle x^k | \widehat{\mathcal{T}}_w^{-1} \hat{\chi} \hat{A}_m^{-1} \in L^2(\mathbb{R}), \tag{98}$$

$$|\tilde{\mathcal{U}}_w^{(k)}\rangle := \hat{A}_m |\mathcal{U}_{L^n(w)}^{(k)}\rangle = \hat{A}_m \hat{\chi}^{-1} \widehat{\mathcal{T}}_w | \frac{1}{k!} \delta^{(k)} \rangle \in L^2(\mathbb{R}), \tag{99}$$

$$\tilde{\Pi}_{k,w,n} = \hat{A}_m \Pi_{k,w,n} \hat{A}_m^{-1} = \hat{A}_m \hat{\chi}^{-1} \widehat{\mathcal{T}}_{L^n(w)} \Pi_k \widehat{\mathcal{T}}_w^{-1} \hat{\chi} \hat{A}_m^{-1} = |\tilde{\mathcal{U}}_{L^n(w)}^{(k)}\rangle \langle \tilde{\mathcal{S}}_w^{(k)}|. \tag{100}$$

Proving (94) is equivalent to proving an expansion for $\hat{Q}^n$ in $L^2(\mathbb{R})$:

$$\left\| \hat{Q}^n - \sum_{w_{0,n}} e^{i\nu\tau_{w_{0,n}} + V_{w_{0,n}}} \sum_{k=0}^{d} e^{-(k+1)J_{w_{0,n}}} \tilde{\Pi}_{k,w,n} \right\|_{L^2} \leq C \nu^{\left(d+\frac{3}{2}\right)} e^{n(\mathrm{Pr}(V-(d+2)J)+R(n))}. \tag{101}$$

For any $n \geq 1$, we have from (69)

$$\begin{aligned} \hat{Q}^n &= \hat{A}_m \hat{\chi}^{-1} \mathcal{L}^n \hat{\chi} \hat{A}_m^{-1} \\ &= \sum_{w_{0,n}} \hat{Q}_{w_{0,n}} \end{aligned} \tag{102}$$

with individual terms

$$\hat{Q}_{w_{0,n}} = \hat{A}_m \hat{\chi}^{-1} \mathcal{L}_{w_{0,n}}^n \hat{\chi} \hat{A}_m^{-1}.$$

Using (70) we get

$$\hat{Q}_{w_{0,n}} = e^{i\nu\tau_{w_{0,n}} + V_{w_{0,n}}} \hat{A}_m \hat{\chi}^{-1} \widehat{\mathcal{T}}_{L^n(w)} \hat{D}_{J_{w_{0,n}}} \widehat{\mathcal{T}}_w^{-1} \hat{\chi} \hat{A}_m^{-1}.$$

In order to use the expansion (83) for $\hat{D}_{J_{w_{0,n}}} \hat{\chi}_0$, let us choose $\chi_0 \in C_0^\infty(\mathbb{R})$ as in Theorem 6.3 such that $\chi_0(y) = 1$ for every $w \in \mathcal{W}$ and $y \in \mathrm{supp}\left(\widehat{\mathcal{T}}_w^{-1} \hat{\chi}\right)$. This choice of χ_0 is possible uniformly with respect to w. Then $\widehat{\mathcal{T}}_w^{-1} \hat{\chi} = \hat{\chi}_0 \widehat{\mathcal{T}}_w^{-1} \hat{\chi}$. So (83) gives that for a given $d \geq 1$, and using notation (95),

$$\begin{aligned} \hat{Q}_{w_{0,n}} &= e^{i\nu\tau_{w_{0,n}} + V_{w_{0,n}}} \left(\sum_{k=0}^{d} e^{-(k+1)J_{w_{0,n}}} \hat{A}_m \hat{\chi}^{-1} \widehat{\mathcal{T}}_{L^n(w)} \Pi_k \widehat{\mathcal{T}}_w^{-1} \hat{\chi} \hat{A}_m^{-1} \right) + R_{w_{0,n}} \\ &= e^{i\nu\tau_{w_{0,n}} + V_{w_{0,n}}} \left(\sum_{k=0}^{d} e^{-(k+1)J_{w_{0,n}}} \tilde{\Pi}_{k,w,n} \right) + R_{w_{0,n}}, \end{aligned} \tag{103}$$

with a remainder $R_{w_{0,n}}$ given by

$$R_{w_{0,n}} = e^{i\nu\tau_{w_{0,n}}+V_{w_{0,n}}} \hat{A}_m \hat{\chi}^{-1} \widehat{\mathcal{T}}_{L^n(w)} \left(\hat{D}_{J_{w_{0,n}}} \hat{\chi}_0 - \sum_{k=0}^{d} e^{-(k+1)J_{w_{0,n}}} \Pi_k \right) \widehat{\mathcal{T}}_w^{-1} \hat{\chi} \hat{A}_m^{-1}.$$

From (83) and (68) its norm is bounded by:

$$\begin{aligned} \left\| R_{w_{0,n}} \right\|_{L^2(\mathbb{R})} &\leq C \left| e^{i\nu\tau_{w_{0,n}}+V_{w_{0,n}}} \right| e^{-\frac{1}{2}J_{w_{0,n}}} \left(\nu e^{-J_{w_{0,n}}} \right)^{d+3/2} \\ &\leq C\nu^{\left(d+\frac{3}{2}\right)} e^{(V-(d+2)J)_{w_{0,n}}} \end{aligned} \tag{104}$$

with some constant $C > 0$ independent of ν, n, and w. Using (160) from the Appendix, the sum of these remainders is bounded by

$$\begin{aligned} \sum_{w_{0,n}} \left\| R_{w_{0,n}} \right\|_{L^2(\mathbb{R})} &\leq C\nu^{\left(d+\frac{3}{2}\right)} \sum_{w_{0,n}} e^{(V-(d+2)J)_{w_{0,n}}} \\ &\leq C\nu^{\left(d+\frac{3}{2}\right)} e^{n(\Pr(V-(d+2)J)+R(n))}, \quad \text{with } R(n) \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

From (102) and (103) we deduce (94).

7. Diagonal Approximation

We have defined the bounded operator $\hat{Q} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ in (20). For $n \geq 1$, let

$$P_n := \left(\hat{Q}^n\right)^* \hat{Q}^n \tag{105}$$

which is a positive bounded self-adjoint operator on $L^2(\mathbb{R})$. In the following Theorem we bound the norm of P_n and this will be used in Sect. 8 to deduce a bound on the spectral radius of the transfer operator (the main result of this paper). $[x]$ will denote the approximation of $x \in \mathbb{R}$ by the closest smaller integer.

Proposition 7.1. *We make the Assumption 4.5 of minimal captivity. Let $\epsilon > 0$, $0 \leq J_c < 2J_{min} - \epsilon$ and $0 < \beta < 1$. There exists $C > 0$, such that for any $\nu > 0$ and n given by*

$$n := \left[\frac{2}{J_c + \epsilon} \log \nu \right] > \frac{1}{J_{min}} \log \nu. \tag{106}$$

We have

$$\|P_n\|_{L^2} \leq C \left(\nu \sum_{w_{0,n}} e^{2(V-J)_{w_{0,N^*}}} e^{(V-J)_{w_{N^*,n}}} \sum_{w'_{N^*,n}} e^{(V-J)_{w'_{N^*,n}}} \right) + C\beta^n \tag{107}$$

with

$$N^* := N^*(w, J_c) := \max\left\{k \leq n, \quad s.t.\ J_{w_{0,k}} < nJ_c\right\}. \tag{108}$$

Remark 7.2. Note that in the inequality (107), the Birkhoff sums of $(V - J)$ can be calculated with any possible extension of the word fragments.

Remark 7.3. In semiclassical analysis, the time $\frac{1}{J_{min}} \log \nu$ in (106) is called the maximal local Ehrenfest time. The definition of N^* in (108) can be written as

$$e^{J_{w_{0,N^*}}} \simeq \nu^2$$

and N^* can be called "**twice the standard local Ehrenfest time**". It is known in specific situations that this particular time is important; see discussions and results in [FND03, Fau07]. For example there are curious phenomena of "quantum revival" or "quantum period" at that time for the quantum cat map as explained in [BD00, FND03].

The rest of Sect. 7 is devoted to the proof of Proposition 7.1. At some point of the proof, i.e. Lemma 7.8, we will use the hypothesis of minimal captivity and obtain that the orbits of length n are "well distributed and separated" on phase space (inside the trapped set $\mathcal{K}$) so that they do not "interfere" with each over, provided the time n is not too long. Using this, the double sum over the orbits that appears in Lemma 7.4, can be reduced to a much smaller sum, which is basically a sum over the diagonal. This step can thus be considered as some kind of "diagonal approximation".

In a first step we use the asymptotic expansion for the transfer operator, Theorem 6.7, in order to write $\|P_n\|_{L^2}$ as a double sum over orbits. The next Lemma shows that this is possible provided we consider time n long enough w.r.t. ν.

Lemma 7.4. *Let $0 < \alpha < J_{min}$ and $0 < \beta < 1$. There exist $d \in \mathbb{N}$ and $C > 0$ such that for any $\nu > 0$ and*

$$n := \left[\frac{1}{\alpha} \log \nu\right] \tag{109}$$

we have

$$\|P_n\|_{L^2} \leq \mathbb{S} + C\beta^n. \tag{110}$$

with

$$\mathbb{S} := \sum_{w'_{0,n}, w_{0,n}} e^{V_{w'_{0,n}} + V_{w_{0,n}}} \sum_{k',k=0}^{d} e^{-(k'+1)J_{w'_{0,n}} - (k+1)J_{w_{0,n}}} \left| \mathrm{Tr}\left(\tilde{\Pi}^*_{k',w',n} \tilde{\Pi}_{k,w,n}\right)\right| \tag{111}$$

Remark 7.5. Later we will provide an upper bound for $\mathbb{S}$ keeping only the terms $k = k' = 0$.

Proof. We use Theorem 6.7 with its formulation (101) and write

$$\hat{Q}^n = S_n + R_n$$

with

$$S_n := \sum_{w_{0,n}} e^{i\nu\tau_{w_{0,n}} + V_{w_{0,n}}} \sum_{k=0}^{d} e^{-(k+1)J_{w_{0,n}}} \tilde{\Pi}_{k,w,n} \tag{112}$$

which is a finite rank operator and

$$\|R_n\|_{L^2} \underset{(94)}{\leq} C\nu^{\left(d+\frac{3}{2}\right)} e^{n(\Pr(V-(d+2)J)+R(n))}. \tag{113}$$

Then

$$P_n \underset{(105)}{=} \left(\hat{Q}^n\right)^* \hat{Q}^n = S_n^* S_n + \underbrace{\left(R_n^* S_n + S_n^* R_n + R_n^* R_n\right)}_{R_n'}, \tag{114}$$

gives

$$\|P_n\|_{L^2} \leq \left\|S_n^* S_n\right\|_{L^2} + \left\|R_n'\right\|_{L^2}. \tag{115}$$

In order to bound the remainder $\left\|R_n'\right\|_{L^2}$ we have to bound $\|R_n\|_{L^2}$ and $\|S_n\|_{L^2}$. Notice that

$$\nu^{\left(k+\frac{1}{2}\right)} \underset{(109)}{\leq} Ce^{n\left(k+\frac{1}{2}\right)\alpha} = Ce^{n\left(k+\frac{1}{2}\right)(\alpha+\epsilon)} e^{-\epsilon n\left(k+\frac{1}{2}\right)} \tag{116}$$

Let

$$\beta_k := e^{(k+\frac{1}{2})(\alpha+\epsilon)+\Pr(V-(k+1)J)}, \tag{117}$$

with $\epsilon > 0$ chosen later. So

$$\begin{aligned} \|R_n\|_{L^2} &\underset{(113)}{\leq} C\nu^{\left(d+\frac{3}{2}\right)} e^{n(\Pr(V-(d+2)J)+R(n))} \\ &\underset{(117),(116)}{\leq} C'\beta_{d+1}^n e^{n\left(R(n)-\epsilon\left(d+\frac{3}{2}\right)\right)} \leq C\beta_{d+1}^n, \end{aligned}$$

and

$$\begin{aligned} \|S_n\|_{L^2} &\underset{(112),(96)}{\leq} C\sum_{k=0}^{d} \nu^{\left(k+\frac{1}{2}\right)} \sum_{w_{0,n}} e^{(V-(k+1)J)_{w_{0,n}}} \\ &\underset{(160)}{\leq} C\sum_{k=0}^{d} \nu^{\left(k+\frac{1}{2}\right)} e^{n(\Pr(V-(k+1)J)+R(n))}, \quad \text{with } R(n) \underset{n\to\infty}{\longrightarrow} 0 \\ &\underset{(117),(116)}{\leq} C\sum_{k=0}^{d} \beta_k^n \end{aligned} \tag{118}$$

Notice that

$$\frac{\beta_{k+1}}{\beta_k} \underset{(117)}{=} e^{\alpha+\epsilon-\delta_k}$$

with

$$\delta_k := \Pr(V-(k+1)J) - \Pr(V-(k+2)J).$$

From Proposition B.5 we have $\forall r$, $\left(\frac{\partial}{\partial r}\Pr(V-rJ)\right)(r) \leq -J_{min}$. Consequently for $\alpha < J_{min}$ we have $k\alpha+\Pr(V-kJ) \underset{k\to\infty}{\to} -\infty$. Hence, if $\epsilon > 0$ is such that $\alpha+\epsilon < J_{min}$ then

$$\beta_k \underset{k\to\infty}{\to} 0. \tag{119}$$

Also $\delta_k \geq \delta_{k+1} \geq J_{min} > \alpha+\epsilon$ for any k hence

$$\frac{\beta_{k+1}}{\beta_k} < 1. \tag{120}$$

In particular $\|S_n\|_{L^2} \underset{(118)}{\leq} C\beta_0^n$ and we record for later use that

$$\sum_{k=0}^{d} \nu^{\left(k+\frac{1}{2}\right)} \sum_{w_{0,n}} e^{(V-(k+1)J)_{w_{0,n}}} \leq C\beta_0^n. \tag{121}$$

We conclude that

$$\left\| R_n' \right\|_{L^2} \leq 2\|R_n\| \, \|S_n\| + \|R_n\|^2 = \|R_n\| \left(2\|S_n\| + \|R_n\|\right) \leq C\beta_{d+1}^n \beta_0^n. \tag{122}$$

We now consider the term $S_n^* S_n$ in (114) given by

$$S_n^* S_n \underset{(112)}{=} \sum_{w_{0,n}', w_{0,n}} e^{\overline{i\nu\tau_{w_{0,n}'} + V_{w_{0,n}'}}} e^{i\nu\tau_{w_{0,n}} + V_{w_{0,n}}} \times \sum_{k',k=0}^{d} e^{-(k'+1)J_{w_{0,n}'} - (k+1)J_{w_{0,n}}} \tilde{\Pi}_{k',w',n}^* \tilde{\Pi}_{k,w,n}. \tag{123}$$

$S_n^* S_n$ is a finite rank positive self-adjoint operator, hence we have the bound

$$\left\| S_n^* S_n \right\|_{L^2} \leq \left\| S_n^* S_n \right\|_{\mathrm{Tr}} = \mathrm{Tr}\left(S_n^* S_n\right) \tag{124}$$

$$\leq \sum_{w_{0,n}', w_{0,n}} e^{V_{w_{0,n}'} + V_{w_{0,n}}} \tag{125}$$

$$\sum_{k',k=0}^{d} e^{-(k'+1)J_{w_{0,n}'} - (k+1)J_{w_{0,n}}} \left| \mathrm{Tr}\left(\tilde{\Pi}_{k',w',n}^* \tilde{\Pi}_{k,w,n} \right) \right| \tag{126}$$

$$\underset{(111)}{=} \mathbb{S}$$

and

$$\|P_n\|_{L^2} \underset{(115),(122),(125)}{\leq} \mathbb{S} + C\left(\beta_{d+1}\beta_0\right)^n.$$

Let $0 < \beta < 1$. Using (119), we can choose d large enough so that $\beta_{d+1}\beta_0 \leq \beta$. We have obtained (110).

Remark 7.6. In the inequality of (124) we have bounded the L^2 norm by a trace norm. This is a crucial step in the paper. This is obviously not an optimal bound. However it makes appear the terms $\left|\mathrm{Tr}\left(\tilde{\Pi}_{k',w',n}^* \tilde{\Pi}_{k,w,n}\right)\right|$ and in the next Proposition we will see that these terms can be neglected for many pairs of trajectories $w_{0,n}$, $w_{0,n}'$.

We first introduce the following notations. For $w, w' \in \mathcal{W}$ and $n \geq 1$, suppose that $w_{0,n} \neq w_{0,n}'$ and let

$$\begin{aligned} n_1\left(w_{0,n}, w_{0,n}'\right) &:= \min\left\{0 \leq k \leq n, \quad w_k \neq w_k'\right\} \\ n_2\left(w_{0,n}, w_{0,n}'\right) &:= \min\left\{0 \leq k \leq n, \quad w_{n-k} \neq w_{n-k}'\right\}. \end{aligned} \tag{127}$$

In other words this means that the words $w_{0,n}$ and $w'_{0,n}$ have equal letters at extremities $w_i = w'_i$ for $i \in [0, n_1[\cup]n - n_2, n]$ and differ for letters: $w_{n_1} \neq w'_{n_1}$, $w_{n-n_2} \neq w'_{n-n_2}$.

Notice that $J_{i,j}(x) > 0$ hence the Birkhoff sum $J_{w_{0,k}}$ (defined in (56)) is an increasing function of k. For some given $w \in \mathcal{W}$, $n \geq 1$, $J_c > 0$, let

$$N_1(w, n, J_c) := \max\left\{0 \leq k \leq n, \quad \text{s.t. } J_{w_{0,k-1}} < \frac{nJ_c}{2}\right\}$$

$$N_2(w, n, J_c) := \max\left\{0 \leq k \leq n, \quad \text{s.t. } J_{w_{n-k+1,n}} < \frac{nJ_c}{2}\right\}.$$

Let us introduce the following Definition.

Definition 7.7. For a given $J_c > 0$, we call a pair of orbits $\left(w'_{0,n}, w_{0,n}\right)$ **separable** if $w'_{0,n} \neq w_{0,n}$ and $n_1\left(w_{0,n}, w'_{0,n}\right) \leq N_1(w, n, J_c)$ or $n_2\left(w_{0,n}, w'_{0,n}\right) \leq N_2(w, n, J_c)$. Otherwise we call the pair $\left(w'_{0,n}, w_{0,n}\right)$ **non-separable**.

Proposition 7.8. *"**Orbit separation**". We make the Assumption 4.5 of minimal captivity. Let $\varepsilon > 0$ and $J_c > 0$. Then for any $M \geq 0$ there exists $C_M > 0$, such that for any $\nu > 0$, $n := \left[\frac{2}{J_c+\varepsilon}\log\nu\right]$, any $w, w' \in \mathcal{W}$, if the pair of orbits $\left(w'_{0,n}, w_{0,n}\right)$ is separable then*

$$\nu^{-(k+k'+1)}\left|\mathrm{Tr}\left(\tilde{\Pi}^*_{k',w',n}\tilde{\Pi}_{k,w,n}\right)\right| \leq C_M e^{-\frac{\varepsilon}{2}nM} \tag{128}$$

Remark 7.9. In other words, Proposition 7.8 says that for a separable pair of orbits $\left(w'_{0,n}, w_{0,n}\right)$, the term $\left|\mathrm{Tr}\left(\tilde{\Pi}^*_{k',w',n}\tilde{\Pi}_{k,w,n}\right)\right|$ will be "negligible". We will see later in Lemma 7.10 that this is because a "separable pair of orbits" is indeed "separated" in phase space so that their Lagrangian states do not overlap.

Proof of Proposition 7.1. The proof of Proposition 7.8 will be given in Sect. 7.1. For now, we continue the proof of Proposition 7.1. Let $J_c < 2J_{min}$. Let $\epsilon > 0$, $\nu > 0$ and $n := \left[\frac{2}{J_c+\varepsilon}\log\nu\right] = \left[\frac{1}{\alpha}\log\nu\right]$ with $\alpha = \frac{1}{2}J_c + \frac{1}{2}\epsilon$. We suppose that $\epsilon > 0$ is small enough so that $J_c + \epsilon < 2J_{min}$. Hence $\alpha < J_{min}$ and we can apply Lemma 7.4. We decompose the double sum (125) over $w'_{0,n}$, $w_{0,n}$ into separable and non-separable pairs:

$$\mathbb{S} = \mathbb{S}_{\text{separable}} + \mathbb{S}_{\text{non-separable}}.$$

We first show that $\mathbb{S}_{\text{separable}}$ is "negligible". We have

$$\begin{aligned}
\mathbb{S}_{\text{separable}} &= \sum_{w_{0,n}, w'_{0,n}\ \text{sep.}} e^{V_{w'_{0,n}}+V_{w_{0,n}}} \sum_{k',k=0}^{d} e^{-(k'+1)J_{w'_{0,n}}-(k+1)J_{w_{0,n}}} \\
&\quad \left|\mathrm{Tr}\left(\tilde{\Pi}^*_{k',w',n}\tilde{\Pi}_{k,w,n}\right)\right| \\
&\underset{(128)}{\leq} C_M e^{-\frac{\varepsilon}{2}nM}\left(\sum_{k=0}^{d}\nu^{\left(k+\frac{1}{2}\right)}\sum_{w_{0,n}} e^{V_{w_{0,n}}-(k+1)J_{w_{0,n}}}\right)^2 \\
&\underset{(121)}{\leq} Ce^{-\frac{\varepsilon}{2}nM}\beta_0^{2n} = C\left(e^{-\frac{\varepsilon}{2}M}\beta_0^2\right)^n
\end{aligned}$$

We deduce that for any given $0 < \beta < 1$ we can choose M large enough so that $e^{-\frac{\varepsilon}{2}M}\beta_0^2 \leq \beta$ hence

$$\mathbb{S}_{\text{separable}} \leq C_M \beta^n,$$

which means that $\mathbb{S}_{\text{separable}}$ is "negligible". We have now to bound from above the "non separable trajectories" for which $n_1 > N_1$ and $n_2 > N_2$. By the definition of J_c and n there exists some $\tilde{\epsilon} > 0$ such $J_c + \epsilon + \tilde{\epsilon} < 2J_{min}$ which implies

$$n \geq \frac{1}{J_{min} - \tilde{\epsilon}} \log(\nu).$$

For every word $w_{0,n}$, we have $J_{w_{0,n}} \geq nJ_{min}$ hence

$$\nu e^{-J_{w_{0,n}}} \leq \nu e^{-nJ_{min}} \leq \left(e^{-\tilde{\epsilon}}\right)^n. \tag{129}$$

We write (with a constant C that is independent of n but whose actual value might change from line to line)

$$\begin{aligned}
\mathbb{S}_{\text{non-separable}} &:= \sum_{w_{0,n}, w'_{0,n}, \text{non-sep}} e^{V_{w'_{0,n}} + V_{w_{0,n}}} \sum_{k',k=0}^{d} e^{-(k'+1)J_{w'_{0,n}} - (k+1)J_{w_{0,n}}} \\
&\quad \left|\mathrm{Tr}\left(\tilde{\Pi}^*_{k',w',n}\tilde{\Pi}_{k,w,n}\right)\right| \\
&\underset{(133)}{\leq} C\nu \sum_{w_{0,n}, w'_{0,n}, \text{non-sep}} e^{(V-J)_{w'_{0,n}} + (V-J)_{w_{0,n}}} \\
&\quad \sum_{k',k=0}^{d} \left(\nu e^{-J_{w'_{0,n}}}\right)^{k'} \left(\nu e^{-J_{w_{0,n}}}\right)^{k} \\
&\underset{(129)}{\leq} C\nu \sum_{w_{0,n}, w'_{0,n}, \text{non-sep}} e^{(V-J)_{w'_{0,n}} + (V-J)_{w_{0,n}}} \sum_{k',k=0}^{d} e^{-n\tilde{\epsilon}(k+k')} \\
&\leq C\nu \sum_{w_{0,n}, w'_{0,n}, \text{non-sep}} e^{(V-J)_{w'_{0,n}} + (V-J)_{w_{0,n}}}
\end{aligned}$$

We will now use the fact that we only sum over non-separable pairs of words. Recall that this requires, that the word $w'_{0,n}$ is equal to $w_{0,n}$ for their first $N_1(w, n, J_c)$ and their last $N_2(w, n, J_c)$ symbols. Accordingly we can write the last expression as

$$\mathbb{S}_{\text{non-separable}} \leq C\nu \sum_{w_{0,n}} \left(e^{(V-J)_{w_{0,n}}} e^{(V-J)_{w_{0,N_1}} + (V-J)_{w_{n-N_2,n}}} \right.$$
$$\left. \sum_{w'_{N_1,n-N_2}, \text{ s.t. } w'_{N_1} = w_{N_1}, w'_{n-N_2} = w_{n-N_2}} e^{(V-J)_{w'_{N_1,n-N_2}}} \right).$$

Note that the last transformation can be done in an exact way (with the same constant C): One can choose appropriate extensions of the words $w'_{0,n}$, $w'_{0,N_1} = w_{0,N_1}$, $w'_{N_1,n-N_2}$ and

$w'_{n-N_2,n} = w_{n-N_2,n}$ such that one has $(V-J)_{w'_{0,n}} = (V-J)_{w_{0,N_1}} + (V-J)_{w'_{N_1,n-N_2}} + (V-J)_{w_{n-N_2,n}}$. Note furthermore, that, since we are only interested in an upper bound, we can remove the restrictions on the initial and last symbol in the second sum and obtain

$$\mathbb{S}_{\text{non-separable}} \leq C\nu \sum_{w_{0,n}} e^{(V-J)_{w_{0,n}}} e^{(V-J)_{w_{0,N_1}} + (V-J)_{w_{n-N_2,n}}} \sum_{w'_{N_1,n-N_2}} e^{(V-J)_{w'_{N_1,n-N_2}}}.$$

Let us finally explain, how to pass from this expression to (107) which involves N^* instead of N_1 and N_2: Let us first hypothetically assume that the symbolic dynamic is complete and that $(V-J)_{w_{0,n}}$ would only depend on the fragment $w_{1,n}$ and not on the extension and the we have $(V-J)_{w_{0,n}} = (V-J)_{w_{0,a}} + (V-J)_{w_{a,n}}$ for any $0 < a < n$. Then we could can rearrange each word $w_{0,n}$ by putting the fragments w_{0,N_1} and $w_{n-N_2,n}$ at the beginning of the new word $\tilde{w}_{0,n}$ such that $N^*(\tilde{w}_{0,n}, J_c) = N_1\,(w, n, J_c) + N_2\,(w, n, J_c)$ and rewrite the last expression by rearranging the combinatorial sum over the words, as

$$\mathbb{S}_{\text{non-separable}} \leq C\nu \sum_{w_{0,n}} e^{(V-J)_{w_{0,n}}} e^{(V-J)_{w_{0,N^*}}} \sum_{w'_{N^*,n}} e^{(V-J)_{w'_{N^*,n}}},$$

without having to modify the constant C. Now both above assumptions are in general not true in the framework in which we are working. Nevertheless we can obtain the above bound by modifying the constant C. This is justified for the following reasons: Firstly the expressions $(V-J)_{w_{0,n}}$ depend on the extension of the word $w_{0,n}$ only in a controlled way (see Lemma B.1) thus we always have $(V-J)_{w_{0,n}} \leq (V-J)_{w_{0,a}} + (V-J)_{w_{a,n}} + c_0$ and the n independent constant c_0 can always be absorbed in multiplicative constant C. The second problem concerns non complete symbolic dynamic: In the sum over $w_{0,n}$ there might occur word fragments w_{0,N_1} and $w_{n-N_2,n}$ that do not occur as the leading and the last letters in some w_{0,N^*}. However, as we demanded a transitive symbolic dynamic we can assure, that the word fragments w_{0,N_1} and $w_{n-N_2,n}$ appear as disjoint fragments of some w_{0,N^*+T} where T is the maximal transition time between two letters. Thus we can bound

$$\mathbb{S}_{\text{non-separable}} \leq C\nu \sum_{w_{0,n}} e^{(V-J)_{w_{0,n}}} e^{(V-J)_{w_{0,N^*+T}}} \sum_{w'_{N^*,n}} e^{(V-J)_{w'_{N^*,n}}},$$

where we absorb the impact of the additional letters that are needed to concatenate w_{0,N_1} and $w_{n-N_2,n}$ in a modified constant C. Finally we can also absorb the last T terms in the Birkhoff sum $(V-J)_{w_{0,N^*+T}}$ in the constant C and obtain

$$\mathbb{S}_{\text{non-separable}} \leq C\nu \sum_{w_{0,n}} e^{(V-J)_{w_{0,n}}} e^{(V-J)_{w_{0,N^*}}} \sum_{w'_{N^*,n}} e^{(V-J)_{w'_{N^*,n}}}.$$

Finally we get

$$\begin{aligned}\mathbb{S} &= \mathbb{S}_{\text{separable}} + \mathbb{S}_{\text{non-separable}}\\ &\leq C_M \beta^n + C\nu \sum_{w_{0,n}} e^{2(V-J)_{w_{0,N^*}}} e^{(V-J)_{w_{N^*,n}}} \sum_{w'_{N^*,n}} e^{(V-J)_{w'_{N^*,n}}}.\end{aligned}$$

Together with (110) we have finished the proof of Proposition 7.1. □

7.1. Proof of Proposition 7.8 about separation of orbits. The following Lemma gives bounds for the quantities $\left|x_{L^n(w)} - x_{L^n(w')}\right|$ and $\min_{x\in K_a}\left(\left|\zeta_w(x) - \zeta_{w'}(x)\right|\right)$ that will appear later in Lemma 7.11, Eq. (134).

Lemma 7.10. *We make the Assumption 4.5 of minimal captivity. Let $w, w' \in \mathcal{W}$, $n \geq 1$ and suppose that $w_{0,n} \neq w'_{0,n}$. Furthermore as in (127), let $n_1, n_2 \in \mathbb{N}$ be such that $w_i = w'_i$ if $i < n_1$ or $i > n - n_2$ and $w_{n_1} \neq w'_{n_1}$, $w_{n-n_2} \neq w'_{n-n_2}$. Then we have*

$$Ce^{-J_{w_{n-n_2+1,n}}} \leq \left|x_{L^n(w)} - x_{L^n(w')}\right|, \tag{130}$$

and for any $x \in K_a \cap I_{w_0}$,

$$Ce^{-J_{w_{0,n_1-1}}} \leq \left|\zeta_w(x) - \zeta_{w'}(x)\right| \leq C'e^{-J_{w_{0,n_1-1}}}, \tag{131}$$

with $C, C' > 0$ independent of w, w', n, x.

Proof. We have

$$x_{L^{n-(n_2-1)}(w)} \in \phi_{w_{n-n_2},w_{n-(n_2-1)}}\left(I_{w_{n-n_2}}\right) \text{ and}$$
$$x_{L^{n-(n_2-1)}(w')} \in \phi_{w'_{n-n_2},w'_{n-(n_2-1)}}\left(I_{w'_{n-n_2}}\right).$$

As we have $w_{n-n_2} \neq w'_{n-n_2}$ we conclude from the strong separation condition (2) we have that

$$\left|x_{L^{n-(n_2-1)}(w)} - x_{L^{n-(n_2-1)}(w')}\right| \geq C \tag{132}$$

with $C > 0$ which is the minimal distance between the intervals $\phi_{i,j}(I_i)$. As $w'_{n-(n_2-1),n} = w_{n-(n_2-1),n}$ we obtain

$$x_{L^n(w)} = \phi_{w_{n-(n_2-1),n}}\left(x_{L^{n-(n_2-1)}(w)}\right) \text{ and } x_{L^n(w')} = \phi_{w_{n-(n_2-1),n}}\left(x_{L^{n-(n_2-1)}(w')}\right).$$

From (132) and the fact, that $\phi_{w_{n_2+1,n}}$ is a diffeomorphism we get

$$\left|x_{L^n(w)} - x_{L^n(w')}\right| \geq \left|x_{L^{n-(n_2-1)}(w)} - x_{L^{n-(n_2-1)}(w')}\right|$$
$$\cdot \min_{x\in I_{w_{n-(n_2-1)}}}\left|\phi'_{w_{n-(n_2-1),n}}(x)\right| \geq Ce^{-J_{w_{n-n_2+1,n}}}.$$

We have obtained the first inequality in (130).

Now we prove the second inequality of (130) which uses Assumption 4.5 of minimal captivity. The minimal captivity assumption is equivalent to the following property: Let $\mathcal{K}_\varepsilon$ be a closed neighborhood of the trapped set as in (43). For any $i \rightsquigarrow j$ and $i \rightsquigarrow k$ with $j \neq k$ we have that

$$\tilde{\phi}_{i,j}^{-1}\left(\mathcal{K}_\varepsilon \cap \pi^{-1}(I_j)\right) \bigcap \tilde{\phi}_{i,k}^{-1}\left(\mathcal{K}_\varepsilon \cap \pi^{-1}(I_k)\right) = \emptyset,$$

because otherwise the dynamics of $\tilde{\phi}$ restricted to $\mathcal{K}_\varepsilon$ is not univalued.

From this we deduce that there exists $C_{\text{mini-capt}} > 0$, such that for any any $i \rightsquigarrow j$ and $i \rightsquigarrow k$ with $j \neq k$, if $\tilde{x} \in I_i$, $(\tilde{x}, \xi), (\tilde{x}, \xi') \in \mathcal{K}_\varepsilon$, $\tilde{\phi}_{i,j}(\tilde{x}, \xi), \tilde{\phi}_{i,k}(\tilde{x}, \xi') \in \mathcal{K}_\varepsilon$ then $\left|\xi - \xi'\right| \geq C_{\text{mini-capt}}$.

Let $x \in I_{w_0} \cap K_a$ and for $m \le n$ define $x_m := \phi_{w_{0,m}}(x)$, $\xi_m := \zeta_{L^m(w)}(x_m)$, $x'_m := \phi_{w'_{0,m}}(x)$ and $\xi'_m := \zeta_{L^m(w')}\left(x'_m\right)$. From Proposition 4.11 one has $\tilde{\phi}_{w_{m,m+1}}(x_m, \xi_m) = (x_{m+1}, \xi_{m+1})$. As $x_m \in K_a$ with a chosen large enough according to (59), we have $(x_m, \xi_m), \left(x'_m, \xi'_m\right) \in \mathcal{K}_\varepsilon$. We have $x_{n_1-1} = x'_{n_1-1}$ from definition of n_1 and we have $\left(x_{n_1}, \xi_{n_1}\right) = \tilde{\phi}_{w_{n_1-1,n_1}}\left(x_{n_1-1}, \xi_{n_1-1}\right)$, $\left(x'_{n_1}, \xi'_{n_1}\right) = \tilde{\phi}_{w'_{n_1-1,n_1}}\left(x'_{n_1-1}, \xi'_{n_1-1}\right)$ and $w_{n_1} \ne w'_{n_1}$ so from above we deduce that $\left|\xi_{n_1-1} - \xi'_{n_1-1}\right| \ge C_{\text{mini-capt}}$. Furthermore by Lemma 4.1 we know that $\left|\xi_{n_1-1} - \xi'_{n_1-1}\right| < \tilde{C}$. Using the definition of the canonical map $\tilde{\phi}$ in (41) we compute

$$\left|\xi_{n_1-1} - \xi'_{n_1-1}\right| = \left|\left(\tilde{\phi}_{w_{0,n}}(x, \xi_0)\right)_\xi - \left(\tilde{\phi}_{w'_{0,n}}(x, \xi'_0)\right)_\xi\right| = e^{J_{w_{0,n}}(x)}\left|\xi_0 - \xi'_0\right|.$$

Now using the bounds for $\left|\xi_{n_1-1} - \xi'_{n_1-1}\right|$ from above as well as the bounded variation estimate from Lemma B.1 we obtain (131).

Lemma 7.11. *For any $m > 0$, there exists $C > 0$, such that for any $0 \le k, k' < m - \frac{3}{2}$, $w, w' \in \mathcal{W}$, $n \ge 1$, $\nu > 0$, we have*

$$\nu^{-(k+k'+1)}\left|\mathrm{Tr}\left(\tilde{\Pi}^*_{k',w',n}\tilde{\Pi}_{k,w,n}\right)\right| \le C. \tag{133}$$

Moreover for any $M_1, M_2 \ge 0$, there exists C_{M_1,M_2}, such that for any $w, w' \in \mathcal{W}$ with $x_{L^n(w)} \ne x_{L^n(w')}$ and $\min_{x\in K_a}\left(\left|\zeta_w(x) - \zeta_{w'}(x)\right|\right) \ne 0$, with K_a given in (59), we have

$$\nu^{-(k+k'+1)}\left|\mathrm{Tr}\left(\tilde{\Pi}^*_{k',w',n}\tilde{\Pi}_{k,w,n}\right)\right| \le C_{M_1,M_2}\left(\frac{\nu^{-1}}{\left|x_{L^n(w)} - x_{L^n(w')}\right|}\right)^{M_1} \left(\frac{\nu^{-1}}{\min_{x\in K_a}\left(\left|\zeta_w(x) - \zeta_{w'}(x)\right|\right)}\right)^{M_2}. \tag{134}$$

Proof. Using Dirac notation (100) we have

$$\tilde{\Pi}^*_{k',w',n}\tilde{\Pi}_{k,w,n} = |\tilde{\mathcal{S}}^{(k')}_{w'}\rangle\langle\tilde{\mathcal{U}}^{(k')}_{L^n(w')}, \tilde{\mathcal{U}}^{(k)}_{L^n(w)}\rangle_{L^2}\langle\tilde{\mathcal{S}}^{(k)}_w|.\rangle_{L^2}$$

so

$$\mathrm{Tr}\left(\tilde{\Pi}^*_{k',w',n}\tilde{\Pi}_{k,w,n}\right) = \langle\tilde{\mathcal{U}}^{(k')}_{L^n(w')}, \tilde{\mathcal{U}}^{(k)}_{L^n(w)}\rangle_{L^2} \cdot \langle\tilde{\mathcal{S}}^{(k)}_w, \tilde{\mathcal{S}}^{(k')}_{w'}\rangle_{L^2}. \tag{135}$$

We first consider the term $\langle\tilde{\mathcal{U}}^{(k')}_{L^n(w')}, \tilde{\mathcal{U}}^{(k)}_{L^n(w)}\rangle_{L^2}$. Since the following estimates are uniform with respect to the words w, w', for simplicity we will estimate $\langle\tilde{\mathcal{U}}^{(k')}_{w'}, \tilde{\mathcal{U}}^{(k)}_w\rangle_{L^2}$ without the action of L^n. In the expression (99) of $|\tilde{\mathcal{U}}^{(k)}_w\rangle$, we have the distribution $\hat{\chi}^{-1}\widehat{\mathcal{T}}_w\frac{1}{k!}\delta^{(k)}$ and the product formula for derivatives gives that

$$\hat{\chi}^{-1}\widehat{\mathcal{T}}_w\frac{1}{k!}\delta^{(k)} = \sum_{l=0}^{k} c_l\nu^{(k-l)}\delta^{(l)}_{x_w}$$

with constants c_l independent on w and ν. Then

$$\langle \tilde{\mathcal{U}}_{w'}^{(k')}, \tilde{\mathcal{U}}_{w}^{(k)} \rangle_{L^2} \underset{(99)}{=} \sum_{l'=0}^{k'} \sum_{l=0}^{k} c_{l'} c_l \nu^{(k+k'-l-l')} \langle \delta_{x_{w'}}^{(l')}, \hat{A}_m^2 \delta_{x_w}^{(l)} \rangle_{L^2}.$$

We have

$$\left| \langle \delta_{x_{w'}}^{(l')}, \hat{A}_m^2 \delta_{x_w}^{(l)} \rangle_{L^2} \right| \leq \left\| \delta_{x_w}^{(l')} \right\|_{H_\nu^{-m}} \left\| \delta_{x_w}^{(l)} \right\|_{H_\nu^{-m}} \underset{(81)}{\leq} C \nu^{l+l'+1},$$

so we have the general bound

$$\left| \langle \tilde{\mathcal{U}}_{w'}^{(k')}, \tilde{\mathcal{U}}_{w}^{(k)} \rangle_{L^2} \right| \leq C.\nu^{(k+k'+1)} \tag{136}$$

Let us now suppose that $x_w \neq x_{w'}$. We use the non stationary phase approximation and get that for any $M_1 \geq 0$,

$$\begin{aligned} \left| \langle \delta_{x_{w'}}^{(l')}, \hat{A}_m^2 \delta_{x_w}^{(l)} \rangle_{L^2} \right| &\underset{(19)}{=} \left| \int \langle \xi \rangle^{-2m} \mathcal{F}_\nu(\delta_{x_w}^{(l)})(\xi) \overline{\mathcal{F}_\nu(\delta_{x_w}^{(l')})(\xi)} d\xi \right| \\ &= \frac{1}{2\pi} \nu^{1+l+l'} \left| \int e^{-i\nu\xi(x_w - x_{w'})} \langle \xi \rangle^{-2m} \xi^{l+l'} d\xi \right| \\ &\leq C_{M_1} \nu^{1+l+l'} \left(\frac{1/\nu}{|x_w - x_{w'}|} \right)^{M_1} \end{aligned}$$

hence

$$\left| \langle \tilde{\mathcal{U}}_{w'}^{(k')}, \tilde{\mathcal{U}}_{w}^{(k)} \rangle_{L^2} \right| \leq C_{M_1} \nu^{(k+k'+1)} \left(\frac{1/\nu}{|x_w - x_{w'}|} \right)^{M_1}$$

with C_{M_1} independent on ν, w. This also gives that for any n:

$$\left| \langle \tilde{\mathcal{U}}_{L^n(w')}^{(k')}, \tilde{\mathcal{U}}_{L^n(w)}^{(k)} \rangle_{L^2} \right| \leq C_{M_1} \nu^{(k+k'+1)} \left(\frac{1/\nu}{\left| x_{L^n(w)} - x_{L^n(w')} \right|} \right)^{M_1}. \tag{137}$$

Let us consider now the second term $\langle \tilde{\mathcal{S}}_w^{(k)}, \tilde{\mathcal{S}}_{w'}^{(k')} \rangle_{L^2}$ in (135). We have $\tilde{\mathcal{S}}_w^{(k)} := \hat{A}_m^{-1} \hat{\chi} \left(\widehat{\mathcal{T}}_w^{-1} \right)^* x^k$ and using the form of $\widehat{\mathcal{T}}_w$ given in Theorem 5.2 we can write

$$\hat{\chi} \left(\widehat{\mathcal{T}}_w^{-1} \right)^* x^k = e^{i\nu\Upsilon_w^{(0)}(x)} a_w(x)$$

with $a_w \in C_0^\infty(\mathbb{R})$ given by

$$a_w(x) = \chi(x) e^{\Upsilon_w^{(1)}(x)} \frac{\left(H_w^{-1}(x - x_w) \right)^k}{\left| H_w' \left(H_w^{-1}(x - x_w) \right) \right|}.$$

Recall that from the choice of χ in Sect. 2.3.1 we have $\mathrm{supp}(a_w) \subset K_a$. Furthermore, from (67) we conclude that a_w and its derivatives are bounded on K_a and that these bounds are uniform with respect to $w \in \mathcal{W}$ and ν. Then

$$\langle \tilde{\mathcal{S}}_w^{(k)}, \tilde{\mathcal{S}}_{w'}^{(k')} \rangle_{L^2} = \langle a_w, e^{-i\nu\Upsilon_w^{(0)}(x)} \hat{A}_m^{-2} e^{i\nu\Upsilon_{w'}^{(0)}(x)} a_{w'}(x) \rangle$$

From Egorov's theorem, since $\hat{A}_m^{-2} = \mathrm{Op}_\nu\left(\langle\xi\rangle^{2m}\right)$, we have

$$\hat{A}_m^{-2} e^{i\nu\Upsilon_{w'}^{(0)}(x)} = e^{i\nu\Upsilon_{w'}^{(0)}(x)}\hat{B}$$

with $\hat{B} \in S\left(\left\langle\xi - \frac{d}{dx}\Upsilon_{w'}^{(0)}(x)\right\rangle^{2m}\right)$. Thus $\hat{B}$ is a continuous operator on $\mathcal{S}(\mathbb{R})$, thus $\tilde{a}_{w'} := \hat{B}a_{w'} \in \mathcal{S}(\mathbb{R})$ with all derivatives uniformly bounded with respect to ν, w. This gives

$$\langle\tilde{\mathcal{S}}_w^{(k)}, \tilde{\mathcal{S}}_{w'}^{(k')}\rangle_{L^2} = \langle a_w, e^{i\nu\left(\Upsilon_{w'}^{(0)} - \Upsilon_w^{(0)}\right)(x)}\tilde{a}_{w'}(x)\rangle \tag{138}$$

We deduce the general bound

$$\left|\langle\tilde{\mathcal{S}}_w^{(k)}, \tilde{\mathcal{S}}_{w'}^{(k')}\rangle_{L^2}\right| \leq C \tag{139}$$

with C independent of ν and w.

Let us now assumed $\min_{x\in K_a}(|\zeta_w(x) - \zeta_{w'}(x)|) \neq 0$ then we want to bound the oscillating integral

$$\left|\langle\tilde{\mathcal{S}}_w^{(k)}, \tilde{\mathcal{S}}_{w'}^{(k')}\rangle_{L^2}\right| = \left|\int_{K_a} \overline{a_w(x)}\tilde{a}_{w'}(x)\, e^{i\nu\left(\Upsilon_{w'}^{(0)} - \Upsilon_w^{(0)}\right)(x)} dx\right|$$

by partial integration. Recall that $\frac{d}{dx}\Upsilon_w^{(0)}(x) = \zeta_w(x)$, thus the differential operator $L := \frac{-i/\nu}{\zeta_w(x) - \zeta_{w'}(x)}\frac{d}{dx}$ fulfills $Le^{i\nu\left(\Upsilon_{w'}^{(0)} - \Upsilon_w^{(0)}\right)(x)} = e^{i\nu\left(\Upsilon_{w'}^{(0)} - \Upsilon_w^{(0)}\right)(x)}$ and we can insert an arbitrary power of this differential operator in front of the oscillating phase. Note, that its L^2-dual is given by

$$L^* = \frac{\frac{i}{\nu}\left(\frac{d}{dx}\zeta_w(x) - \frac{d}{dx}\zeta_{w'}(x)\right)}{(\zeta_w(x) - \zeta_{w'}(x))^2} + \frac{i/\nu}{\zeta_w(x) - \zeta_{w'}(x)}\frac{d}{dx}.$$

Note that without any additional knowledge about the ζ_w partial integration would only allow us to obtain remainder terms of the form

$$\left(\frac{\nu^{-1/2}}{\min_{x\in K_a}(|\zeta_w(x) - \zeta_{w'}(x)|)}\right)^{M_2},$$

where the term $\nu^{-1/2}$ comes from a non stationary phase estimate. We can however improve this estimate crucially if we take into account the regularity of the invariant foliation of $\tilde{\phi}$. Let us explain this in more detail:

Recall that $\mathcal{K}$ was the hyperbolic set of the map $\tilde{\phi}$ (cf. Remark 4.6(4)). Further more this hyperbolic set has a precise description via the symbolic dynamics, i.e. for any $(x,\xi) \in \mathcal{K}$ there is $w \in \mathcal{W}$ such that $(x,\xi) = (x_{w_-}, \zeta_{w_+}(x_{w_-}))$. Recall furthermore (cf. Remark 4.12), that the stable manifold through such a point $(x_{w_-}, \zeta_{w_+}(x_{w_-}))$ is locally given by $\{(x, \zeta_{w_+}(x)), x \in I_{w_{-a,0}}\}$. Now for general hyperbolic C^∞ diffeomorphisms, the regularity theory of the invariant manifolds implies, that the stable manifolds are C^∞ and that they depend Hölder continuously on the base point w.r.t. the C^∞-topology [HP70]. Let us make precise what this means using our notation: If we fix the value of x, i.e. if we restrict ourselves to an unstable manifold, then the map

$$g : \begin{cases} \mathcal{K}\cap\{x = x_{w_-}\} & \to C^\infty(I_{w_{-a,0}}, \mathbb{R}) \\ (x_{w_-}, \zeta_{w_+}(x_{w_-})) & \mapsto \zeta_{w_+} \end{cases}$$

that associates to a point in $\mathcal{K}$ the function describing the unstable manifold, is Hölder continuous, where we put the metrizable $C^\infty-$topology on the right side. As the hyperbolic map $\tilde{\phi}$ acts on a two dimensional space one even knows that the Hölder regularity is $2-\varepsilon$ for any $\varepsilon > 0$ (This is a direct consequence of the more general regularity estimate in terms of bunching coefficients, see [Has02, Proposition 2.3.3]). In particular the map g is Lipschitz[15], thus for any k, there is $C_k > 0$ such that for any $x \in I_{w_{-a,0}}$

$$\left|\left(\frac{d}{dx}\right)^k \zeta_{w_+}(x) - \left(\frac{d}{dx}\right)^k \zeta_{w'_+}(x)\right| \leq C_k \left|\zeta_{w_+}(x_{w_-}) - \zeta_{w'_+}(x_{w_-})\right| \underset{(131)}{\leq} C'_k \min_{y\in I_{w_{-a,0}}} \left|\zeta_{w_+}(y) - \zeta_{w'_+}(y)\right|.$$

As K_a is a finite union of $I_{w_{-a,0}}$ we obtain

$$\max_{x\in K_a}\left|\left(\frac{d}{dx}\right)^k \zeta_{w_+}(x) - \left(\frac{d}{dx}\right)^k \zeta_{w'_+}(x)\right| \leq C_k \min_{x\in K_a}\left|\zeta_{w_+}(x) - \zeta_{w'_+}(x)\right|.$$

Using this estimate, partial integration of (138) with respect to L yields that for any $M_2 \geq 0$,

$$\left|\langle \tilde{\mathcal{S}}_w^{(k)}, \tilde{\mathcal{S}}_{w'}^{(k')}\rangle_{L^2}\right| \leq C_{M_2}\cdot \left(\frac{\nu^{-1}}{\min_{x\in K_a}\left(|\zeta_w(x) - \zeta_{w'}(x)|\right)}\right)^{M_2} \tag{140}$$

with C_{M_2} independent on ν, w. Finally (136), (139) and (135) give (133). Equations (137), (140) and (135) give (134).

Proof of Proposition 7.8. Let us summarize what we have obtained so far. Lemma 7.10 gives us lower bounds

$$|x_{L^n(w)} - x_{L^n(w')}| \geq Ce^{-J_{w_{n-n_2+1,n}}}, \quad \min_{x\in K_a}|\zeta_w(x) - \zeta_{w'}(x)| \geq Ce^{-J_{w_{0,n_1-1}}},$$

and in (134) we have the terms

$$\frac{\nu^{-1}}{\left|x_{L^n(w)} - x_{L^n(w')}\right|} \leq \frac{1}{C\nu}e^{J_{w_{n-n_2+1,n}}}$$
$$\frac{1/\nu}{\min_{x\in K_a}\left(|\zeta_w(x) - \zeta_{w'}(x)|\right)} \leq \frac{1}{C\nu}e^{J_{w_{0,n_1-1}}}$$

that we would like to be "small". Therefore, for any $J_c > 0$, any $\varepsilon > 0$, $\nu > 0$ we take $n = \left[\frac{2}{J_c+\varepsilon}\log\nu\right]$ (equivalently $\nu \asymp e^{n\frac{J_c+\varepsilon}{2}}$). Then the condition $n_2 \leq N_2(w, n, J_c)$ implies

$$\frac{1}{\nu}e^{J_{w_{n-n_2+1,n}}} \leq Ce^{-n\frac{J_c+\varepsilon}{2}}e^{n\frac{J_c}{2}} = e^{-\frac{\varepsilon}{2}n}$$

and $n_1 \leq N_1(w, n, J_c)$ implies

$$\frac{1}{\nu}e^{-J_{w_{0,n_1-1}}} \leq Ce^{-n\frac{J_c+\varepsilon}{2}}e^{n\frac{J_c}{2}} = e^{-\frac{\varepsilon}{2}n}.$$

[15] Note that we in fact only need Lipschitz continuity of the stable foliation which might be an important observation for generalizations to higher dimensional settings.

Consider a pair of words $w, w' \in \mathcal{W}$ with $w_{0,n} \neq w'_{0,n}$ with $n_1 \leq N_1$ or $n_2 \leq N_2$. From (134), (130) and definitions of N_1, N_2 we get

$$\left|\mathrm{Tr}\left(\tilde{\Pi}^*_{k',w',n}\tilde{\Pi}_{k,w,n}\right)\right| \leq C_{M_1,M_2}\nu^{(k+k'+1)}\left(\frac{1}{\nu}e^{J_{w_{n-n_2+1,n}}}\right)^{M_1}\left(\frac{1}{\nu}e^{J_{w_{0,n_1-1}}}\right)^{M_2}$$
$$\leq C_M\nu^{(k+k'+1)}e^{-\frac{\varepsilon}{2}nM}$$

where, if $n_1 \leq N_1$ we have set $M_2 = M$ and $M_1 = 0$ otherwise we have set $M_1 = 0$ and $M_2 = M$. This finishes the proof of Proposition 7.8. □

8. Proof of the Main Theorems 3.3 and 3.6

8.1. Proof of Theorem 3.3. Let

$$\gamma_{\text{asympt.}} := \limsup_{\nu\to+\infty}\left(\log\left(r_s\left(\mathcal{L}_{\nu,\chi\restriction H_\nu^{-m}}\right)\right)\right).$$

We will proceed in few steps in order to bound from above $\gamma_{\text{asympt.}}$. The following Proposition gives an upper bound γ_{up} with a complicated expression that will be simplified later.

Proposition 8.1. *Under the assumption of minimal captivity* (43) *we have*

$$\gamma_{\text{asympt.}} \leq \gamma_{\text{up}}$$

with

$$\gamma_{up} := \inf_{0\leq J_c<2J_{min}}\left(\gamma\left(J_c\right)\right), \tag{141}$$

$$\gamma\left(J_c\right) := \frac{J_c}{4} + \lim_{n\to\infty}\frac{1}{2n}\log\left(\sum_{w_{0,n}} e^{2(V-J)_{w_{0,N^*}}}e^{(V-J)_{w_{N^*,n}}}\sum_{w'_{N^*,n}} e^{(V-J)_{w'_{N^*,n}}}\right).$$

$$N^* := N^*\left(w_{0,n}, J_c\right) := \max\left\{k \leq n, \quad s.t.\ J_{w_{0,k}} < nJ_c\right\}. \tag{142}$$

Proof. In order to estimate the spectral radius of the transfer operator we use that for any $n \geq 1$:

$$r_s\left(\mathcal{L}_{\nu,\chi\restriction H_\nu^{-m}}\right) = r_s\left(\hat{Q}_{\restriction L^2}\right) \leq \left\|\left(\hat{Q}^n\right)^*\hat{Q}^n\right\|_{L^2}^{1/(2n)} = \|P_n\|_{L^2}^{1/(2n)}. \tag{143}$$

Now we use Proposition 7.1 for some arbitrary $\varepsilon > 0$, $0 < \beta < 1$, $0 \leq J_c \leq 2J_{min} - \varepsilon$, $\nu > 0$ and $n = \left[\frac{2}{J_c+\varepsilon}\log\nu\right]$ and calculate

$$\log\left(r_s\left(\mathcal{L}_{\nu,\chi\restriction H_\nu^{-m}}\right)\right) \leq \frac{1}{2n}\log\|P_n\|_{L^2}$$

$$\underset{(107)}{\leq}\frac{1}{2n}\log\left(C\nu\sum_{w_{0,n}} e^{2(V-J)_{w_{0,N^*}}}e^{(V-J)_{w_{N^*,n}}}\sum_{w'_{N^*,n}} e^{(V-J)_{w'_{N^*,n}}} + C\beta^n\right)$$

$$\underset{(106)}{=} \frac{J_c + \varepsilon}{4} + \frac{1}{2n} \log (C)$$

$$+\frac{1}{2n} \log \left(\sum_{w_{0,n}} e^{2(V-J)_{w_{0,N^*}}} e^{(V-J)_{w_{N^*,n}}} \sum_{w'_{N^*,n}} e^{(V-J)_{w'_{N^*,n}}} + \frac{1}{\nu} \beta^n \right).$$

As we can choose $\varepsilon > 0$ and $0 < \beta < 1$ arbitrarily small we deduce Proposition 8.1. The fact that the limit exists is explained in Remark B.3.

8.1.1. Expression of $\gamma(J_c)$ in terms of topological pressure. We will now express $\gamma(J_c)$ in Eq. (142) in a concise way that will finally give the formulation in Theorem 3.3. For this we will use the topological pressure $\Pr(.)$ defined in (159).

Remark 8.2. Let us first remark that if the Jacobian J is equal to (or even cohomologous to) a constant J_0 , then the expression of $\gamma(J_c)$ and of γ_{up} in Theorem 3.3 are obtained very easily: in this case $\lim_{n\to\infty} \frac{1}{n} J_{w_{0,n}} = J_0$ and by choosing $J_c = J_0$ one obtains $\gamma(J_0) = \frac{J_0}{4} + \frac{1}{2}\Pr(2(V - J))$ which is precisely the upper bound (28) for $\gamma_{\mathrm{asympt.}}$ in Theorem 3.3. The rest of this section will be devoted to derive (28) in the case where J is not cohomologous to a constant, which we will suppose from now on.

Let $0 \le J_c < 2J_{min}$ and

$$N_{min} := n\frac{J_c}{J_{max}} = \frac{2}{J_{max}} \log(\nu) \frac{J_c}{J_c + \varepsilon}, \quad N_{max} := n\frac{J_c}{J_{min}} = 2\frac{1}{J_{min}} \log(\nu) \frac{J_c}{J_c + \varepsilon}.$$

Note that $N_{min} \le N_{max}$. As $\frac{J_c}{J_c+\varepsilon} \approx 1$ we can interpret them as twice the Ehrenfest time for the most expanding and less expanding trajectory respectively. For every word $w_{0,n}$ we have $N^*\left(w_{0,n}, J_c\right) \in [N_{min}; \min(N_{max}; n)]$. Let us sort the words $w_{0,n}$ in the sum (142) according to their values $N^*\left(w_{0,n}, J_c\right)$:

$$\gamma(J_c)$$

$$= \frac{J_c}{4} + \lim_{n\to\infty} \frac{1}{2n} \log \left(\sum_{\overline{N}=N_{min}}^{\min(N_{max};n)} \sum_{w_{0,\overline{N}} \text{ s.t. } J_{w_{0,\overline{N}}} \le nJ_c < J_{w_{0,\overline{N}+1}}} e^{2(V-J)_{w_{0,\overline{N}}}} e^{2\left(n-\overline{N}\right)\Pr(V-J)} \right.$$

$$\left. + \sum_{w_{0,n} \text{ s.t. } N^*=n} e^{2(V-J)_{w_{0,n}}} \right).$$

In the above formula, $\Pr(V - J)$ appears by using (159). This gives

$$\gamma(J_c) = \frac{J_c}{4} + \max(\mathcal{A}; \mathcal{B}) \tag{144}$$

with

$$\mathcal{A} := \lim_{n\to\infty} \frac{1}{2n} \log \left(\sum_{\overline{N}=N_{min}}^{\min(N_{max};n)} \left(\sum_{w_{0,\overline{N}} \text{ s.t. } J_{w_{0,\overline{N}}} \le nJ_c < J_{w_{0,\overline{N}+1}}} e^{2(V-J)_{w_{0,\overline{N}}}} \right) e^{2\left(n-\overline{N}\right)\Pr(V-J)} \right)$$

$$\mathcal{B} := \lim_{n\to\infty} \frac{1}{2n} \log \left(\sum_{w_{0,n} \text{ s.t. } J_{w_{0,n}} < nJ_c} e^{2(V-J)_{w_{0,n}}} \right).$$

Recall that we are interested in determining $\inf_{0 \leq J_c < 2J_{min}} \gamma(J_c)$. Let us first discuss what happens if $J_c > J_{max}$. In this case $\mathcal{A} = -\infty$ and $\mathcal{B} = \frac{1}{2} Pr(2(V - J))$ are both independent of J_c. Consequently, through the additional $\frac{J_c}{4}$ term in (144), the quantity $\gamma(J_c)$ becomes monotonously increasing in this regime. Thus in order to find $\inf_{0 \leq J_c < 2J_{min}} \gamma(J_c)$ we can from no on suppose, that we only consider $J_c \leq J_{max}$.

Next we want to give precise expressions for the terms $\mathcal{A}$ and $\mathcal{B}$ in term of the topological pressure functions. For this purpose we use some large deviations results presented in Appendix B:

For the term $\mathcal{B}$ we use the formula (169) giving

$$\mathcal{B} = \frac{1}{2} \max_{\overline{J} \in [J_{min}; J_c]} \left(-v_1\left(\overline{J}\right)\right)$$

with

$$\begin{aligned} v_1\left(\overline{J}\right) &:= \beta\left(\overline{J}\right)\overline{J} - u\left(\beta\left(\overline{J}\right)\right) \\ u\left(\beta\right) &:= \Pr\left(2\left(V - J\right) + \beta J\right). \end{aligned}$$

and with $\beta\left(\overline{J}\right)$ such that

$$\overline{J} = u'\left(\beta\left(\overline{J}\right)\right). \tag{145}$$

Observe that $v_1\left(\overline{J}\right)$ is the Legendre transform of the convex and increasing function $u\left(\beta\right)$. For the term $\mathcal{A}$ we change the variable $\overline{N}$ by $\overline{J} = \frac{nJ_c}{\overline{N}}$ and we also use the formula (169) giving

$$\begin{aligned} \mathcal{A} &= \lim_{n\to\infty} \frac{1}{2n} \log\left(\int_{J_c}^{J_{max}} d\overline{J} e^{\frac{nJ_c}{\overline{J}}\left(\Pr\left(2(V-J)+\beta(\overline{J})J\right)-\beta(\overline{J})\overline{J}\right)} e^{2\left(n-\frac{nJ_c}{\overline{J}}\right)\Pr(V-J)}\right) \\ &= \frac{1}{2} \max_{\overline{J} \in [J_c; J_{max}]} \left(-v_2\left(\overline{J}\right)\right) \end{aligned}$$

with

$$v_2\left(\overline{J}\right) := \frac{J_c}{\overline{J}}\left(v_1\left(\overline{J}\right) + 2\Pr\left(V - J\right)\right) - 2\Pr\left(V - J\right).$$

In summary we have that

$$\gamma\left(J_c\right) = \frac{J_c}{4} - \frac{1}{2} \min_{\overline{J} \in [J_{min}; J_{max}]} v\left(\overline{J}\right) \tag{146}$$

with the function $v\left(\overline{J}\right)$ defined in two parts:

$$\begin{aligned} v\left(\overline{J}\right) &:= v_1\left(\overline{J}\right) \text{ if } \overline{J} \in [J_{min}; J_c] \\ &= v_2\left(\overline{J}\right) \text{ if } \overline{J} \in [J_c; J_{max}]. \end{aligned} \tag{147}$$

8.1.2. Minimization of $\gamma\left(J_c\right)$ to deduce γ_{up}. Considering (141) we finally want to minimize $\gamma(J_c)$ in order to obtain the final expression (28) for γ_{up}. Note that this minimization

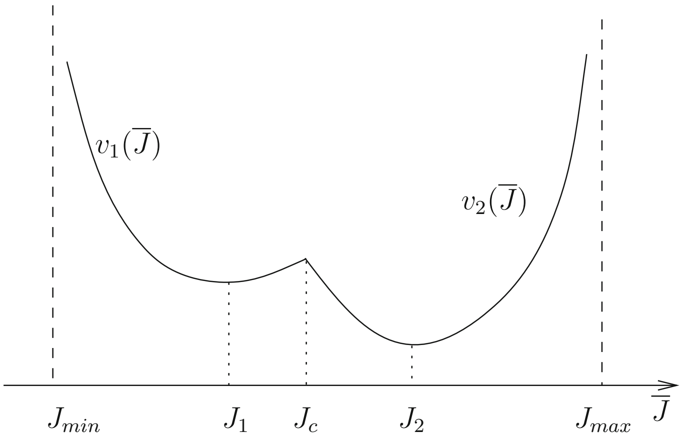

Fig. 5. The function $v\left(\overline{J}\right)$ defined in (147) depends on the parameter J_c and is defined piece-wise. We look for its global minimum $\min_{\overline{J}\in[J_{min};J_{max}]} v\left(\overline{J}\right)$. It will finally turn out that the optimal value of J_c is between the local minima J_1 and J_2 of the functions v_1 and v_2

demands two steps: first for a given J_c, $\gamma(J_c)$ is given by (146) as a minimum over some parameter $\overline{J}$. In a second step we then have to minimize $\gamma(J_c)$ for $J_c \leq 2J_{min}$.

Let us start with the first step and fix $J_c < 2J_{min}$ for the moment. The function $v\left(\overline{J}\right)$ depends on the parameter J_c and $v_2\left(J_c\right) = v_1\left(J_c\right)$ hence $v\left(\overline{J}\right)$ is continuous on $]J_{min}, J_{max}[$. The function $v\left(\overline{J}\right)$ is depicted on Fig. 5. Note that the function v_1 itself does not depend on J_c. Our goal is to minimize the composite function $v(\bar{J})$ that is piece-wise defined via the functions v_1, v_2. However the functions v_1 and v_2 are themselves both well defined on the whole interval $[J_{min}, J_{max}]$ and as a first step we look for the minima of v_1 and v_2 on the whole interval $[J_{min}, J_{max}]$.

Recall that by Remark 8.2 and Proposition B.5, the function $v_1\left(\overline{J}\right)$ is strictly convex and we compute that $v_1'\left(\overline{J}\right) = \beta\left(\overline{J}\right)$ hence its minimum is for $\overline{J} = J_1$ such that $\beta\left(J_1\right) = 0$ giving

$$\min_{\overline{J}} v_1\left(\overline{J}\right) = v_1\left(J_1\right) = -u\left(0\right) = -\mathrm{Pr}\left(2\left(V - J\right)\right)$$

Notice also that

$$\frac{d\beta\left(\overline{J}\right)}{d\overline{J}} = \frac{d^2 v_1}{d\overline{J}^2} > 0 \tag{148}$$

so $\beta\left(\overline{J}\right)$ is increasing.

For the function $v_2\left(\overline{J}\right)$ we compute its derivative

$$\begin{aligned} v_2'\left(\overline{J}\right) &= -\frac{J_c}{\overline{J}^2}\left(v_1\left(\overline{J}\right) + 2\mathrm{Pr}\left(V - J\right)\right) + \frac{J_c}{\overline{J}} v_1'\left(\overline{J}\right) \\ &= \frac{J_c}{\overline{J}^2}\left(-v_1\left(\overline{J}\right) - 2\mathrm{Pr}\left(V - J\right) + \overline{J}\beta\left(\overline{J}\right)\right) \\ &= \frac{J_c}{\overline{J}^2}\left(u\left(\beta\left(\overline{J}\right)\right) - 2\mathrm{Pr}\left(V - J\right)\right). \end{aligned}$$

Let $J_2 \in [J_{min}; J_{max}]$ be such that $v_2'(J_2) = 0$, i.e.

$$u(\beta(J_2)) = 2\mathrm{Pr}(V - J).$$

We have

$$u(\beta(J_2)) = 2\mathrm{Pr}(V - J) > \mathrm{Pr}(2(V - J)) = u(\beta(J_1))$$

hence $J_2 > J_1$ and from (148) $\beta(J_2) > \beta(J_1) = 0$. Plugging J_2 into v_2 we get the minimum

$$\begin{aligned}\min_{\overline{J}} v_2\left(\overline{J}\right) = v_2(J_2) &= \frac{J_c}{J_2}\left(v_1(J_2) + 2\mathrm{Pr}(V - J)\right) - 2\mathrm{Pr}(V - J)\\ &= J_c\beta(J_2) - 2\mathrm{Pr}(V - J).\end{aligned}$$

Remark 8.3. The function v_1 hence its minimum J_1 also do not depend on J_c. The value J_2 does not depend on J_c neither but $v_2(J_2)$ depends on J_c.

The two local minima coincide $v_1(J_1) = v_2(J_2)$ for the parameter $J_c = \langle J\rangle$ given by

$$\langle J\rangle := \frac{2\mathrm{Pr}(V - J) - \mathrm{Pr}(2(V - J))}{\beta(J_2)}. \tag{149}$$

Since $u(\beta)$ is strictly convex and $\beta\left(\overline{J}\right)$ is increasing we have that

$$u'(\beta(J_1)) < \frac{u(\beta(J_2)) - u(\beta(J_1))}{\beta(J_2) - \beta(J_1)} < u'(\beta(J_2))$$

and using (145) and (149) this gives

$$J_1 < \langle J\rangle < J_2.$$

Note that up to now we considered the minima of v_1 and v_2 independently. Now let us consider the minimum of the composite function v which will give us a concrete value for $\gamma(J_c) = \frac{J_c}{4} - \frac{1}{2}\inf_{\overline{J}} v\left(\overline{J}\right)$:

We finally deduce for (146) that

- If $J_c \leq \langle J\rangle$ then

$$\begin{aligned}\gamma(J_c) &= \frac{J_c}{4} - \frac{1}{2}\inf_{\overline{J}} v_2\left(\overline{J}\right)\\ &= \frac{J_c}{4} - \frac{1}{2}J_c\beta(J_2) + \mathrm{Pr}(V - J).\end{aligned} \tag{150}$$

- If $J_c \geq \langle J\rangle$ then

$$\begin{aligned}\gamma(J_c) &= \frac{J_c}{4} - \frac{1}{2}\inf_{\overline{J}} v_1\left(\overline{J}\right)\\ &= \frac{J_c}{4} + \frac{1}{2}\mathrm{Pr}(2(V - J))\end{aligned}$$

  that is minimal for $J_c = \langle J\rangle$.

From (150), we have $\gamma'(J_c)_{/J_c\leq\langle J\rangle} = \frac{1}{2}\left(\frac{1}{2} - \beta(J_2)\right)$ hence if $\beta(J_2) > \frac{1}{2}$ then

$$\begin{aligned} \gamma_{\text{up}} &\underset{(141)}{=} \inf_{J_c\leq\langle J\rangle} (\gamma(J_c)) = \gamma(\langle J\rangle) \\ &= \frac{\langle J\rangle}{4} + \frac{1}{2}\Pr(2(V-J)), \end{aligned}$$

otherwise if $\beta(J_2) < \frac{1}{2}$ then

$$\gamma_{\text{up}} = \gamma(0) = \Pr(V-J).$$

We have finished the proof of the main Theorem 3.3.

8.2. Proof of Theorem 3.6. From (105) and (97) we have

$$\frac{1}{n}\log\left\|\mathcal{L}^n_{\nu,\chi}\right\|_{H_\nu^{-m}} = \frac{1}{2n}\log\|P_n\|_{L^2}.$$

In Sect. 8 we have shown that for n related to ν by $n = \left[\frac{2}{\langle J\rangle+\varepsilon}\log\nu\right]$ we have for $\nu\to\infty$,

$$\frac{1}{2n}\log\|P_n\|_{L^2} \leq \gamma_{\text{up}} + o(1).$$

We deduce that for any $\epsilon > 0$, $\exists\nu_0 > 0$, $\forall\nu > \nu_0$,

$$\begin{aligned} \frac{1}{n}\log\left\|\mathcal{L}^n_{\nu,\chi}\right\|_{H_\nu^{-m}} &\leq \gamma_{\text{up}} + \epsilon \\ \left\|\mathcal{L}^n_{\nu,\chi}\right\|_{H_\nu^{-m}} &\leq e^{n(\gamma_{\text{up}}+\epsilon)}. \end{aligned}$$

In [AFW13, thm 2.9, or proof of thm 2.11] we have shown that for any r,

$$\left\|\mathcal{L}^r_{\nu,\chi}\right\|_{H_\nu^{-m}} \leq C_0 e^{r(\gamma_{\text{sc}}+\epsilon)}.$$

Let us suppose that $\gamma_{\text{up}} < \gamma_{\text{sc}}$ in order to improve this bound (otherwise Theorem 3.6 is covered by [AFW13, thm 2.9]). For any $t\in\mathbb{N}$, we write $t = Nn + r$ with $r\leq n$, $N\in\mathbb{N}$, and we have

$$\begin{aligned} \left\|\mathcal{L}^t_{\nu,\chi}\right\|_{H_\nu^{-m}} &\leq \left\|\mathcal{L}^n_{\nu,\chi}\right\|^N_{H_\nu^{-m}} \left\|\mathcal{L}^r_{\nu,\chi}\right\|_{H_\nu^{-m}} \\ &\leq e^{Nn(\gamma_{\text{up}}+\epsilon)} C_0 e^{r(\gamma_{\text{sc}}+\epsilon)} \\ &\leq C_0 e^{t(\gamma_{\text{up}}+\epsilon)} e^{r(\gamma_{\text{sc}}-\gamma_{\text{up}})} \\ &\leq C_0 e^{t(\gamma_{\text{up}}+\epsilon)} e^{n(\gamma_{\text{sc}}-\gamma_{\text{up}})} \\ &\leq C_0 e^{t(\gamma_{\text{up}}+\epsilon)} \nu^{\frac{2}{\langle J\rangle+\epsilon}(\gamma_{\text{sc}}-\gamma_{\text{up}})}. \end{aligned}$$

The relation $\left(z-\mathcal{L}_{\nu,\chi}\right)^{-1}=z^{-1}\sum_{t\geq 0}\left(\frac{\mathcal{L}_{h,\chi}}{z}\right)^{t}$ gives that

$$\left\|\left(z-\mathcal{L}_{\nu,\chi}\right)^{-1}\right\|_{H_{\nu}^{-m}}\leq|z|^{-1}\sum_{t\geq 0}\frac{\left\|\mathcal{L}_{\nu,\chi}^{t}\right\|_{H_{\nu}^{-m}}}{|z|^{t}}\leq|z|^{-1}C_{0}\nu^{\frac{2}{\langle J\rangle+\epsilon}(\gamma_{\mathrm{sc}}-\gamma_{\mathrm{up}})}\sum_{t\geq 0}\frac{e^{t(\gamma_{\mathrm{up}}+\epsilon)}}{|z|^{t}}$$

$$=\frac{C_{0}\nu^{\frac{2}{\langle J\rangle+\epsilon}(\gamma_{\mathrm{sc}}-\gamma_{\mathrm{up}})}}{|z|-e^{(\gamma_{\mathrm{up}}+\epsilon)}}\leq C_{1}\nu^{\frac{2}{\langle J\rangle+\epsilon}(\gamma_{\mathrm{sc}}-\gamma_{\mathrm{up}})}$$

We have finished the proof of Theorem 3.6.

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A. Examples

In this section we compare the upper bound on the spectral gap γ_{up} from Theorem 3.3 with the so far known bounds $\gamma_{\mathrm{Gibbs}}=\mathrm{Pr}\,(V-J)$ and $\gamma_{\mathrm{sc}}=\mathrm{tsup}\left(V-\frac{1}{2}J\right)$ for two explicit examples of IFS according to Definition 2.1: a "two branched linear IFS" and the "truncated Gauss map" which plays an important role in the study of continued fraction expansion.

A.1. Linear IFS. Within the class of dynamical systems, treated in this article, the linear IFS is perhaps the most simple, nevertheless non trivial example.

A.1.1. Definition of the model. See Fig. 6.

Definition A.1. "Linear IFS". Let $0<a_1<b_1<a_2<b_2<1$. Consider the intervals $I_1:=[a_1,b_1]$ and $I_2:=[a_2,b_2]$ and the adjacency matrix $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$. The contracting maps are the linear functions:

$$\phi_{i,j}:\begin{cases}I_i & \to I_j\\ x & \mapsto a_j+\left(b_j-a_j\right)x\end{cases}.$$

The Jacobian function $J_{i,j}\left(x\right):=-\log\frac{d\phi_{i,j}}{dx}$ is constant on intervals $x\in I_i$:

$$J_j:=J_{i,j}\left(x\right)=-\log(b_j-a_j)>0.$$

The topological pressure takes a particularly simple form:

Lemma A.2. *Let $\phi_{i,j}$ be a linear IFS with Jacobians $J_1,J_2>0$. Then the topological pressure function* (161) *is given by*

$$P\left(\beta\right)=\log\left(e^{-\beta J_1}+e^{-\beta J_2}\right). \tag{151}$$

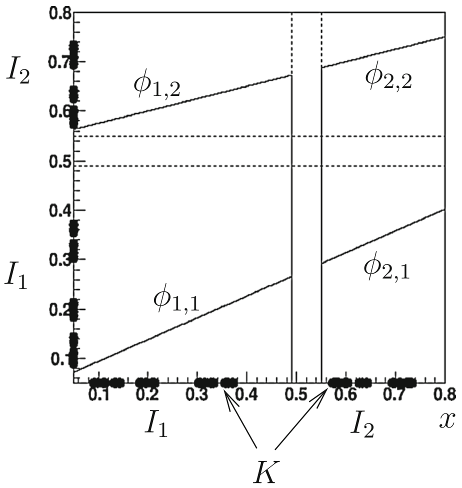

Fig. 6. Graphs of $\phi_{i,j}$ a the linear IFS of Definition A.1 with interval $I_1 = [0.05, 0.49]$, $I_2 = [0.55, 0.8]$ giving Jacobians $J_1 = 0.821\ldots$ and $J_2 = 1.38\ldots$. The trapped set K defined in (5) is a dyadic Cantor set of Hausdorff dimension $\delta = \dim_H(K) = 0.643\ldots$ given by (162)

Proof. We have

$$P(\beta) \underset{(161)}{=} \Pr(-\beta J) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{w_{0,n}} e^{-\beta J_{w_{0,n}}(x_w)}\right).$$

The fact that $J(x)$ is constant in I_1 and I_2 gives $J_{w_{0,n}}(x) = \sum_{k=1}^{n} J_{w_k}$ and since the adjacency matrix A is full,

$$\sum_{w_{0,n}} e^{-\beta J_{w_{0,n}}(x_w)} = \sum_{w_{0,n}} e^{-\beta \sum_{k=1}^{n} J_{w_k}} = \left(e^{-\beta J_1} + e^{-\beta J_2}\right)^n.$$

In order to apply Theorem 3.3 we choose a roof function $\tau : I \to \mathbb{R}$ such that the minimal captivity assumption is fulfilled. This can be achieved by a piece-wise linear function.

Lemma A.3. *Let $\phi_{i,j}$ be a linear IFS as defined in Definition A.1 and suppose that $0 < J_1 \leq J_2$. Let*

$$\tau(x) = \tau_i \cdot x, \quad x \in I_i, \tag{152}$$

with $\tau_1 := 0$ and $\tau_2 := 1$. Then the minimal captivity assumption (*Assumption 4.5*) *is fulfilled.*

Proof. With the above definitions, the canonical map takes a particularly simple form

$$(x', \xi') = \tilde{\phi}_{i,j}(x, \xi) \underset{(40)}{=} (\phi_{i,j}(x), e^{J_j}\xi + \tau_j).$$

Let $x \in I$, $\xi > 0$. We have $\xi' \geq e^{J_1}\xi$ hence any trajectory starting from positive ξ escapes to infinity. This implies that the trapped set is $\mathcal{K} \subset I \times [-\infty, 0]$.

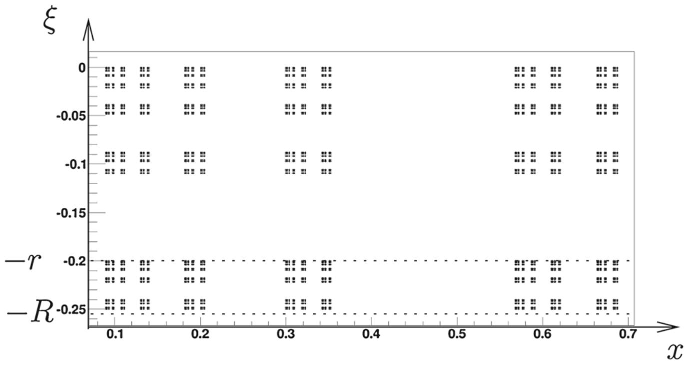

Fig. 7. Trapped set $\mathcal{K}$ in phase space $T^*\mathbb{R}$ for the linear IFS of Fig. 6 constructed with a linear function τ given in (152). One clearly sees its Cantor set nature. The dashed lines indicate the values $-R$, $-r$ that appear in the proof of Lemma A.3

Let $R := \frac{1}{e^{J_2}-1} > 0$. For $j = 1$ we get $\xi' = e^{J_1}\xi$ and for $j = 2$ we get

$$\left(\xi' + R\right) = \left(e^{J_2}\xi + 1 + \frac{1}{e^{J_2} - 1}\right) = e^{J_2}\left(\xi + R\right).$$

This implies that all trajectories starting with $\xi < -R$ escape towards minus infinity and $\mathcal{K} \subset I \times [-R, 0]$.

Let $(x, \xi) \in \mathcal{K}$ and i such that $x \in I_i$. In order to prove minimal captivity we have to show that either $(x_1', \xi_1') = \tilde{\phi}_{i,1}(x, \xi) \notin \mathcal{K}$ or $(x_2', \xi_2') = \tilde{\phi}_{i,2}(x, \xi) \notin \mathcal{K}$. Let $r := e^{-J_2} < R$. If $-r < \xi \leq 0$ then $\xi_2' = e^{J_2}\xi + 1 > 0$ which implies $(x_2', \xi_2') \notin \mathcal{K}$. If $-R \leq \xi \leq -r$ then $\xi_1' = e^{J_1}\xi \leq -e^{J_1 - J_2}$. We use the constraint[16] $I_1 \cup I_2 \subset [0, 1]$ that gives

$$e^{-J_1} + e^{-J_2} < 1. \tag{153}$$

We deduce $\xi_1' < -R$ and $(x_1', \xi_1') \notin \mathcal{K}$.

A.1.2. Estimates for the asymptotic spectral gap $\gamma_{\text{asympt.}}$. Let us now consider the asymptotic spectral radius of the family of transfer operators $\mathcal{L}_\nu$ for a linear IFS with unstable Jacobians $0 < J_1 \leq J_2$ and τ as in Lemma A.3 and with a potential function of the form

$$V(x) = (1 - a)\, J(x), \quad a \in \mathbb{R}. \tag{154}$$

We recall that the value $a = 0$, giving $V = J$ is interesting for counting orbits (180) and that $a = 1/2$ is the "quantum case" [FT15]. The different upper bound estimates for the asymptotic spectral gap $\gamma_{\text{asympt.}}$ can be expressed very explicitly as follows.

$$\gamma_{\text{Gibbs}} \underset{(29)}{:=} \Pr(V - J) = P(a) = \log\left(e^{-aJ_1} + e^{-aJ_2}\right)$$

$$\gamma_{\text{sc}} \underset{(30)}{:=} \operatorname{tsup}\left(V - \frac{1}{2}J\right) = \left(\frac{1}{2} - a\right) J_1 \text{ if } a \geq \frac{1}{2}, \quad \left(\frac{1}{2} - a\right) J_2 \text{ if } a \leq \frac{1}{2}.$$

16 Obviously this constraint is not necessary for the proof but simplifies the choice of the constants.

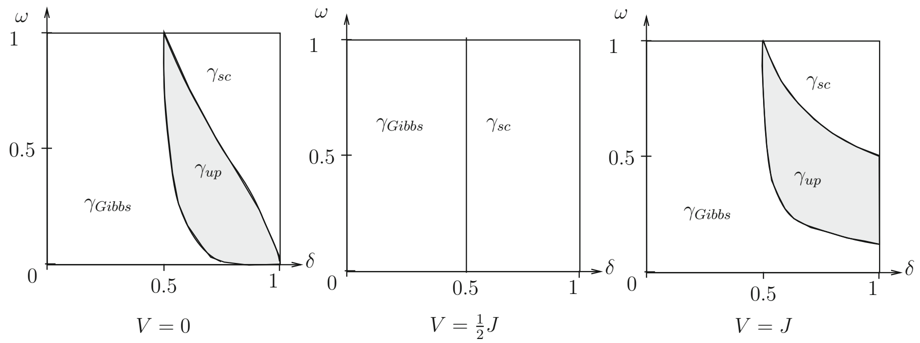

Fig. 8. "**Phase diagram**" in the domain $(\delta,\omega)\in]0,1[^2$ with $\delta=\dim_H(K)\in]0,1[$ being the Hausdorff dimension of the trapped set K given by (162) and $\omega:=\exp(J_1-J_2)\in]0,1]$ that measures the homogeneity of the Jacobian (we have $\omega=1$ if $J_1=J_2$). The three different plots correspond to three different potential functions V. For each of the three potentials and each value (δ,ω) and a, we indicate by a numerical calculation, which value $\gamma_{\text{Gibbs}},\gamma_{\text{sc}},\gamma_{\text{up}}$ is the lowest

$$\gamma_{\text{conj}}\underset{(185)}{:=}\frac{1}{2}\text{Pr}\left(2\left(V-J\right)\right)=\frac{1}{2}P\left(2a\right)$$

$$\gamma_{\text{up}}\underset{(28)}{:=}\gamma_{\text{conj}}+\frac{1}{4}\left\langle J\right\rangle,\quad \left\langle J\right\rangle:=\frac{2}{\beta_0}\left(\gamma_{\text{Gibbs}}-\gamma_{\text{conj}}\right)\tag{155}$$

where β_0 solves the equation

$$P\left(2a-\beta_0\right)=2P\left(a\right).\tag{156}$$

Let us introduce

$$\omega:=\exp\left(J_1-J_2\right)\in]0,1]$$

that measures the non homogeneity of the Jacobian. Note that up to dynamical equivalence the linear IFS is uniquely determined by δ and ω where δ is the Hausdorff dimension of the trapped set K defined in (162). Thus given a fixed potential and a set of parameters (δ,ω) we can ask the question which of the known estimates $\gamma_{\text{sc}},\gamma_{\text{Gibbs}}$ and γ_{up} is the best i.e. lowest one. This leads to a partition of the δ,ω parameter space which is shown in Fig. 8 for three different choices of V. One observes that γ_{up} obtained in this paper gives the best (lowest) result (in grey domain) for intermediate values of δ and for $a\neq 1/2$. γ_{Gibbs} is better for small values of δ (i.e. very open system) or very small values of ω (i.e. very in-homogeneous Jacobian) whereas γ_{sc} is better for large values of δ and ω (i.e. closed system with homogeneous Jacobian).

Figure 9 is a plot of $\gamma_{\text{sc}},\gamma_{\text{Gibbs}},\gamma_{\text{up}},\gamma_{conj}$ as functions of $\delta=\dim_H(K)\in]0,1[$ and for $\omega=0.5$.

Proposition A.4. *For a linear IFS with roof function τ given in (152) we have the following three properties that appear on Fig. 8:*

1. *For any $a\in\mathbb{R}$, potential $V=(1-a)J$, and $\omega=1$ (i.e. homogeneous case $J=J_1=J_2$) we have $\gamma_{\text{Gibbs}}<\min(\gamma_{\text{sc}},\gamma_{\text{up}})$ if $\delta<0.5$ and $\gamma_{\text{sc}}<\min\left(\gamma_{\text{Gibbs}},\gamma_{\text{up}}\right)$ if $\delta>0.5$. For $\delta=0.5$ we have $\gamma_{\text{sc}}=\gamma_{\text{Gibbs}}=\gamma_{\text{up}}$.*
2. *For the potential $V(x)=0$, for any $\omega\in]0,1[$ there exists $\delta=\delta(\omega)$ such that $\gamma_{\text{up}}<\gamma_{Gibss}=\gamma_{\text{sc}}$.*

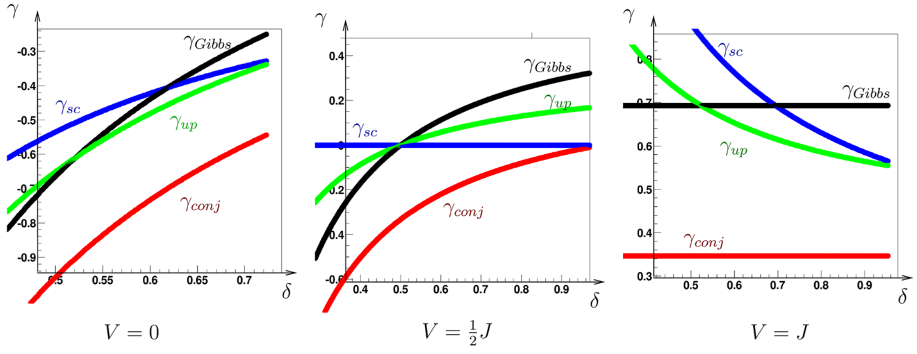

Fig. 9. Plot of various estimates $\gamma_{\text{sc}}, \gamma_{\text{Gibbs}}, \gamma_{\text{up}}, \gamma_{conj}$ defined in Sect. (3) for the linear IFS model with $\omega = 0.5$, as a function of $\delta = \dim_H(K) \in]0, 1[$

3. *For the potential* $V(x) = \frac{1}{2}J(x)$, *for any* $\omega \in]0, 1[$, *we have* $\gamma_{\text{Gibbs}} < \min(\gamma_{\text{sc}}, \gamma_{\text{up}})$ *if* $\delta < 0.5$ *and* $\gamma_{\text{sc}} < \min(\gamma_{\text{Gibbs}}, \gamma_{\text{up}})$ *if* $\delta > 0.5$.

Proof. Proof of case (1). Suppose $J = J_1 = J_2$. This is the case $\omega := \exp(J_1 - J_2) = 1$ on Fig. 8. We have

$$P(\beta) \underset{(151)}{=} \log\left(2e^{-\beta J}\right) = h_{top} - \beta J$$

with $h_{top} \underset{(163)}{=} P(0) = \log 2$ being the topological entropy. Then the Hausdorff dimension $\delta = \dim_H(K)$ of the trapped set K, given by $P(\delta) \underset{(162)}{=} 0$, is $\delta = \frac{h_{top}}{J}$. We get

$$\begin{aligned}
\gamma_{\text{Gibbs}} &= h_{top}\left(1 - \frac{a}{\delta}\right) \\
\gamma_{\text{sc}} &= h_{top}\left(\frac{1}{2} - a\right) \cdot \frac{1}{\delta} \\
\gamma_{\text{up}} &= h_{top}\left(\frac{1}{2} + \left(\frac{1}{4} - a\right)\frac{1}{\delta}\right), \qquad \beta_0 \underset{(156)}{=} \delta, \quad \langle J \rangle = J \\
\gamma_{\text{conj}} &= h_{top}\left(\frac{1}{2} - \frac{a}{\delta}\right)
\end{aligned}$$

We deduce that for $\delta < \frac{1}{2}$ then $\gamma_{\text{Gibbs}} < \min(\gamma_{\text{sc}}, \gamma_{\text{up}})$, for $\delta > \frac{1}{2}$ then $\gamma_{\text{sc}} < \min(\gamma_{\text{Gibbs}}, \gamma_{\text{up}})$ and for $\delta = \frac{1}{2}$ then $\gamma_{\text{sc}} = \gamma_{\text{Gibbs}} = \gamma_{\text{up}}$.

Proof of case (2). Suppose $V = 0$. For a given $0 < \omega < 1$ we choose $0 < J_1 < 2h_{top} = 2\log 2$ such that

$$e^{J_1/2} = 1 + \omega. \tag{157}$$

From $\omega = \exp(J_1 - J_2)$ this gives a value of J_2 and δ. Equation (155) gives $\gamma_{\text{Gibbs}} = \gamma_{\text{sc}}$. According to the statement of Theorem 3.3, if we show that $\beta_0 > 1/2$, this implies that $\gamma_{\text{up}} < \gamma_{\text{Gibbs}}$. From (156) the condition for β_0 is

$$P(2 - \beta_0) - 2P(1) = 0.$$

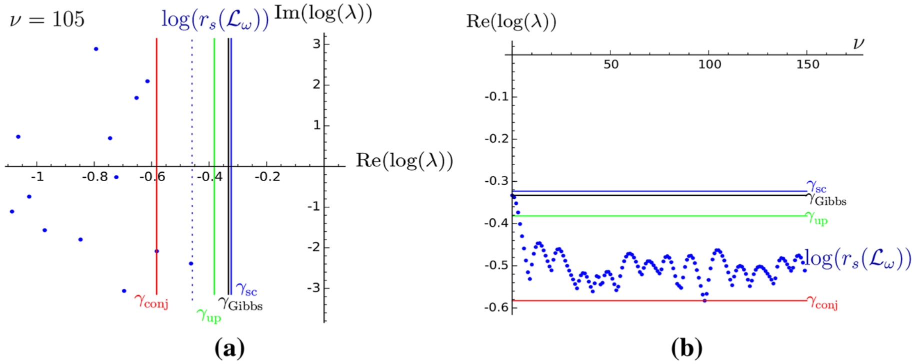

Fig. 10. Model of linear IFS. **a** Shows the Ruelle–Pollicott eigenvalues $\lambda_j \in \mathbb{C}$ (blue points) of the operator $\mathcal{L}_\nu$ for parameters $\nu = 105$, $V = 0$, $J_2 = J_1 + 1$ and $\delta = 0.65$. Vertical lines show $\gamma_{conj}, \gamma_{\rm up}, \gamma_{\rm Gibbs}, \gamma_{\rm sc}$ and $\log(r_s(\mathcal{L}_\nu)) = \max_j \left(\mathrm{Re}\left(\log\left(\lambda_j\right)\right)\right)$ in dotted line. **b** Shows $\log(r_s(\mathcal{L}_\nu))$ with blue points, as a function of ν (color figure online)

As $P(2-\beta) - 2P(1)$ is strictly increasing in β it is sufficient to show that $P\left(2-\frac{1}{2}\right) - 2P(1) < 0$. Using that $P(1) = -\frac{1}{2}J_1$ from our choice $\gamma_{\rm Gibbs} = \gamma_{\rm sc}$, we compute

$$\begin{aligned} P(3/2) - 2P(1) &\underset{(151)}{=} \log\left(e^{-3/2J_1}(1+e^{-3/2(J_2-J_1)})\right) + J_1 \\ &= \log\left(e^{-1/2J_1}(1+e^{-3/2(J_2-J_1)})\right) \\ &< \log\left(e^{-1/2J_1}(1+\omega)\right) \\ &\underset{(157)}{=} 0. \end{aligned}$$

A.1.3. Numerical observations for the Ruelle–Pollicott resonances and $\gamma_{\rm asympt.}$ and discussion. As the linear branches of the IFS can be extended from the intervals I_i to disks in the complex plane, the linear IFS can also be considered as a holomorphic IFS and its Ruelle–Pollicott spectrum can be calculated using a dynamical zeta function approach, introduced by Jenkinson and Pollicott [JP02] (see also [GLZ04,Bor14,BW16,Wei15, BFW14] for applications and further details).

Figure 10a shows the Ruelle–Pollicott spectrum of $\mathcal{L}_\nu$ for a given value of ν. Figure 10b shows the value $\log(r_s(\mathcal{L}_\nu)) = \max_j \left(\mathrm{Re}\left(\log\left(\lambda_j\right)\right)\right)$ as a function of ν, that we want to bound for $\nu \to \infty$. It can be observed, that $\log(r_s(\mathcal{L}_\nu))$ decays rather quickly starting from $\gamma_{\rm Gibbs}$ and then oscillates in a wide range. Each "bump" is due to an individual eigenvalue. The numerical results indicate that **the new bound $\gamma_{\rm up}$ is not an optimal bound** of $\log(r_s(\mathcal{L}_\nu))$. Furthermore the conjecture $\gamma_{conj} = \frac{1}{2}\Pr(2(V-J))$ proposed (185) is not observed to be an upper bound in this range of ν. However the value of $\log(r_s(\mathcal{L}_\nu))$ performs "large fluctuations" touching the value of γ_{conj} several times. A similar phenomenon has been observed for the related question of the asymptotic spectral gap for the Laplacian on Schottky surfaces (see [BW16, Figure 13]). The conjecture $\gamma_{\rm asympt.} = \gamma_{conj}$ could thus hold if one suspects, that the "large fluctuations" of $\log(r_s(\mathcal{L}_\nu))$ die out in the semiclassical limit.

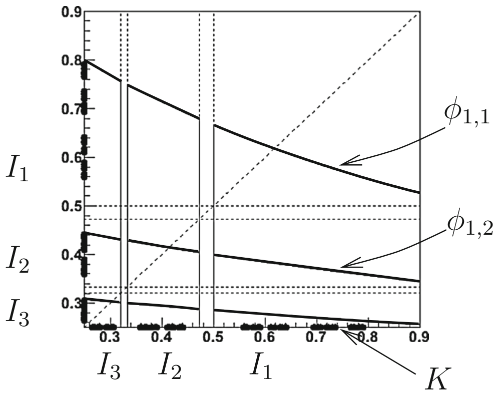

Fig. 11. The iterated functions system (IFS) defined from the truncated Gauss map (158). Here we have $N = 3$ branches. The maps ϕ: $\phi_{i,j} : I_i \to I_j$, $i, j = 1 \dots N$ are contracting and given by $\phi_{i,j}(x) = \frac{1}{x+j}$. The trapped set K defined in (5) is a N-adic Cantor set

A.2. Truncated Gauss map. The model of transfer operators considered here is constructed from the Gauss map and has simple expressions. The Gauss map is important in number theory in relation with continued fractions. The Gauss map is defined by

$$G : \begin{cases}]0, 1] & \to]0, 1[\\ y & \to \frac{1}{y} \bmod 1 \end{cases}. \tag{158}$$

As this map has an infinite number of branches it does not fit into the Definition 2.1 of an IFS. However if we restrict ourselves to a finite number of branches we get a well defined IFS. For more details on this construction we refer to [AFW13, Section 2.1 and 7.1].

Definition A.5. Let $N \geq 1$. We consider the finite number of inverse branches of the Gauss map given for $1 \leq j \leq N$ by $(G^{-1})_j(x) := 1/(x + j)$. Now for $1 \leq i \leq N$ let $a_i = 1 + i$ and b_i such that $(G^{-1})_i(\frac{1}{N+1}) < b_i < \frac{1}{i}$. Then we set the intervals of the truncated Gauss IFS to be $I_i = [a_i, b_i]$. We take the full $N \times N$ matrix as adjacency matrix and define the maps

$$\phi_{i,j}(x) := (G^{-1})_j(x) := \frac{1}{x + j}, \quad 1 \leq i, j \leq N.$$

See Fig. 11.

The dynamical properties of such truncated Gauss IFS play an important role in the study of continued fraction expansions (see e.g. [Hen92, MU99]). In [AFW13, Prop.7.1] it has been shown, that the minimal captivity assumption is fulfilled for roof function $\tau(x) = -J(x)$. So Theorem 3.3 can be applied.

Figure 12 shows $\gamma_{sc}, \gamma_{Gibbs}, \gamma_{up}, \gamma_{conj}$ as a function of δ for $V = 0$ and $V = J$. Figure 13 shows numerical results for $\log(r_s(\mathcal{L}_\nu))$. We can make the same observations and comments as in Sect. A.1.3.

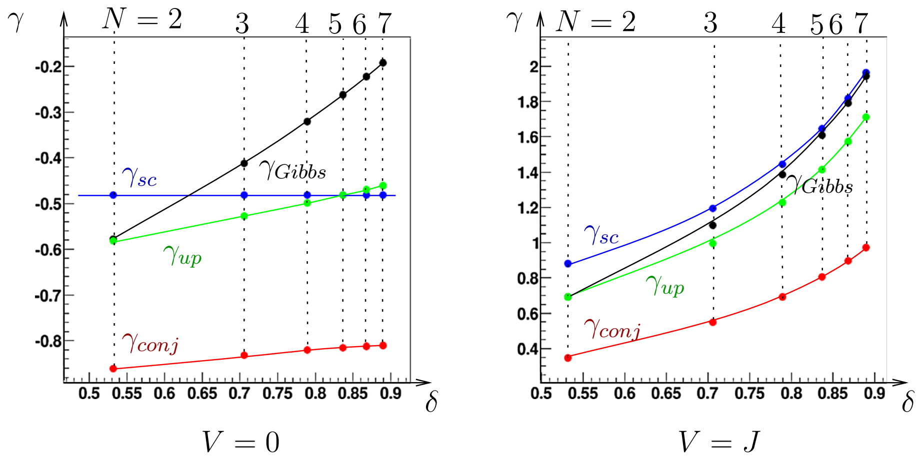

Fig. 12. Plot of various estimates $\gamma_{\mathrm{sc}}, \gamma_{\mathrm{Gibbs}}, \gamma_{\mathrm{up}}, \gamma_{conj}$ defined in Sect. 3.1 for the truncated Gauss map as a function of $\delta = \dim_H(K) \in]0, 1[$ and for $N = 2, 3 \ldots 7$ branches. We put some tiny lines in color between the dots to help the reading (color figure online)

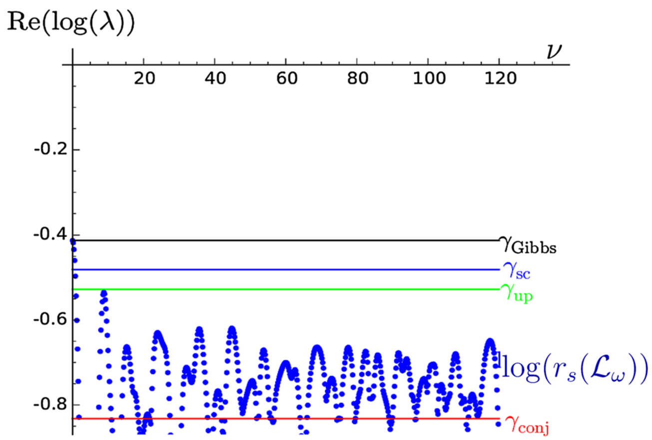

Fig. 13. Numerical values of $\log(r_s(\mathcal{L}_\nu))$ (blue points) as a function of ν for the truncated Gauss map model with $N = 3$ branches and $V = 0$

B. Topological Pressure

B.1. Definition and basic properties. We use the notations introduced in Sect. 4.2. For a given admissible word $w_{0,n}$ of length $n + 1$, let $w \in \mathcal{W}$ be an arbitrary extension of $w_{0,n}$. Let $x_w := S(w_-) \in K$ according to Definition 4.7. For a function $g \in C(I; \mathbb{R})$, we define $g_{w_{0,n}}(x_w) := \sum_{k=1}^{n} g\left(\phi_{w_{0,k}}(x_w)\right)$ its Birkhoff sum. Note that $g_{w_{0,n}}(x_w)$ is not completely determined by $w_{0,n}$ but depends also on its extension, to a bi-infinite word $w \in \mathcal{W}$. However this dependence is well controllable for Lipschitz functions:

Lemma B.1. *If $g \in C(I; \mathbb{R})$ is Lipschitz, then there is a constant C such that for any $n \in \mathbb{N}$ any $w_{0,n}$ and two arbitrary points $x, y \in I_{w_0}$ we have*

$$\left|g_{w_{0,n}}(x) - g_{w_{0,n}}(y)\right| \leq C.$$

In particular for two arbitrary extensions $w, w' \in \mathcal{W}$ of $w_{0,n}$ we have

$$\left| g_{w_{0,n}}(x_w) - g_{w_{0,n}}(x_{w'}) \right| \leq C.$$

Proof. The statement follows directly from a geometric series argument using the fact that $\phi_{i,j}$ are uniformly contracting and that g is Lipschitz (see e.g. [Fal97, Proposition 4.1])

Definition B.2. The **topological pressure** of a function $g \in C(I; \mathbb{R})$ which is Lipschitz continuous is defined as

$$\Pr(g) := \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{w_{0,n}} e^{g_{w_{0,n}}(x_w)} \right). \tag{159}$$

Equivalently

$$\sum_{w_{0,n}} e^{g_{w_{0,n}}(x_w)} = e^{n(\Pr(g)+R(n))}, \quad R(n) \underset{n \to \infty}{\longrightarrow} 0\,. \tag{160}$$

Remark B.3. Lemma B.1 assures that $\Pr(g)$ is independent on the extensions of the words. The fact, that the limit $n \to \infty$ exists can be seen as follows: If we set $a_n := \log\left(\sum_{w_{0,n}} e^{g_{w_{0,n}}(x_w)}\right)$, then using Lemma B.1 we deduce that there is a constant $c > 0$ such that $a_{k+m} \geq a_k + a_m - c$. Consequently $\tilde{a}_k = a_k - c$ is a superadditive sequence (i.e. $\tilde{a}_{k+m} \geq \tilde{a}_k + \tilde{a}_m$) thus the limit $\lim_{n \to \infty} \frac{\tilde{a}_n}{n} = \lim_{n \to \infty} \frac{a_n}{n}$ exists in $\mathbb{R} \cup \{\infty\}$ from Fekete's Lemma. The fact that the limit is finite is deduced from the crude bound $\sum_{w_{0,n}} e^{g_{w_{0,n}}(x_w)} \leq N^n e^{n \sup_I g}$.

Remark B.4. The expression of $\Pr(g)$, Eq. (159) is similar to the Helmholtz free energy in statistical physics.

A particular useful example of a topological pressure is with the choice of function $g = -\beta J$ where $\beta \in \mathbb{R}$ and J is the unstable Jacobian (24):

$$P(\beta) := \Pr(-\beta J) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{w_{0,n}} e^{-\beta J_{w_{0,n}}(x_w)} \right). \tag{161}$$

The Bowen formula [Fal97, p.77] gives the **Hausdorff dimension** $\delta = \dim_H K \in [0, 1]$ of the trapped set K, (5), as the unique solution of

$$P(\delta) = 0. \tag{162}$$

The **topological entropy** counts the exponential rate of number of trajectories with respect to time n:

$$h_{top} := P(0) = \lim_{n \to \infty} \frac{1}{n} \log\left(\sharp \left\{ w_{0,n} \text{ admissible} \right\}\right) \tag{163}$$

B.2. Distribution of time averages of f weighted by g. The theory of large deviations has originally been developed in the context of stochastic processes and has later been adapted for hyperbolic dynamical systems (see e.g. [You90, Kif92, Kif94]). In this section we will shortly collect a few of these results in the context of our systems and give self contained proofs for the sake of completeness.

Let $f, g \in C(I, \mathbb{R})$ be two functions. For a given $n \geq 1$, we use the function g to define a probability measure p_g on the set of admissible words (or trajectories) $w_{0,n}$ with a given length $n+1$:

$$p_g\left(w_{0,n}\right) := \frac{1}{Z_n(g)} e^{g_{w_{0,n}}(x_w)} \tag{164}$$

where $Z_n(g) := \sum_{w_{0,n}} e^{g_{w_{0,n}}(x_w)}$ is the normalization factor (called "partition function" in physics). We are interested in the distribution of time Birkhoff averages of the function f for large time n, namely the values $\left(\frac{1}{n} f_{w_{0,n}}(x_w)\right)_{w_{0,n}}$ where each value $\frac{1}{n} f_{w_{0,n}}(x_w)$ is weighted by the probability $p_g\left(w_{0,n}\right)$. Let

$$f_{min} := \lim_{n\to\infty} \inf_{w_{0,n}} \left(\frac{1}{n} f_{w_{0,n}}(x_w)\right), \quad f_{max} := \lim_{n\to\infty} \sup_{w_{0,n}} \left(\frac{1}{n} f_{w_{0,n}}(x_w)\right)$$

be the limit values of the distribution. The average of this distribution is

$$\langle f\rangle_{n,g} := \sum_{w_{0,n}} p_g\left(w_{0,n}\right)\left(\frac{1}{n} f_{w_{0,n}}(x_w)\right),$$

and its variance is

$$\mathrm{Var}_{n,g}(f) := \sum_{w_{0,n}} p_g\left(w_{0,n}\right)\left(\left(\frac{1}{n} f_{w_{0,n}}(x_w)\right) - \langle f\rangle_{n,g}\right)^2.$$

To express some results concerning this distribution, let us introduce the function

$$u : \beta \in \mathbb{R} \to u(\beta) := \Pr(g + \beta f) \in \mathbb{R} \tag{165}$$

Proposition B.5. *The function u is convex. We have*

$$\lim_{n\to\infty} \langle f\rangle_{n,g} = \left(\frac{du}{d\beta}\right)(0), \quad \lim_{n\to\infty} n\mathrm{Var}_{n,g}(f) = \left(\frac{d^2u}{d\beta^2}\right)(0).$$

We also have

$$f_{min} = \lim_{\beta\to-\infty}\left(\frac{du}{d\beta}\right)(\beta), \quad f_{max} = \lim_{\beta\to+\infty}\left(\frac{du}{d\beta}\right)(\beta).$$

Proof. Write $S_n(\beta) := \sum_{w_{0,n}} e^{g_{w_{0,n}}(x_w)+\beta f_{w_{0,n}}(x_w)}$ and $u_n(\beta) := \frac{1}{n}\log S_n(\beta)$. We have $\left(\frac{du_n}{d\beta}\right)(0) = \frac{1}{n}\frac{S_n'(0)}{S_n(0)} = \langle f\rangle_{n,g}$ and $\left(\frac{d^2u_n}{d\beta^2}\right)(0) = \frac{1}{n}\left(\frac{S_n''(0)}{S_n(0)} - \left(\frac{S_n'(0)}{S_n(0)}\right)^2\right) = n\mathrm{Var}_{n,g}(f)$. We deduce that $\left(\frac{d^2u}{d\beta^2}\right)(0) \geq 0$. We can replace g by $g+\beta f$ and deduce that $\left(\frac{d^2u}{d\beta^2}\right)(\beta) \geq 0$. So u is convex.

We will now consider "large deviations" of the distribution. Note that the variance of the distribution is of order $1/n$ so being at a distance $\asymp 1$ from the expectation value is already a "large deviation". Thus we consider for an interval $\mathcal{I} \subset]f_{min}, f_{max}[$ the quantity

$$P(n, \mathcal{I}) := \sum_{w_{0,n} \text{ s.t. } \frac{1}{n} f_{w_{0,n}} \in \mathcal{I}} p_g\left(w_{0,n}\right)$$

which represents the probability that $\left(\frac{1}{n} f_{w_{0,n}}\right) \in \mathcal{I}$. In particular for $f_{min} < t < f_{max}$ let

$$\begin{aligned}\Omega(t) &:= \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log\left(P\left(n, [t-\varepsilon, t+\varepsilon[\right)\right) \quad \in \mathbb{R} \cup \{-\infty\}. \\ &\underset{(164)}{=} -\Pr(g) + \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log\left(\sum_{w_{0,n} \text{ s.t. } \frac{1}{n} f_{w_{0,n}} \in [t-\varepsilon, t+\varepsilon[} e^{g_{w_{0,n}}(x_w)}\right) \end{aligned} \tag{166}$$

be the exponential rate of the probability as $n \to \infty$ for a small interval around t. In the last expression, the limit $n \to \infty$ exists from a superadditivity argument analogous to the argument given in Remark B.3 above. The limit $\varepsilon \to 0$ exists because one obtains a monotonously decreasing sequence.

Note that if f is cohomologous to a constant c (i.e. $f = c + \eta - \eta \circ \phi^{-1}$ with some function η), then there is another constant C such that for any word $w_{0,n}$ we have $\left|f_{w_{0,n}} - nc\right| \leq C$. In particular the complete distribution of $f_{w_{0,n}}$ is contained in the interval $[c - C/n, c + C/n]$, so the question of studying large deviations becomes trivial in this case. We therefore assume from now on, that f is not cohomologous to a constant, which implies, that the pressure function $u(\beta)$ is strictly convex.

Proposition B.6. ***"Large deviations"***. *Let $t \in \mathbb{R}$ and $\beta(t)$ be defined by*

$$t = \frac{d}{d\beta} \Pr(g + \beta f)_{/\beta = \beta(t)} = \frac{du}{d\beta}_{/\beta = \beta(t)},$$

and

$$v(t) := \beta(t) \cdot t - \Pr(g + \beta(t) f)$$

be the Legendre transform [Arn76, p.61] *of the function u, Eq.* (165). *Then for $f_{min} < t < f_{max}$ we have*

$$\Omega(t) = -\Pr(g) - v(t). \tag{167}$$

Remark B.7. We have

$$\left(\frac{dv}{dt}\right)(t) = \beta(t). \tag{168}$$

The functions u and v are convex. We deduce:

Corollary B.8. *Let* $t_0 \in]f_{min}, f_{max}[$ *such that* $\left(\frac{dv}{dt}\right)(t_0) = \beta(t_0) = 0$. *For any interval* $\mathcal{I} = [t_a, t_b]$ *with* $f_{min} < t_a < t_b < f_{max}$ *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log(P(n, \mathcal{I})) = -\Pr(g) + \sup_{t\in\mathcal{I}}(-v(t))$$

equivalently

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{w_{0,n}\ s.t.\ \frac{1}{n} f_{w_{0,n}} \in \mathcal{I}} e^{g_{w_{0,n}}(x_w)}\right) = \sup_{t\in\mathcal{I}}(-v(t)) \tag{169}$$

with

$$\sup_{t\in\mathcal{I}}(-v(t)) = \begin{cases} -v(t_0) & \text{if } t_0 \in [t_a, t_b] \\ -v(t_b) & \text{if } t_0 \geq t_b \\ -v(t_a) & \text{if } t_0 \leq t_a \end{cases} \tag{170}$$

Proof of Proposition B.6. We are grateful to Mark Pollicott and Richard Sharp for explaining Proposition B.6 and Corollary B.8 to us. Based on ideas from Kifer [Kif92, Kif94] these kind of formulas can be derived from the work of Pollicott and Sharp [PS96,Pol95,Sha92] using the variational approach to the pressure function. In the sequel we provide a self-contained proof, which fits into the periodic orbit definition of the topological pressure which we use in this article.

For any two functions $f, g \in C(I, \mathbb{R})$, for any $t \in \mathbb{R}$ and any $\varepsilon > 0$ let us define the following quantity

$$K_{g,\varepsilon}(t) := \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{w_{0,n}\ \text{s.t.}\ \frac{1}{n} f_{w_{0,n}} \in [t-\varepsilon, t+\varepsilon[} e^{g_{w_{0,n}}(x_w)}\right)$$

Recall that we denoted by N the number of letters, so we get the very crude estimate $-\infty \leq K_{g,\varepsilon}(t) \leq \log N + \cdot \max_{x\in I} g$. We also deduce from that fact, that $e^{g_{w_{0,n}}} > 0$ and the monotonicity of the logarithm, that for any fixed $t \in \mathbb{R}$ and for $\varepsilon \to 0$ the expression $K_{g,\varepsilon}(t)$ is monotonously decreasing. Thus we can define

$$K_g(t) := \lim_{\varepsilon\to 0} K_{g,\varepsilon}(t) \in \mathbb{R} \cup -\infty.$$

Notice that $\Omega(t) \underset{(166)}{=} -\Pr(g) + K_g(t)$. In a first step let us show the following Lemma

Lemma B.9. *The function* $t \to K_g(t)$ *is an upper semi-continuous concave function.*

Proof. The upper semi-continuity follows easily from the definition of K_g: for a given t_0 and $\varepsilon > 0$ take $\delta > 0$ such that $0 \leq K_{g,\delta}(t_0) - K_g(t_0) \leq \varepsilon$. Then for any t such that $|t - t_0| < \delta$ we get, that $K_g(t) \leq K_{g,\delta}(t_0) \leq K_g(t_0) + \varepsilon$. For every $\varepsilon > 0$, $K_{g,\varepsilon}$ is midpoint concave because for any $t_1, t_2 \in [f_{min}, f_{max}]$ we have

$$\begin{aligned}
K_{g,\varepsilon}\left(\frac{t_1+t_2}{2}\right) &:= \lim_{n\to\infty}\frac{1}{2n}\log\left(\sum_{w_{0,2n}\text{ s.t. }\frac{1}{2n}f_{w_{0,2n}}\in[\frac{t_1+t_2}{2}-\varepsilon,\frac{t_1+t_2}{2}+\varepsilon[} e^{g_{w_{0,2n}}}\right)\\
&\geq \lim_{n\to\infty}\frac{1}{2n}\log\left(\sum_{w_{0,2n}\text{ s.t. }\frac{1}{n}f_{w_{0,n}}\in[t_1-\varepsilon,t_1+\varepsilon[\text{ and }\frac{1}{n}f_{w_{n,2n}}\in[t_2-\varepsilon,t_2+\varepsilon[} e^{g_{w_{0,n}}+g_{w_{n,2n}}}\right)\\
&= \lim_{n\to\infty}\frac{1}{2n}\log\left(\left(\sum_{w_{0,n}\text{ s.t. }\frac{1}{n}f_{w_{0,n}}\in[t_1-\varepsilon,t_1+\varepsilon[} e^{g_{w_{0,n}}}\right)\right.\\
&\qquad\left.\left(\sum_{w_{0,n}\text{ s.t. }\frac{1}{n}f_{w_{0,n}}\in[t_2-\varepsilon,t_2+\varepsilon[} e^{g_{w_{0,n}}}\right)\right)\\
&= \frac{1}{2}\left(K_{g,\varepsilon}(t_1)+K_{g,\varepsilon}(t_2)\right)
\end{aligned}\tag{171}$$

Taking the limit $\varepsilon\to 0$ we deduce that K_g is midpoint concave. As upper semi-continuity implies Lebesgue measurable we deduce that K_g is concave.

Remark B.10. Note that in (171) we crucially use the transitivity of the adjacency matrix as assumed in Definition 2.1. Without this assumption the statement that $K_g(t)$ is concave becomes obviously false: Assume for example the case with $N=2$ intervals and the non-transitive adjacency matrix $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. If now f is piece-wise constant, with $f_{/I_1}\neq f_{/I_2}$then for any word $w_{0,n}$, either $f_{w_{0,n}}=nf_{/I_1}$ or $f_{w_{0,n}}=nf_{/I_2}$ and consequently $K_g(t)\geq 0$ if $t=f_{/I_1}$or $t=f_{/I_2}$ and $K_g(t)=-\infty$ else.

We continue the proof of Proposition B.6. Let us now show that $K_g(t)=-v(t)$. Recall from (168) that $\frac{d}{dt}v(t)=\beta(t)$ hence for any t_0 we have $\frac{d}{dt}\left(\beta(t_0)t-v(t)\right)_{|t=t_0}=0$ or in other words $\beta(t_0)t-v(t)$ has a maximum at $t=t_0$ given by

$$\max_t\left(\beta(t_0)t-v(t)\right)=\beta(t_0)t_0-v(t_0)=\Pr\left(g+\beta(t_0)f\right)=u\left(\beta(t_0)\right).\tag{172}$$

Recall the definition (159) of the topological pressure. For $K\in\mathbb{N}$ and $\Delta_K:=\frac{f_{max}-f_{min}}{K}$ we write

$$\begin{aligned}
&\Pr(g+\beta f)\\
&=\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{k=0}^{K-1}\left(\sum_{w_{0,n}\text{ s.t. }\frac{1}{n}f_{w_{0,n}}\in[f_{min}+k\Delta_K,f_{min}+(k+1)\Delta_K[} e^{g_{w_{0,n}}+\beta f_{w_{0,n}}}\right)\right)\\
&=\max_{k=0,\dots K-1}\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{w_{0,n}\text{ s.t. }\frac{1}{n}f_{w_{0,n}}\in[f_{min}+k\Delta_K,f_{min}+(k+1)\Delta_K[} e^{g_{w_{0,n}}+\beta f_{w_{0,n}}}\right)
\end{aligned}$$

Recalling the definition of $K_{g,\varepsilon}(t)$ above we get for any $k = 0, \ldots, K-1$

$$\left| \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{w_{0,n} \text{ s.t. } \frac{1}{n} f w_{0,n} \in [f_{min}+k\Delta_K, f_{min}+(k+1)\Delta_K[} e^{g w_{0,n} + \beta f w_{0,n}} \right) - \left(K_{g,\Delta_K/2}(f_k) + \beta f_k \right) \right| \\ \leq \beta \Delta_K$$

where $f_k = f_{min} + (k + \frac{1}{2})\Delta_K$. Taking the limit $K \to \infty$, we get

$$u(\beta) = \Pr(g + \beta f) = \max_t \left(\beta t + K_g(t) \right)$$

which is the same expression that we have obtained for $-v(t)$ in (172). As we have shown that K_g is upper semi continuous and concave, the Fenchel-Moreau theorem implies that $K_g = -v$. We have finished the proof of Proposition B.6.

C. Discussion About $\gamma_{\text{asympt.}}$ in Hyperbolic Dynamics

C.1. Motivation to study $\gamma_{\text{asympt.}}$. Let us consider the case of an Anosov flow $\phi^t = e^{tX}$, $t \in \mathbb{R}$ (also called uniformly hyperbolic flow) generated by a **Anosov vector field** X on a closed manifold M. A typical example is the geodesic vector field X associated to a Riemannian manifold $(\mathcal{M}, g)$ with (variable) negative curvature: X is the Hamiltonian vector field on $M = T_1^*\mathcal{M}$ (the unit cotangent bundle). This example is special because the flow preserves the canonical Liouville contact one form α on M. More generally Anosov flows that preserve a contact one form are called "contact Anosov flows". We introduce an arbitrary smooth function $V \in C^\infty(M; \mathbb{R})$ called the **potential function** and consider the operator

$$A := -X + V.$$

A has intrinsic discrete spectrum (of finite multiplicity) in certain anisotropic Sobolev spaces $\mathcal{H}(M)$ [BL07,FS11,FT17] and the set of eigenvalues $(z_j)_j \subset \mathbb{C}$ of A are called the **Ruelle–Pollicott resonances of** A **for positive time** $t \geq 0$. The operator A is the generator of $\mathcal{L}^t := e^{tA}$, $t \geq 0$, called the transfer operator giving transport of functions $u \in C^\infty(M)$ by

$$\mathcal{L}^t u = e^{tA} u = e^{V_{[-t,0]}} \cdot (u \circ \phi_{-t}) \tag{173}$$

with $V_{[-t,0]} := \int_{-t}^0 V \circ \phi^s ds$.

We define

$$\gamma_{\text{asympt.}} := \limsup_{\nu\to\infty} \sup_j \left\{ \operatorname{Re}(z_j),\ \text{s.t.}\ \left|\operatorname{Im}(z_j)\right| \geq \nu \right\} \tag{174}$$

i.e. $\gamma_{\text{asympt.}}$ is such that for any $\epsilon > 0$ there are only finitely many Ruelle–Pollicott resonances on the right of the line $\operatorname{Re}(z) = \gamma_{\text{asympt.}} + \epsilon$. To express the importance of the quantity $\gamma_{\text{asympt.}}$, we will assume the following two properties about the spectrum of A. We will see many examples in Sect. C.2 where these assumptions are satisfied.

Assumption C.1. We will assume

1. The Ruelle–Pollicott spectrum of the operator A has a single and simple dominant real eigenvalue γ_{Gibbs}[17].
2. "Uniform control of the norm of the resolvent": there exists $\epsilon > 0$, $\nu > 0$, $C > 0$ such that $\gamma_{\text{asympt.}} + \epsilon < \gamma_{\text{Gibbs}}$ and

$$\forall z \in \mathbb{C} \text{ s.t. } \operatorname{Re}(z) \geq \gamma_{\text{asympt.}} + \epsilon, |\operatorname{Im}(z)| \geq \nu, \quad \left\| (z - A)^{-1} \right\|_{\mathcal{H}(M)} \leq C. \quad (175)$$

Equivalently to (175) one has [EN99]

$$\exists C, \forall t \geq 0, \quad \left\| \mathcal{L}^t - \sum_{j \text{ s.t } \operatorname{Re}(z_j) \geq \gamma_{\text{asympt.}} + \epsilon} \mathcal{L}^t \Pi_j \right\|_{\mathcal{H}(M)} \leq C e^{(\gamma_{\text{asympt.}} + \epsilon) t} \quad (176)$$

where Π_j is the spectral projector of finite rank associated to z_j. If the eigenvalue z_j is simple then $\mathcal{L}^t \Pi_j = e^{z_j t} \Pi_j$. The sum in (176) is finite. In particular

$$\exists \epsilon > 0, \exists C, \forall t \geq 0, \quad \left\| \mathcal{L}^t - e^{\gamma_{\text{Gibbs}} t} \Pi_{\text{Gibbs}} \right\|_{\mathcal{H}(M)} \leq C e^{(\gamma_{\text{Gibbs}} - \epsilon) t} \quad (177)$$

From the construction of $\mathcal{H}(M)$, one has $C^\infty(M) \subset \mathcal{H}(M) \subset \mathcal{D}'(M)$. We define the dual space $\mathcal{H}'(M)$ by

$$\mathcal{H}'(M) := \left\{ u \in \mathcal{D}'(M), \text{ s.t. } v \in \mathcal{H}(M) \rightarrow \langle u, v \rangle_{L^2(M;dx)} \in \mathbb{C} \text{ is bounded} \right\},$$

where dx is an arbitrary smooth volume on M (in case of contact Anosov flow dx is inherited from the contact structure). Equation (176) implies some expansions of time correlation of functions (as in [Tsu10, Corollary 1.2][NZ15, Corollary 5]):

$$\forall u \in \mathcal{H}'(M), v \in \mathcal{H}(M), \quad \langle u, \mathcal{L}^t v \rangle_{L^2(M,dx)} = \sum_{j \text{ s.t.} \operatorname{Re}(z_j) \geq \gamma_{\text{asympt.}} + \epsilon} \langle u, \mathcal{L}^t \Pi_j v \rangle_{L^2} + O\left(e^{(\gamma_{\text{asympt.}} + \epsilon) t} \right). \quad (178)$$

C.1.1. Gibbs measure.

Remark C.2. Equation (178) shows that Ruelle–Pollicott resonances describe the correlation functions w.r.t. Lebesgue measure dx for the dynamics weighted with the potential V. We will see later in Corollary C.4 that the same spectrum describes the correlation functions w.r.t. a Gibbs measure defined from V but for the pure flow dynamics, i.e. without potential V.

The Atiyah-Bott flat trace formula [AB67,Gui77] gives that

$$\gamma_{\text{Gibbs}} = \Pr(V - J) \in \mathbb{R}$$

where[18]

$$J := -\operatorname{div} X /_{E_s} > 0.$$

[17] i.e. other eigenvalues $z_j \in \mathbb{C}$ satisfy $\operatorname{Re}(z_j) < \gamma_{\text{Gibbs}}$.

[18] The choice $A = X + V$ would have give instead $\gamma_{\text{Gibbs}} = \Pr(V - \operatorname{div} X /_{E_u})$.

$\mathrm{div}X/_{E_s} < 0$ is the expansion rate along the stable direction E_u. We have $\mathrm{div}X/_{E_s} + \mathrm{div}X/_{E_u} = \mathrm{div}X$ hence for a volume preserving flow, $\mathrm{div}X = 0$, one has $J := -\mathrm{div}X/_{E_s} = \mathrm{div}X/_{E_u}$. $\mathrm{Pr}(\varphi)$ is the topological pressure of a function $\varphi \in C(M)$ and is defined for flows using a sum over periodic orbits γ as follows:

$$\mathrm{Pr}(\varphi) := \lim_{t\to\infty} \frac{1}{t} \log \left(\sum_{p.o.\gamma \text{ s.t. } |\gamma|\in[t,t+1]} e^{\int_\gamma \varphi} \right). \tag{179}$$

We denote Π_{Gibbs} the rank one spectral projector associated to the eigenvalue γ_{Gibbs}. It defines the so called "**Gibbs equilibrium measure**"[19] associated to the potential V by

$$\mu_{\mathrm{Gibbs}} : \varphi \in C^\infty(M) \to \mathrm{Tr}\left(\mathcal{M}_\varphi \Pi_{\mathrm{Gibbs}}\right) \in \mathbb{R}$$

where $\mathcal{M}_\varphi u = \varphi u$ denotes the multiplication operator by a function $\varphi \in C^\infty(M)$. $\mathcal{M}_\varphi : \mathcal{H}(M) \to \mathcal{H}(M)$ is a bounded operator. The operator $\mathcal{M}_\varphi \Pi_{\mathrm{Gibbs}}$ is finite rank hence trace class in $\mathcal{H}(M)$. The Gibbs measure μ_{Gibbs} has the following properties:

Lemma C.3. ***"Invariance of Gibbs measure under the flow"***. *We have*

$$\forall \varphi \in C^\infty(M), \forall t \geq 0, \quad \mu_{\mathrm{Gibbs}}\left(\varphi \circ \phi^{-t}\right) = \mu_{\mathrm{Gibbs}}(\varphi).$$

Proof. Let us write $\mathcal{L}_0^t = e^{-tX}$ and $\mathcal{M}_\varphi u = \varphi u$ the multiplication operator by φ. We have the relations $\mathcal{L}^t \Pi_{\mathrm{Gibbs}} = \Pi_{\mathrm{Gibbs}} \mathcal{L}^t = e^{t\gamma_{\mathrm{Gibbs}}} \Pi_{\mathrm{Gibbs}}$, $\mathcal{L}^t = \mathcal{M}_{e^{V_{[-t,0]}}} \mathcal{L}_0^t$, $\mathcal{M}_u \mathcal{M}_v = \mathcal{M}_v \mathcal{M}_u$, $\mathcal{L}^t \mathcal{M}_u = \mathcal{M}_{\mathcal{L}_0^t u} \mathcal{L}^t$ and circularity of $\mathrm{Tr}(.)$ and deduce

$$\begin{aligned}
\mu_{\mathrm{Gibbs}}\left(\varphi \circ \phi^{-t}\right) &= \mathrm{Tr}\left(\mathcal{M}_{\mathcal{L}_0^t \varphi} \Pi_{\mathrm{Gibbs}}\right) = e^{-t\gamma_{\mathrm{Gibbs}}} \mathrm{Tr}\left(\mathcal{M}_{\mathcal{L}_0^t \varphi} \mathcal{L}^t \Pi_{\mathrm{Gibbs}}\right) \\
&= e^{-t\gamma_{\mathrm{Gibbs}}} \mathrm{Tr}\left(\mathcal{L}^t \mathcal{M}_\varphi \Pi_{\mathrm{Gibbs}}\right) = e^{-t\gamma_{\mathrm{Gibbs}}} \mathrm{Tr}\left(\mathcal{M}_\varphi \Pi_{\mathrm{Gibbs}} \mathcal{L}^t\right) \\
&= \mathrm{Tr}\left(\mathcal{M}_\varphi \Pi_{\mathrm{Gibbs}}\right) \\
&= \mu_{\mathrm{Gibbs}}(\varphi).
\end{aligned}$$

The expansion (178) implies some expansion for correlation functions expressed with the Gibbs measure (that is more usual in dynamical systems theory) as follows.

[19] μ_{Gibbs} is a **positive measure** (i.e. distribution of order 0) because of the following argument. Let $\varphi \in C^\infty\left(M; \mathbb{R}^+\right)$. Let us denote δ_x the Dirac measure at $x \in M$. The Atiyah-Bott flat trace of an operator A is $\mathrm{Tr}^\flat(A) := \int_M \langle \delta_x, A\delta_x \rangle dx$, see [Gui77]. The Schwartz kernel of the operator $\mathcal{L}^t$ is positive hence for any $t \geq 0$,

$$\begin{aligned}
\mathrm{Tr}^\flat\left(\mathcal{M}_\varphi e^{-\gamma_{\mathrm{Gibbs}} t} \mathcal{L}^t\right) &:= \int_M \varphi(x) \langle \delta_x, e^{-\gamma_{\mathrm{Gibbs}} t} \mathcal{L}^t \delta_x \rangle dx \leq |\varphi|_{C^0} \int_M \langle \delta_x, e^{-\gamma_{\mathrm{Gibbs}} t} \mathcal{L}^t \delta_x \rangle dx \\
&= |\varphi|_{C^0} \mathrm{Tr}^\flat\left(e^{-\gamma_{\mathrm{Gibbs}} t} \mathcal{L}^t\right).
\end{aligned}$$

We make $t \to +\infty$. Using (177) and additional arguments that can be found in [FT16, Appendix B] one obtains

$$\mu_{\mathrm{Gibbs}}(\varphi) \leq |\varphi|_{C^0} \mathrm{Tr}\left(\Pi_{\mathrm{Gibbs}}\right) = |\varphi|_{C^0}.$$

Corollary C.4. ***"Decay of correlations for the Gibbs measure"***. *Under Assumption C.1, we have*

$$\forall\varphi_1,\varphi_2\in C^{\infty}(M),\forall t\geq 0,\quad \mu_{\text{Gibbs}}\left(\left(\varphi_1\circ\phi^{-t}\right).\varphi_2\right)=\mu_{\text{Gibbs}}(\varphi_1)\,\mu_{\text{Gibbs}}(\varphi_2)$$
$$+\sum_{j\ s.t\ \mathrm{Re}(z_j)>\gamma_{\text{asympt.}}+\epsilon,\quad z_j\neq\gamma_{\text{Gibbs}}} e^{-t\gamma_{\text{Gibbs}}}\mathrm{Tr}\left(\mathcal{M}_{\varphi_2}\left(\mathcal{L}^t\Pi_j\right)\mathcal{M}_{\varphi_1}\Pi_{\text{Gibbs}}\right)$$
$$+O\left(e^{-\left(\gamma_{\text{Gibbs}}-\left(\gamma_{\text{asympt.}}+\epsilon\right)\right)t}\right)$$

Remark C.5. 1. The quantity $\left(\gamma_{\text{Gibbs}}-\left(\gamma_{\text{asympt.}}+\epsilon\right)\right)$ that governs the exponential decay of the remainder is called the "**asymptotic spectral gap**".

2. We have assumed that γ_{Gibbs} is dominant eigenvalue i.e. $\gamma_{\text{Gibbs}}>\mathrm{Re}\left(z_j\right)$ for $z_j\neq\gamma_{\text{Gibbs}}$. Then the second line decays and one gets **exponential mixing** property for the Gibbs measure:

$$\mu_{\text{Gibbs}}\left(\left(\varphi_1\circ\phi^{-t}\right).\varphi_2\right)=\mu_{\text{Gibbs}}(\varphi_1)\,\mu_{\text{Gibbs}}(\varphi_2)+O\left(e^{-\left(\gamma_{\text{Gibbs}}-\max_j\mathrm{Re}(z_j)\right)t}\right).$$

3. If eigenvalues z_j are simple then $\mathcal{L}^t\Pi_j=e^{z_jt}\Pi_j$ and each term on the second line writes as

$$e^{-t\gamma_{\text{Gibbs}}}\mathrm{Tr}\left(\mathcal{M}_{\varphi_2}\left(\mathcal{L}^t\Pi_j\right)\mathcal{M}_{\varphi_1}\Pi_{\text{Gibbs}}\right)=e^{-t\left(\gamma_{\text{Gibbs}}-z_j\right)}\mathrm{Tr}\left(\mathcal{M}_{\varphi_2}\Pi_j\mathcal{M}_{\varphi_1}\Pi_{\text{Gibbs}}\right).$$

Proof. Recall the relations at the beginning of proof of Lemma C.3. We have

$$\begin{aligned}\mu_{\text{Gibbs}}\left(\left(\varphi_1\circ\phi^{-t}\right).\varphi_2\right)&=\mathrm{Tr}\left(\mathcal{M}_{\mathcal{L}_0^t\varphi_1}\mathcal{M}_{\varphi_2}\Pi_{\text{Gibbs}}\right)\\&=e^{-t\gamma_{\text{Gibbs}}}\mathrm{Tr}\left(\mathcal{M}_{\mathcal{L}_0^t\varphi_1}\mathcal{M}_{\varphi_2}\mathcal{L}^t\Pi_{\text{Gibbs}}\right)\\&=e^{-t\gamma_{\text{Gibbs}}}\mathrm{Tr}\left(\mathcal{M}_{\varphi_2}\mathcal{L}^t\mathcal{M}_{\varphi_1}\Pi_{\text{Gibbs}}\right)\\&=e^{-t\gamma_{\text{Gibbs}}}\sum_{j\ \mathrm{s.t}\ \mathrm{Re}(z_j)>\gamma_{\text{asympt.}}+\epsilon}\mathrm{Tr}\left(\mathcal{M}_{\varphi_2}\left(\mathcal{L}^t\Pi_j\right)\mathcal{M}_{\varphi_1}\Pi_{\text{Gibbs}}\right)\\&\quad+O\left(e^{-\left(\gamma_{\text{Gibbs}}-\left(\gamma_{\text{asympt.}}+\epsilon\right)\right)t}\right).\end{aligned}$$

From the fact that Π_{Gibbs} is a rank one projector we deduce that the first term of the sum is

$$\begin{aligned}e^{-t\gamma_{\text{Gibbs}}}\mathrm{Tr}\left(\mathcal{M}_{\varphi_2}\left(\mathcal{L}^t\Pi_{\text{Gibbs}}\right)\mathcal{M}_{\varphi_1}\Pi_{\text{Gibbs}}\right)&=\mathrm{Tr}\left(\mathcal{M}_{\varphi_2}\Pi_{\text{Gibbs}}\mathcal{M}_{\varphi_1}\Pi_{\text{Gibbs}}\right)\\&=\mu_{\text{Gibbs}}(\varphi_1)\,\mu_{\text{Gibbs}}(\varphi_2).\end{aligned}$$

C.1.2. Special choice $V=J=-\mathrm{div}X/_{E_s}$. In particular, by choosing[20] the potential $V=J$ we get the **topological entropy** h_{top}:

$$\gamma_{\text{Gibbs}}=\mathrm{Pr}(V-J)=\mathrm{Pr}(0)=\lim_{t\to\infty}\frac{1}{t}\log\mathcal{N}(t)=:h_{top}\tag{180}$$

where $\mathcal{N}(t):=\sharp\{\gamma\text{ periodic orbit s.t. }|\gamma|\leq t\}$ counts the periodic orbits of the flow of period less than t. In this case the Gibbs measure is called "**the Bowen Margulis measure of maximal entropy**" $\mu_{\text{Gibbs}}=:\mu_{B.M.}$.

[20] In general J is only Hölder continuous so it requires some special arguments, namely considering the extension of the transfer operator on a Grassmanian bundle [FT15,FT16].

C.1.3. Special choice $V = 0$. Let $\mathbf{1}(x) = 1$ be the constant function 1 on M. One has $X(\mathbf{1}) = 0$. If we choose potential $V = 0$ then $A = -X$, $A(\mathbf{1}) = 0$, $\gamma_{\text{Gibbs}} = 0$ and

$$\Pi_{\text{Gibbs}} = \mathbf{1}\langle\mu|.\rangle_{L^2},$$

with $\mu \in \mathcal{H}'(M)$. The Gibbs measure is $\mu_{\text{Gibbs}} = \mu dx =: \mu_{S.R.B.}$ and is called "**the Sinai–Ruelle–Bowen measure**". If the flow is volume preserving, i.e. $\text{div}_{dx} X = 0$ then $\mu_{S.R.B.} = dx$.

C.2. Known results about $\gamma_{\text{asympt.}}$.

C.2.1. Contact Anosov flows. For contact Anosov flows, it has been shown by C. Liverani [Liv04] that in the case $V = 0$,

$$\exists\epsilon > 0, \quad \gamma_{\text{asympt.}} < \gamma_{\text{Gibbs}} - \epsilon. \tag{181}$$

M. Tsujii [Tsu12] (and [NZ15] for a generalization to other semiclassical operators), has shown an explicit upper bound for $\gamma_{\text{asympt.}}$ (his method works for any smooth potential V):

$$\gamma_{\text{asympt.}} \leq \gamma_{\text{sc}} := \text{tsup}(D) := \lim_{t\to\infty} \sup_{x\in M} \left(\frac{1}{t}\int_0^t D \circ \phi^{-s}(x)\, ds\right) \tag{182}$$

with the so called damping function $D := V - \frac{1}{2}J$ and where the linear functional tsup() called "time-averaged-sup" is defined from the last expression. The proof uses semiclassical analysis with $\nu := |\text{Im}(z)| \to \infty$ being the frequency in the neutral direction. For this we consider the flow ϕ^t lifted on the cotangent bundle $\tilde{\phi}_t : T^*M \to T^*M$. Since ϕ^t preserves the contact one form α, the trapped set $\mathcal{K}$ (i.e. non wandering set) for the lifted flow $\tilde{\phi}_t$ is the line bundle $\mathcal{K} = \mathbb{R}\alpha \subset T^*M$. A crucial property is that $\mathcal{K}\backslash\{0\}$ is a smooth symplectic submanifold of T^*M and transversally the dynamics of $\tilde{\phi}_t$ is hyperbolic. From this and using semiclassical techniques, one deduces (182) and also a band structure of the spectrum [FT13].

In particular for the special choice $V = \frac{1}{2}J = \frac{1}{2}\text{div}X/_{E_u}$ called "**semi-classical potential**" it is shown in [FT16] that

$$\gamma_{asympt} = \gamma_{\text{sc}} = 0.$$

i.e. there is an accumulation of Ruelle resonances on the imaginary axis. In that case $\gamma_{\text{Gibbs}} > 0$.

C.2.2. Anosov flows in dimension 3. M. Tsujii has shown in [Tsu16] that for generic volume preserving Anosov flow in dimension 3 and $V = 0$, there exists $\gamma_{Tsujii} < 0$ such that

$$\gamma_{\text{asympt.}} \leq \gamma_{Tsujii} < \gamma_{\text{Gibbs}} = 0$$

and one has a uniform control of the resolvent on $\text{Re}(z) \geq \gamma_{Tsujii}$ that gives decay of correlations.

M. Tsujii considers in [Tsu17] the case $V = J$ for an expanding semi-flow and gives a bound for $\gamma_{\text{asympt.}}$ that improves previous known results.

C.2.3. Open hyperbolic dynamics. To the authors best knowledge an analog of (182) for open hyperbolic flows (i.e. Axiom A flow) is not known. However in [AFW13] the authors proved an analog of (182) together with resolvent estimates for the $\mathbb{R}$-extensions of IFS which can be considered as a toy model of an Axiom A flow.

The present article concerns open dynamics. Its purpose is to improve the established bounds γ_{sc} and γ_{Gibbs}. We directly work on a model for open dynamical systems such that there is hope, that our methods and results can be useful for the study of Ruelle–Pollicott resonances of open hyperbolic flows, such as Axiom A flows.

C.2.4. Quantum hyperbolic dynamics. In quantum mechanics similar questions concerning the asymptotic spectral gap of an operator arises as follows. On a negative curvature smoothed closed manifold $(\mathcal{M}, g)$, consider the operator[21]

$$P := \begin{pmatrix} 0 & \mathrm{Id} \\ -\Delta & 2iD \end{pmatrix}$$

on $H^1(\mathcal{M}) \oplus L^2(\mathcal{M})$ where Δ is the Laplace Beltrami operator and $D \in C^\infty(\mathcal{M}; \mathbb{R})$ is a smooth function [Sjö00]. P has discrete spectrum $(z_j)_j \subset \mathbb{C}$ that belongs to the band $\mathrm{Im}(z_j) \in [\inf D, \sup D]$ (for $\mathrm{Re} z_j > 0$). One defines

$$\gamma_{\mathrm{asympt.}} := \lim_{\nu \to +\infty} \sup \sup_j \left\{ \mathrm{Im}(z_j),\ \text{s.t.} \mathrm{Re}(z_j) \geq \nu \right\}. \tag{183}$$

G. Lebeau [Leb96] has shown that

$$\gamma_{\mathrm{asympt.}} \leq \gamma_{\mathrm{sc}} := \mathrm{tsup}(D) := \lim_{t \to \infty} \sup_{(x,\xi) \in T_1^*\mathcal{M}} \left(\frac{1}{t} \int_0^t D \circ \phi^{-s}(x, \xi)\, ds \right) \tag{184}$$

where $\phi^s : T_1^*\mathcal{M} \to T_1^*\mathcal{M}$ is the geodesic flow and D is trivially extended to $M = T_1^*\mathcal{M}$ by $D(x, \xi) := D(x)$. The bound (184) is similar to the bound (182).

For "open quantum dynamics" Dyatlov and Zahl have recently established [DZ15] a new bound for the asymptotic spectral gap for resonances of the Laplacian on convex co-compact manifolds of constant negative curvature. Although their model is different, it would be interesting to compare their results methods and concepts with ours.

In particular it has been shown recently that there is an exact relation between Ruelle Pollicott resonances and quantum resonances on convex co-compact hyperbolic surfaces [GHW16]. For these models it has also been shown that $\gamma_{\mathrm{asympt}} < \gamma_{\mathrm{sc}}$ in [BD16].

C.3. Conjecture for γ_{asympt}. In this section we discuss a conjecture for the asymptotic spectral gap $\gamma_{\mathrm{asympt.}}$.

This conjecture is motivated from the expression (38) that appears in the sketch of proof of Theorem 3.3 and that leads us to our result $\gamma_{\mathrm{asympt.}} \leq \gamma_{\mathrm{up}}$.

In (38) we have a sum of complex numbers over pairs of orbits w, w'. This sum has the form $\sum_{w,w'} e^{(V-J)_w + (V-J)_{w'} + i\nu(\tau_w - \tau_{w'})} T_{w',w}$. In this double sum, we are not able to control the phases $e^{i\nu(\tau_w - \tau_{w'})}$ of non diagonal terms so we have considered a time

[21] If we put $\Phi = (\psi, \varphi) \in L^2(\mathcal{M}) \oplus L^2(\mathcal{M})$ then the Schrodinger equation $i\partial_t \Phi = P\Phi$ is equivalent to the "damped wave equation" $\partial_t^2 \psi = \Delta\psi - 2D\partial_t\psi$ with $\varphi = i\partial_t\psi$.

$n \sim 2\frac{\log \nu}{\langle J \rangle}$ for which these non diagonal terms vanish (because $T_{w',w} \sim 0$). Then the last term in (39) gives the remainder $\langle J \rangle /4$ in our result (28).

If we bound all the phases by 1 and consider the limit $n \to \infty$ one obtains the bound $\gamma_{\text{asympt.}} \leq \gamma_{\text{Gibbs}}$. However if one were able to show that phases behave as "random phases" (this could hold generically), then the non diagonal terms in (38) become negligible. Consequently we can make the diagonal approximation for arbitrary long time and if we take time $n = A\frac{\log(\nu)}{\langle J \rangle}$ with $A \gg 1$ arbitrary large then the last term in (39) becomes $\frac{1}{2A} \langle J \rangle \ll 1$ and is negligible. One obtains the conjecture that:

Conjecture C.6. *For a generic system,*

$$\gamma_{\text{asympt.}} = \gamma_{conj} := \frac{1}{2}\text{Pr}\left(2\left(V - J\right)\right). \tag{185}$$

This conjecture can been found in [DP98, p.9]. It makes sense for a general hyperbolic dynamics (Anosov flow, Axiom A flow,...), even for *quantum* systems as those discussed in Sect. C.2.4 for which the conjecture is[22]

$$\gamma_{\text{asympt.}}^{(quantum)} = \gamma_{conj}^{(quantum)} := \frac{1}{2}\text{Pr}\left(2\left(D - \frac{1}{2}J\right)\right) = \frac{1}{2}\text{Pr}\left(2D - J\right)$$

- In particular if we choose the potential $V = J$ (this choice is used for counting periodic orbits) the conjecture is

$$\gamma_{\text{asympt.}} = \gamma_{conj} = \frac{1}{2}\text{Pr}\left(0\right) = \frac{1}{2}h_{top},$$

  where $h_{top} = \text{Pr}\left(0\right)$ is the topological entropy.
- In particular for $D = 0$ we have $\gamma_{conj}^{(quantum)} = \frac{1}{2}\text{Pr}\left(-J\right)$ and for hyperbolic surfaces this gives $\gamma_{conj}^{(quantum)} = \frac{\delta - 1}{2}$ where δ is the Hausdorff dimension of the limit set [Bor07]. This conjecture has been made in [JN12] for convex co-compact hyperbolic surfaces.

Some numerical observations are in favor of this conjecture, e.g. Figs. 10 and 13. With some other numerical observations the value $\frac{1}{2}\text{Pr}\left(2\left(V - J\right)\right)$ describes rather the maximum of the distribution of concentration of eigenvalues γ_{max} and not $\gamma_{\text{asympt.}}$ [LSZ03, figure 2], [BWP+13, figure 4], [Bor14, figure 27], [BW16, Section 5.3]. One could conjecture that both coincide in the semiclassical limit $\nu \to +\infty$ (and for generic hyperbolic systems), i.e. that $\gamma_{max} = \gamma_{\text{asympt.}} = \gamma_{conj}$.

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