# Small perturbations of Dirichlet Laplacians are QUE

Peyresq, May 2016

[CG]: S. Chatterjee and J. Galkowski, arxiv:1603.00597

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**Quantum Ergodicity** theorem (Shnirelman, Zelditch, Colin de Verdière): $M$-compact; $g$-Riemannian metric on $M$; $G^t$ - geodesic flow for $g$. If $G^t$ is *ergodic* (e.g. $M$ has negative sectional curvatures), then "almost all" Laplace eigenfunctions, $\Delta_g \phi_j + \lambda_j \phi_j = 0$ and the corresponding microlocal lifts (Wigner functions) become uniformly distributed as $\lambda \to \infty$: For $a \in C(M)$ (observable depending only on position):

$$\int_M |\phi_j(x)|^2 a(x) dx \to \int_M a(x) dx$$

for *almost all* eigenfunctions $\phi_j$ (except possibly for a subsequence of density zero).

For $A$ a pseudodifferential operator (observable depending on position and momentum)

$$\langle \phi_j, A\phi_j \rangle \to \int_{S^*M} \sigma_A(x,\xi)\, dx d\xi,$$

where $\sigma_A$ is the principal symbol of $A$.
In particular, for $B \subset M$, $\int_B |\phi|^2$, for almost all $\phi_\lambda$,

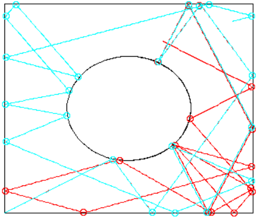
$$\lim_{\lambda\to\infty} \frac{\int_B |\phi_\lambda|^2}{\int_M |\phi_\lambda|^2} = \frac{\mathrm{vol}(B)}{\mathrm{vol}(M)}$$

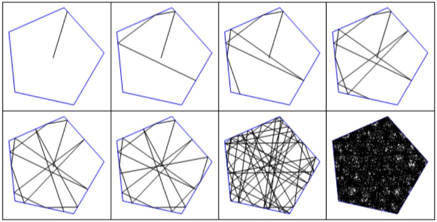
**Quantum Unique Ergodicity:**
There may be *exceptional* sequences of eigenfunctions that *do not* become uniformly distributed ("strong scars"), but these sequences are "thin."
If *all* eigenfunctions become uniformly distributed (no exceptions!), then *quantum unique ergodicity* (or QUE) holds.
Example: $S^1$.
**Conjecture** (Rudnick, Sarnak): QUE holds on negatively-curved manifolds.

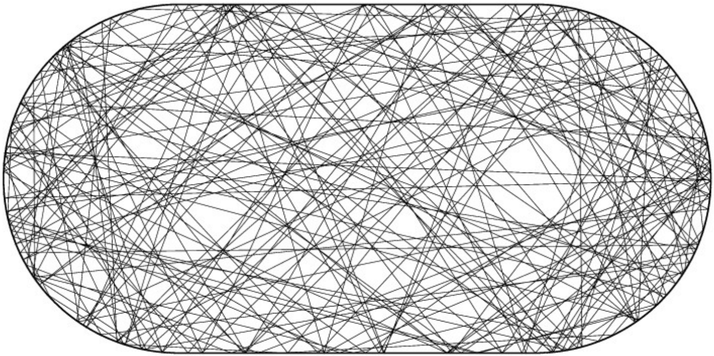
**Theorem** (Lindenstrauss; Soundararajan, Holowinsky): QUE holds for Hecke eigenfunctions on *arithmetic* hyperbolic surfaces. Higher dimensions: Silberman, Venkatesh, Anantharaman et al
Arithmetic hyperbolic surfaces correspond to *arithmetic* groups $\Gamma$: groups whose *commensurator* is dense. Commensurator is the set of all $g$ such that $g\Gamma g^{-1} \cap \Gamma$ has finite index in both $\Gamma$ and $g\Gamma g^{-1}$. Those results make crucial use of ergodic properties of the action of *Hecke operators*.
• QE results were established for *Eisenstein series* (eigenfunctions that correspond to the continuous spectrum of $\Delta$ for $M$ of finite hyperbolic area): QE for general $\Gamma$ was established by Zelditch, Z-Bonthonneau; QUE for $\Gamma = PSL_2(\mathbf{Z})$ by Luo, Sarnak and D.J. Related results were proved for hyperbolic manifolds of dimension $n \geq 3$, for Eisenstein series for Fuchsian groups of the 2nd kind ($M$ has infinite volume) etc: Guillarmou, Naud, Dyatlov et al

**Billiards:**
QE theorem holds for billiards (bounded domains in $\mathbf{R}^2$); proved by Gerard-Leichtnam, Zelditch-Zworski. Geodesic flow is replaced by the *billiard flow:* move along straight line until the boundary; at the boundary, angle of incidence equals angle of reflection.

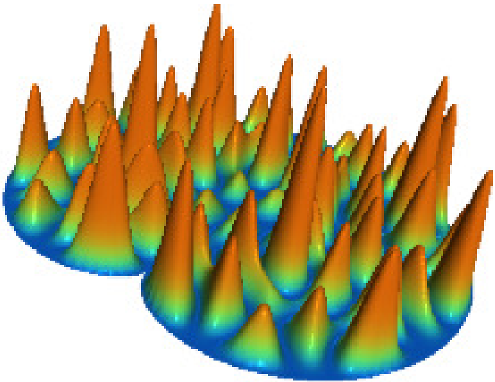
# Ergodic planar billiards: Sinai billiard and Bunimovich stadium



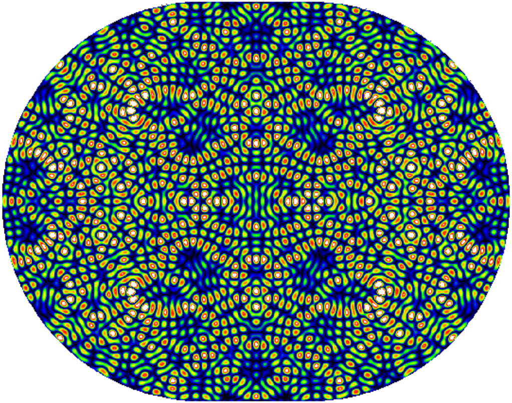




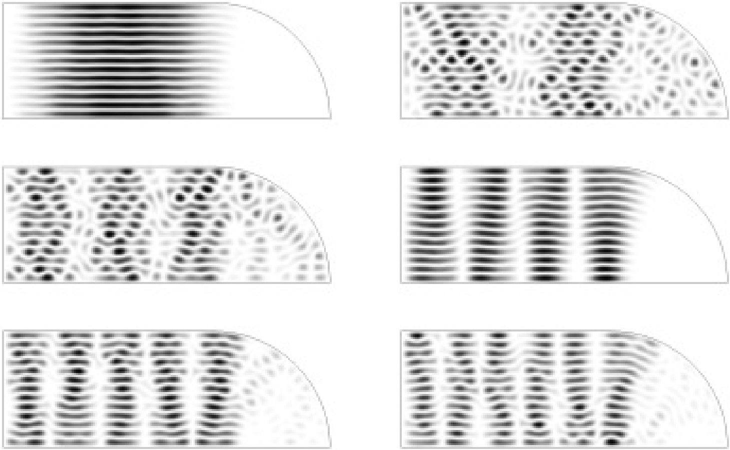
Ergodic eigenfunction on a cardioid billiard.



Ergodic eigenfunction on the stadium billiard:



**Theorem** (Hassell): QUE conjecture *does not* hold for *almost any* (Bunimovich) stadium billiard.
**Exceptions:** “bouncing ball” eigenfunctions, (they have density 0 among all eigenfunctions, so QE still holds).



**Generic** domains/metrics are conjectured to be QUE
**Random spherical harmonics:** QE (Zelditch, 1992), QUE (Vanderkam, 1997)
Shiffman, Zelditch: random sections of high powers of positive line bundles (2003)
Burq, Lebeau: applications to PDE with random initial conditions (2011)
Random bases of high dimensional subspaces of $L^2(M)$ are QE a.s. (Zelditch, 2012); are QUE a.s. (Maples, 2013)
Random bases defined by more general Wigner RM ensembles are QE (R. Chang, 2015)

Riviére, Eswarathasan: QE for random perturbations of semiclassical Schrödinger equation on negatively curved surfaces (2014)
“Loschmit echo” eigenfunctions: Canzani, Jakobson, Toth

**Chatterjee-Galkowski:** QUE for random perturbations of the Dirichlet Laplacian for Euclidean domains with mild regularity properties.
Petrubations for general domains have small $L^2$ norm. For more regular domains, QUE is shown for perturbations of small $L^2 \to H^\gamma$ norm, where $\gamma$ depends on the domain.

**Regularity assumptions** for the domain $\Omega \subset \mathbf{R}^d$:
**Definition:** $\Omega$ is called *regular* if

($a$) $\Omega$ is nonempty, bounded, open, connected; $Vol(\partial\Omega) = 0$.

($b$) For any $x \in \partial\Omega$, $\mathbb{P}^x(\tau_\Omega = 0) = 1$, where $\mathbb{P}^x$ is the law of the Brownian motion started at $x$, and $\tau_\Omega$ is the exit time of the standard BM in $\mathbf{R}^d$ from $\Omega$.

Condition (b) is a sharp condition (in an appropriate sense) for existence of solutions to the Dirichlet problem on $\Omega$. If $\partial\Omega$ is smooth enough, then $\Omega$ is regular.

**Defect measures.**
Let $\{f_n\} \in L^2(\mathbf{R}^d)$. Given $a \in C_c^\infty(S^*\mathbf{R}^d)$, let

$$\widetilde{a}(x,\xi) = a(x, \xi/|\xi|)(1-\chi(\xi))$$

where $\chi \in C_c^\infty(\mathbf{R}^d)$, and $\chi \equiv 1$ in a neighborhood of 0.
Let $\mu_n \in \mathcal{D}'(S^*\mathbf{R}^d)$ be defined by

$$\mu_n(a) = \langle \widetilde{a}(x,D) f_n, f_n \rangle.$$

Let $\mathcal{M}(f_n)$ be the set of limit points of $\{f_n\}$, i.e. the set of measures $\mu$ s.t. for a subsequence $f_{n_k}$ and for all operators $A$ with a symbol $\sigma(A)$ compactly supported in $x$, we have

$$\langle A f_{n_k}, f_{n_k} \rangle \; \to \; \int_{S^*\mathbf{R}^d} \sigma(A) d\mu,$$

the set of defect measures associated to $\{f_n\}$.

$H$ - linear operator from a subspace of $L^2(\overline{\Omega})$ into $L^2(\overline{\Omega})$.

**Definition 1.** $H$ has QUE eigenfunctions if for any sequence of $L^2$ normalized eigenfunctions $\{f_n\}$ of $H$, we have

$$\mathcal{M}(1_{\overline{\Omega}} f_n) = \left\{ \frac{1}{Vol(\Omega)} 1_{\overline{\Omega}} dx d\sigma(\xi) \right\},$$

where $\sigma$ is the normalized surface measure on $S^{d-1}$. In particular,

$$\langle A 1_{\overline{\Omega}} f_n, 1_{\overline{\Omega}} f_n \rangle \ \to \ \frac{1}{Vol(\Omega)} \int_{S^* \mathbf{R}^d} \sigma(A) 1_{\overline{\Omega}} dx d\sigma(\xi).$$

**Definition 2.** $H$ has *uniquely equidistributed* (UD) eigenfunctions if $|f_n(x)|^2 dx \to dx / Vol(\Omega)$ (QUE "on the base.")

**Main results:** Let $-\Delta$ be the Dirichlet Laplacian with the domain $\mathcal{F}_\Delta$ defined below; if $\partial\Omega$ is $C^2$, then $\mathcal{F}_\Delta = H_0^1(\Omega) \cap H^2(\Omega)$.

**Theorem 1.** $\Omega$ - regular domain. $\forall\epsilon > 0$, $\exists S_\epsilon : L^2(\overline{\Omega}) \to L^2(\overline{\Omega})$ such that

(i) $||S_\epsilon||_{L^2 \to L^2} \leq \epsilon$.

(ii) $H = H_\epsilon = -(I + S_\epsilon)\Delta$ is a positive operator on $\mathcal{F}_\Delta$.

(iii) $\exists$ an ONB of $L^2(\overline{\Omega})$ consisting of eigenfunctions of $H$ that belong to $\mathcal{F}_\Delta$ and the corresponding eigenvalues can be ordered as $0 < \lambda_1 \leq \lambda_2 \leq \ldots \to +\infty$.

(iv) $H$ has QUE eigenfunctions.

If $\partial\Omega$ is smooth, then $\forall\gamma < 1$, there exists $S_\epsilon : L^2(\overline{\Omega}) \to H^\gamma(\overline{\Omega})$ with the norm $||S_\epsilon||_{L^2(\overline{\Omega}) \to H^\gamma(\overline{\Omega})} \leq \epsilon$. If, in addition, the set of periodic billiard trajectories has measure 0, then one can choose any $\gamma \leq 1$.

**Theorem 2; local Weyl law for regular domains:** $\Omega$ - regular domain. Let $\Delta u_j + \mu_j u_j = 0$ be a complete ONB of the Dirichlet Laplacian on $\Omega$. Then for an operator $A$ with $\sigma(A)$ supported in a compact subset of $\Omega$ and for any $E > 1$,

$$\sum_{\mu_j \in [\mu, \mu E]} \langle A 1_{\overline{\Omega}} f_n, 1_{\overline{\Omega}} f_n \rangle = \frac{\mu^d}{(2\pi)^d} \int \int_{1 \leq |\xi| \leq E} \sigma(A) 1_{\overline{\Omega}} dx d\xi \, + \, o(\mu^d). \tag{1}$$

Let $\alpha = \alpha(\mu)$ be non-increasing.
**Definition 3:** Domain $\Omega$ is *average QE* (AQE) at scale $\alpha$ if an analogue of (1) holds for *A* from a dense set of operators, and with $[\mu, \mu E]$ replaced by $[\mu, \mu(1 + \alpha(\mu))]$:

$$\sum_{\mu_j \in [\mu, \mu(1+\alpha(\mu))]} \langle A 1_{\overline{\Omega}} f_n, 1_{\overline{\Omega}} f_n \rangle = \tag{2}$$

$$\frac{\mu^d}{(2\pi)^d} \int \int_{1 \le |\xi| \le 1 + \alpha(\mu)} \sigma(A) 1_{\overline{\Omega}} dx d\xi \; + \; o(\alpha(\mu)\mu^d).$$

If $\partial\Omega$ is smooth, then (Duistermaat, Guillemin, Safarov, Vassiliev) $\Omega$ is AQE at scale $\mu^{-\gamma}$, for any $\gamma < 1$; if the set of periodic trajectories has measure 0, then $\Omega$ is AQE at scale $\mu^{-1}$.

The following Theorem 3 implies Theorem 1.
Let $\gamma \in [0, 2]$. $\mathcal{F}_{\Delta}^{\gamma}$ - complex interpolation space $(L^2(\Omega), \mathcal{F}_{\Delta})_{\gamma/2}$.
**Theorem 3:** $\Omega$ - regular domain, AQE at scale $\alpha(\mu) = \mu^{-\gamma}$.
Then an analogue of Theorem 1 holds with $S_{\epsilon} : L^2(\overline{\Omega}) \to \mathcal{F}_{\Delta}^{\gamma}$,
where the corresponding norm satisfies $||S_{\epsilon}|| \leq \epsilon$.
Also, AQE at scale $O(\mu^{-\gamma})$ implies the existence of an ONB of $L^2$-normalized quasi-modes

$$(-\alpha_n^{-2}\Delta - 1)f_n = O_{L^2}(\alpha_n^{-\gamma})$$

that are QUE, Corollary 2.8 in [CG].
On Riemannian manifolds s.t. the measure of the set of closed geodesics is $= 0$, AQE holds at scale $\mu^{-1}$. It follows that there exists and ONB of $o_{L^2}(\alpha_n^{-1})$ quasimodes “on the base” (uniquely equidistributed), Corollary 2.9 in [CG].

Many earlier results: Fix a basis $\{u_n\}$ of eigenfunctions of $\Delta$. Prove QE for the "rotated" basis $\{Uu_n\}$, where

$$U = \oplus_k U_k$$

is a "block diagonal" unitary operator, $\dim \mathrm{Ran} U_k < \infty$ and $\dim \mathrm{Ran} U_k \to \infty$ polynomially in $k$.
Zelditch, VanderKam, Maples - use Haar measure to choose $U_k$. Chang - uses more general (Wigner) measures to construct $U_k$, and uses results of Bourgade and Yau (2013).
Consider the operator $P = -U\Delta U^*$, where $U \approx \mathrm{Id}$. Then $P = -(I + S)\Delta$, where $||S||_{L^2 \to L^2}$ is small. In [CG], the operators $U_k$ are replaced by nearly unitary operators; Hanson-Wright inequality is used in place of LLN. This allows to use smaller spectral windows, show that the perturbation is often regularizing, and prove QUE.

**Sketch of the proof of Theorem 2:** compare the heat trace of $\Delta$ on $\mathbf{R}^d$ with the heat trace on $\Omega$, as in Gerard-Leichtnam, 1993. Key estimate: let $k(t,x,y)$ be the heat kernel for the "free" laplacian, and $k_D(t,x,y)$ the heat kernel for the Dirichlet Laplacian on $\Omega$. Key estimate:

$$|\partial_x^\alpha(k(t,x,y) - k_D(t,x,y))| \leq C_\delta t^{-N_\alpha} e^{-c_\delta/t}, \qquad d(x,\partial\Omega) > \delta. \tag{3}$$

Proof uses results about Brownian motion and works for domains that are only regular.
$\mathbb{E}^x(f(B_t); t < \tau_\Omega) = \int_\Omega p(t,x,y)f(y)dy$, where $f : \Omega \to \mathbf{R}$, $\mathbb{E}^x$ - expectation w.r. to the law of BM started at $x$;
$\tau_\Omega = \{\inf t > 0 : b_t \notin \Omega\}$.
Here $p(t,x,y) = \sum_i e^{-t\mu_i^2/2}\phi_i(x)\phi_i(y)$ - heat kernel for the Dirichlet $\Delta$ in $\Omega$, $-\Delta\phi_i = \mu_i^2\phi_i$.
Let

$$J_{\epsilon,\lambda} := \{i : \mu_i \in [\lambda, \lambda(1+\epsilon)]\}.$$

The following result ([CG], Theorem 4.2) is establishes for $A \in \Psi(\mathbf{R}^d)$ with $\sigma(A)$ compactly supported in $\Omega$; here $\overline{A}$ = average of $\sigma(A)$:

**Theorem 4:** Fix $\epsilon > 0$. Then $J_{\epsilon,\lambda}$ is nonempty for large $\lambda$ and

$$\lim_{\lambda \to \infty} \left| \frac{1}{|J_{\epsilon,\lambda}|} \sum_{i \in J_{\epsilon,\lambda}} \langle (A - \overline{A}) 1_{\overline{\Omega}} \phi_j, 1_{\overline{\Omega}} \phi_j \rangle \right| = 0.$$

Also,

$$\frac{|J_{\epsilon,\lambda}|}{\lambda^d} \to \frac{(1+\epsilon)^d - 1}{(4\pi)^{d/2} \Gamma(d/2+1)}.$$

Theorem 4 implies Theorem 2. Theorem 4 is proved using trace asymptotics (Lemma 4.3 in [CG]) and a Tauberian theorem (Lemma 4.4 in [CG]). In the proof of the Tauberian theorem, the estimate (3) is established using the Markov property of BM.

Next, Hanson-Wright inequality (the version in Rudelson-Vershynin, 2013) is used to show that random (Haar) rotations of small groups of eigenfunctions are QUE; the size of the group depends on the remainder in Weyl's law.
Let $u = (u_1, \dots, u_n)$ - ON set of bounded functions in $L^2(\overline{\Omega})$;
$Q = q_{ij}$ - random (Haar) $n \times n$ orthogonal matrix. Let
$v = (v_1, \dots, v_n)$ be given by

$$v_i(x) = \sum_j q_{ij} u_j(x);$$

*random rotation* of $u$.

**Theorem 5:** Let $A : L^2(\Omega) \to L^2(\Omega)$ be a bounded operator. Then

$$\mathbb{P}\left(\left|\langle Av_i, v_i\rangle - \frac{1}{n}\sum_{i=1}^{n}\langle Au_i, u_i\rangle\right| \geq t\right) \leq$$

$$C_1 \exp[-C_2(||A||)\min\{t^2, t\}n],$$

where $C_1$ only depends on $d$ and $\Omega$, and $C_2(||A||)$ depends on $d, \Omega$ and the operator norm $||A||$.

The proof uses the following version of Hanson-Wright inequality (a la Rudelson-Vershynin). A r.v. $X$ is *sub-Gaussian* iff the norm

$$||X||_{\psi_2} := \sup_{p \geq 1} \frac{(\mathbb{E}|X|^p)^{1/p}}{p^{1/2}}.$$

is finite.

Let $M = m_{ij}$ be $n \times n$ matrix; $X_1, \ldots, X_n$ - independent RV-s with mean 0 and $||X_j||_{\psi_2} \leq K$. Let $R = \sum_{i,j} m_{ij} X_i X_j$. The $\forall t \geq 0$,

$$\mathbb{P}(|R - \mathbb{E}(R)| \geq t) \leq 2 \ \exp\left[-C \min\left\{\frac{t^2}{K^4 ||M||_{HS}^2}, \frac{t}{K^2 ||M||}\right\}\right], \tag{4}$$

where $||M||$ denotes the operator norm, and $||M||_{HS}$ denotes the Hilbert-Schmidt norm of $M$.

**Applying HW:** Let $q_i$ be the $i$-th row of $Q$. Let $z$ be the standard $n$-dimensional Gaussian; then $z/||z||$ is uniformly distributed in $S^{n-1}$ (like $q_i$); and is independent of $||z||$. Let $r_i$ have the same distribution as $||z||$, for all $i$; then $w_{ij} = r_i q_{ij}$ are iid standard Gaussians.
Let $H = (h_{jk}) = \langle Au_j, u_k \rangle$. Let

$$A_i = \langle Av_i, v_i \rangle = \sum_{j,k} q_{ij} q_{ik} h_{jk}.$$

Let $A_i' = r_i^2 A_i = \sum_{j,k} w_{ij} w_{ik} h_{jk}$. The operator $H$ in $u_j$ coordinates is given by $\Pi A \Pi$ where $\Pi$ is the projection of $L^2$ onto span of $\{u_i\}$, so $||H|| \leq ||A||$. Also, $||H||_{HS} \leq ||A||\sqrt{n}$.

Apply HW with $X_i = r_i q_i$, i.e. $(X_i)_j = r_i q_{ij}$, and $M = H$, and $R = A'_i$. Then $K = C \cdot ||A||$, $||M|| \leq ||A||$ and $||M||_{HS} \leq ||A||\sqrt{n}$. Application of HW gives

$$\mathbb{P}(|A'_i - \mathbb{E}(A'_i)| \geq t) \leq 2\exp\left[-C(||A||)\min\{t^2/n, t\}\right],$$

where $C(||A||) = O(\min\{||A||^{-1}, ||A||^{-2}\})$.
Next, HW implies that

$$\mathbb{P}(|r_i^2 - n| \geq t) \leq 2\exp\left[-C\min\{t^2/n, t\}\right],$$

Also note that $|A_i| \leq ||A||_{L^2\to L^2}||v||^2 = ||A||_{L^2\to L^2}$, and also $\mathbb{E}(A'_i) = n \cdot \mathbb{E}(A_i) := nB$, where $B = (\sum_i A_i)/n$.

Combining the above observations, we see that

$$
\begin{aligned}
&\mathbb{P}(|A_i - B| \geq t) \leq \mathbb{P}(|nA_i - A_i'| \geq \frac{nt}{2}) + \mathbb{P}(|A_i' - \mathbb{E}(A_i')| \geq \frac{nt}{2}) \\
&\leq \mathbb{P}(|(r_i^2 - n)A_i| \geq \frac{nt}{2}) + \mathbb{P}(|A_i' - \mathbb{E}(A_i')| \geq \frac{nt}{2}) \\
&\leq \mathbb{P}(|(r_i^2 - n)| \geq \frac{nt}{2K}) + \mathbb{P}(|A_i' - \mathbb{E}(A_i')| \geq \frac{nt}{2}) \\
&\leq C_1 \exp[-C_2(||A||) \min\{t^2, t\} n].
\end{aligned}
$$

QED

**Spectral preliminaries:**
Let $\Psi = (\psi_i)$ be an ONB of $L^2(\overline{\Omega})$. Let $\Lambda = (\lambda_i)_{i \geq 1} \subset \mathbf{R}$. For $s \geq 0$

$$\mathcal{F}^s(\Psi, \Lambda) := \left\{ f \in L^2(\overline{\Omega}) : \sum (1 + |\lambda_i|^2)^{s/2} |\langle f, \psi_i \rangle|^2 < \infty \right\}.$$

For $s < 0$, let $\mathcal{F}^s(\Psi, \Lambda) := (\mathcal{F}^{-s}(\Psi, \Lambda))^*$. Also,

$$T_{\Psi,\Lambda} f = \sum_i \lambda_i \langle f, \psi_i \rangle \psi_i.$$

**Lemma 1:** Let $\Lambda' = (\lambda'_i)$. If $|\lambda_i - \lambda'_i| < \epsilon(1 + |\lambda_i|^2)^{(1-\gamma)/2}$, then the norms $|| \cdot ||_{\mathcal{F}^s(\Psi,\Lambda)}$ and $|| \cdot ||_{\mathcal{F}^s(\Psi,\Lambda')}$ are equivalent, and

$$||T_{\Psi,\Lambda'} - T_{\Psi,\Lambda}||_{\mathcal{F}^s(\Psi,\Lambda) \to \mathcal{F}^{s-1+\gamma}(\Psi,\Lambda)} \leq \epsilon.$$

**Lemma 2:** Assume $|\lambda_i| > c > 0, |\lambda_i| \to \infty$; let $\Gamma = (1/\lambda_i)$. Then $L^2(\overline{\Omega}) \subset \mathcal{F}(\Phi, \Gamma)$, $\mathrm{Ran}\, T_{\Phi,\Gamma} \subset \mathcal{F}(\Phi, \Lambda)$ and $T_{\Phi,\Lambda} T_{\Phi,\Gamma} = I$.

Let $\Phi = \{\phi_i\}$ be a (Dirichlet) ONB of $L^2(\overline{\Omega})$, with eigenvalues $\lambda_i = \mu_i^2$.
**Lemma 3:** Notation as above, $\mathcal{F}(\Phi, \Lambda) = \mathcal{F}_\Delta$ (the domain of the Dirichlet $\Delta$) and $T_{\Phi,\Gamma}\Delta f = -f$.

**Constructing the perturbation:** recall that $\Omega$ is assumed to be AQE at scale $\mu^{-\gamma}$. Let $\bar{\gamma} = \gamma/2$.
Divide the real line into intervals $\mathbf{R}^+ = J_0 \cup J_1 \cup \ldots$ as follows:

$$[0, (1+\epsilon) \cup [(1+\epsilon), (1+\epsilon)^2) \cup \ldots$$

Divide each of the intervals $J_n = [(1+\epsilon)^n, (1+\epsilon)^{n+1})$ into $N_n = \lceil (1+\epsilon)^{n\bar{\gamma}} \rceil$ disjoint intervals, denoted by $\mathrm{I}_{n,1}, \mathrm{I}_{n,2}, \ldots, \mathrm{I}_{n,N_n}$ where $\mathrm{I}_{n,j} = [a_{n,j}, b_{n,j})$ is given by

$$\left[(1+\epsilon)^n \left(1 + \frac{j\epsilon}{N_n}\right), (1+\epsilon)^n \left(1 + \frac{(j+1)\epsilon}{N_n}\right)\right).$$

**Step 1:** Don't move the eigenvalues in $J_0$. For all eigenvalues in $\lambda_i \in \mathrm{I}_{n,j}$, let $\lambda_i' = a_{n,j}$: move them to $a_{n,j}$. Let $\Lambda' = (\lambda_i')$. Lemma 1 $\Longrightarrow \mathcal{F}^s(\Phi, \Lambda') = \mathcal{F}^s(\Phi, \Lambda)$ and $||T_{\Phi,\Lambda'} - T_{\Phi,\Lambda}||_{\mathcal{F}^s \to \mathcal{F}^{s-1+\bar{\gamma}}} \le 2\epsilon$.

**Step 2:** Perform a random rotation of all $\{\phi_i : \lambda_i = a_{n,j}\}$; call the resulting functions $\phi_i'$, they still form a basis $\Phi'$.

**Step 3:** Next, make all the eigenvalues in $J_0$ distinct; decrease them by at most $1/(1-\epsilon)$ if necessary. Also, make all the $\lambda_i'$ distinct again, but keep them all in $\mathrm{I}_{n,j}$ (the interval that contained the corresponding $\lambda_i$-s). Call the corresponding eigenvalues $\lambda_i''$, and their union $\Lambda''$.
Lemma 1 $\Longrightarrow$

$$\mathcal{F}^s(\Phi', \Lambda'') = \mathcal{F}^s(\Phi', \Lambda') = \mathcal{F}^s(\Phi, \Lambda') = \mathcal{F}^s(\Phi, \Lambda)$$

and

$$||T_{\Phi',\Lambda''} - T_{\Phi,\Lambda'}||_{\mathcal{F}^s \to \mathcal{F}^{s-1+\bar{\gamma}}} \le \epsilon, ||T_{\Phi',\Lambda''} - T_{\Phi,\Lambda}||_{\mathcal{F}^s \to \mathcal{F}^{s-1+\bar{\gamma}}} \le 10\epsilon$$

Let $\Gamma = \{\lambda_i^{-1}\}$, and let $G = T_{\Phi,\Gamma}$. Lemma 2 $\Longrightarrow$ $TG = I$. Let $T = T_{\Phi,\Lambda}$, $T'' = T_{\Phi',\Lambda''}$. Let

$$S := (T'' - T)G = T''G - I.$$

Then $S : L^2 \to \mathcal{F}^{\bar{\gamma}}$.

**Proof of Theorem 3:** The operator $S$ satisfies the conclusion of Theorem 3.

First, $||Sf||_{\mathcal{F}^{\bar{\gamma}}} \leq 10\epsilon ||Gf||_{\mathcal{F}} \leq C\epsilon ||f||$. This + Lemma 3 $\Longrightarrow$ $||S|| \leq \epsilon$, part (i) of Theorems 1 and 3.

Next, $-(I + S)\Delta = T''$ - positive on $\mathcal{F}_\Delta$, part (ii) of Theorems 1 and 3.

Part (iii) of Theorems 1 and 3 follows from the fact that $\Phi'$ is still a Dirichlet basis of $L^2(\overline{\Omega})$.

It remains to prove the (most important) QUE part (iv).

Denote by $|\mathrm{I}_{n,j}|$ the number of $\lambda_i \in \mathrm{I}_{n,j}$. By the AQE assumption at scale $\alpha(\mu) = O(\mu^{-\gamma} = \mu^{-2\bar{\gamma}})$,

$$\lim_{n\to\infty} \frac{1}{|\mathrm{I}_{n,j}|} \left| \sum_{i\in \mathrm{I}_{n,j}} \left\langle (A - \overline{\sigma_A}) 1_{\overline{\Omega}} \phi_i, \phi_i \right\rangle \right| = 0. \tag{5}$$

Here

$$\overline{\sigma_A} = \frac{1}{Vol(1 \leq |\xi| \leq 1 + r_+)} \int\int_{1\leq|\xi|\leq 1+r_+} \sigma(A)(x,\xi) 1_{\overline{\Omega}} dx d\xi$$

where $r_+ = \sqrt{\frac{1+(j+1)\epsilon/N_n}{1+j\epsilon/N_n}} = 1 + \frac{\epsilon}{2N_n} + O(\epsilon^2/N_n)$.

Theorem 5 $\implies$ that $\forall t \in (0,1)$,

$$\mathbb{P}\left(\left|\langle A1_{\overline{\Omega}}\phi_i', 1_{\overline{\Omega}}\phi_i'\rangle - \frac{1}{|\mathrm{I}_{n,j}|}\sum_{\lambda_i \in \mathrm{I}_{n,j}} \langle A1_{\overline{\Omega}}\phi_i, 1_{\overline{\Omega}}\phi_i\rangle\right| \geq t\right) \leq$$

$$C_1 \exp[-C_2(||A||)\min\{t^2, t\}|\mathrm{I}_{n,j}|].$$

It follows that

$$\mathbb{P}\left(\max_{i \in \mathrm{I}_{n,j}}\left|\langle A1_{\overline{\Omega}}\phi_i', 1_{\overline{\Omega}}\phi_i'\rangle - \frac{1}{|\mathrm{I}_{n,j}|}\sum_{\lambda_i \in \mathrm{I}_{n,j}} \langle A1_{\overline{\Omega}}\phi_i, 1_{\overline{\Omega}}\phi_i\rangle\right| \geq t\right) \leq$$

$$C_1|\mathrm{I}_{n,j}| \exp[-C_2(||A||)\min\{t^2, t\}|\mathrm{I}_{n,j}|].$$

It follows from Weyl's law that

$$\sum_n \sum_{1 \leq j \leq N_n} C_1|\mathrm{I}_{n,j}| \exp[-C_2(||A||)\min\{t^2, t\}|\mathrm{I}_{n,j}|] < \infty.$$

It follows from Borel-Cantelli Lemma that

$$\mathbb{P}\left(\left|\langle A1_{\overline{\Omega}}\phi'_i, 1_{\overline{\Omega}}\phi'_i\rangle - \frac{1}{|\mathrm{I}_{n,j}|}\sum_{\lambda_i\in\mathrm{I}_{n,j}}\langle A1_{\overline{\Omega}}\phi_i, 1_{\overline{\Omega}}\phi_i\rangle\right| \geq t :\right.$$

$$\left.\lambda_i \in \mathrm{I}_{n,j} \textit{ for inf. many } n, j\right) = 0.$$

Application of (5) show that $\forall\delta > 0$,

$$\mathbb{P}\left(\limsup_{i\to\infty}\left|\langle A1_{\overline{\Omega}}\phi'_i, 1_{\overline{\Omega}}\phi'_i\rangle - \overline{\sigma_A}\right| \geq \delta\right) = 0.$$

Applying that argument for a dense set of operators $A \in C_0(S^*\Omega)$ finishes the proof of Theorem 3, and hence also of Theorem 1.
QED

The existence of $O(\alpha_n)^{-\gamma})$ quasimodes (Corollary 2.8 in [CG]) follows from $-(I+S)\Delta f_n = T'' f_n = \alpha_n^2 f_n$ (where $f_n$ coincides with one of the $\phi'_{n_j}$s) and hence
$||Sf_n|| = ||(T''-T)\phi'_{n_j}|| \leq C\epsilon(1+|\alpha_n|^2)^{-\gamma/2}$.
On closed manifolds, we consider functions in $L_0^2(M)$, orthogonal complement to constants. On that space, the proof of Theorem 1 proceeds as before. Note that $\gamma = 1, \bar{\gamma} = 1/2$ in that case.
The proof of Corollary (2.9) in [CG] uses local Weyl's law with remainder on $M$, and dividing the spectrum into slightly smaller intervals: $\mathrm{I}_n$ is now divided into $N_n := \lceil (1+\epsilon)^{n/2} \rceil \beta_n$, where $\beta_n \to \infty$ slowly enough.