HECKE OPERATORS AND SPECTRAL GAPS ON COMPACT LIE GROUPS.

FRÉDÉRIC NAUD, EXTENDED VERSION OF TALK AT PEYRESQ, MAY 22TH 2016

Abstract. In these notes we review some of the recent progress on questions related to spectral gaps of Hecke operators on compact Lie groups. In particular, we discuss the historical Ruziewicz problem that motivated the discovery of the first examples, followed by the work of Gamburd-Jakobson-Sarnak [5], Bourgain-Gamburd [3],[2] and also the recent paper of Benoist-Saxcé [1].

1. Introduction

Let $\mathbf{G}$ be a compact connected Lie Group whose normalized Haar measure is denoted by m. Let μ be a *probability measure* on $\mathbf{G}$. On the Hilbert space $L^2(\mathbf{G}, dm)$, we define a convolution operator by

$$T_\mu(f)(x) := \int_{\mathbf{G}} f(xg) d\mu(g).$$

We can check easily that $T_\mu : L^2 \to L^2$ is bounded linear with norm 1. The constant function $\mathbf{1}$ is an obvious eigenfunction related to the eigenvalue 1. If in addition μ is symmetric i.e. $\mu(A^{-1}) = \mu(A)$ for all Borel set A, then T_μ is self-adjoint. The main purpose of these notes is the following question. Set

$$L^2_0(\mathbf{G}) := \left\{ f \in L^2(\mathbf{G}) \ : \ \int_{\mathbf{G}} f dm = 0 \right\}.$$

Clearly L^2_0 is a closed T_μ-invariant space. If we have (ρ_{sp} is the spectral radius)

$$\rho_{sp}(T_\mu|_{L^2_0}) < 1,$$

then we say that T_μ has a *spectral gap*. There is an easy case: if μ is symmetric and has a *continuous density with respect to Haar*, then by Ascoli-Arzuela T_μ is a *compact operator* and one can check that 1 is a simple eigenvalue while -1 is not in the spectrum.

Therefore we will focus on non absolutely continuous examples such as

$$\mu_S := \frac{1}{|S|} \sum_{g \in S} \delta_g,$$

where δ_g is the dirac mass and S is a finite symmetric set

$$S = \{\gamma_1, \gamma_2, \dots, \gamma_p, \gamma_1^{-1}, \dots, \gamma_p^{-1}\}.$$

In that case we simply have

$$T_\mu(f)(x) = \frac{1}{|S|} \sum_{g \in S} f(xg),$$

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which is called a "Hecke operator". This operators were born out of various problems among which the Ruziewicz measure problem which is a historical motivation, see below. There is one fundamental observation that we must mention. Hecke operators on torii *can never have a spectral gap*. Indeed, let's have a look at $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$, endowed with the normalized Haar measure. Let

$$S := \{e^{i\theta_1}, \dots, e^{i\theta_p}\},$$

and consider the family of functions f_k, $k \in \mathbb{N} \setminus \{0\}$ with

$$f_k(z) := z^k.$$

Obviously we have $\|f\|_{L^2} = 1$ and $\int f_k dm = 0$, while

$$\|T_\mu(f_k)\|^2_{L^2} = \frac{1}{2\pi p^2} \int_0^{2\pi} \left| \sum_{\ell=1}^p e^{ik(\theta+\theta_\ell)} \right|^2 d\theta = \frac{1}{p^2} \left| \sum_{\ell=1}^p e^{ik\theta_\ell} \right|^2 .$$

By the *Dirichlet Box Principle*, for all $\epsilon > 0$, one can find $k(\epsilon) >> 1$ such that

$$\left| \sum_{\ell=1}^p e^{ik\theta_\ell} \right| \geq p - \epsilon.$$

This shows that $\|T_\mu|_{L^2_0}\| = 1$. More generally, the same idea gives

$$\|T^N_\mu|_{L^2_0}\| = 1$$

for all N large and therefore

$$\rho_{sp}(T_\mu|_{L^2_0}) = \lim_{N \to +\infty} \left(\|T^N_\mu|_{L^2_0}\| \right)^{1/N} = 1.$$

The same type of argument works on general torii. The lesson of this observation is that *non-commutativity* must play a key role in the mechanism that produces a spectral gap for Hecke operators.

One way to attack this problem is to remember that convolutions operators are leaving invariant coefficients of representations. Combining this with Peter-Weyl Theorem, we can restrict T_μ to the representation spaces: if ρ is an irreducible complex representation of $\mathbf{G}$ acting unitarily on the finite dimensional representation space V_ρ, we have to study

$$T_{\mu,\rho} := \int_{\mathbf{G}} \rho(g) d\mu(g) : V_\rho \to V_\rho.$$

If μ is symmetric, then this is a self-adjoint matrix acting on V_π. It is not difficult to see that the spectral gap problem is equivalent to prove that there exists a uniform $r < 1$ such that for all ρ non trivial,

$$\rho_{sp}(T_{\mu,\rho}) \leq r.$$

In the case of Hecke operators, it is equivalent to show that the sum of unitary operators (which is self adjoint if S is a symmetric set),

$$\frac{1}{|S|} \sum_{g \in S} \rho(g) : V_\rho \to V_\rho$$

has a norm which is uniformly bounded away from 1 when ρ ranges over all non trivial representations.

2. The Ruziewicz problem

In this section, we will outline the genesis of the spectral gap problem through the (now solved) Ruziewicz measure problem. Our basic references for this matter are the books of Sarnak [7] and Lubotzky [6]. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. We denote by λ the Lebesgue measure, defined on its complete σ-algebra. As usual $L^\infty(S^n)$ denotes the space of Lebesgue-measurable, essentially bounded functions on S^n.

Definition 1. *A linear functional $\nu : L^\infty(S^n) \to \mathbb{R}$ is called an invariant mean iff we have:*

- *$f \geq 0$ implies that $\nu(f) \geq 0$.*
- *$\nu(\mathbf{1}) = 1$.*
- *For all $t \in SO(n+1)$ and $f \in L^\infty(S^n)$, we have $\nu(f_t) = \nu(f)$, where $f_t(x) = f(t.x)$.*

To each invariant mean ν one can associate a *finitely additive* measure on Lebesgue sets by setting

$$\nu(A) := \nu(\chi_A)$$

where χ_A is the characteristic function of set A. On the other hand, we know by the Riesz-Kakutani representation theorem that ν, which is obviously a continuous linear functional on $C^0(S^n)$ is a Borel measure, hence countably additive on Borel sets.

It is tempting to believe that such an invariant mean is automatically a *countably additive measure* on Lebesgue sets (and therefore proportional to Lebesgue measure), but this is **false** in general !

Theorem 2. *(Rudin) There exists an invariant mean ν on S^1 which is not Lebesgue measure.*

Proof. Let $A \subset S^1$ be an open and dense set such that $\lambda(A) < 1$. Consider

$$H = \mathrm{Span}\{f_t - f \ : \ t \in S^1, \ f \in L^\infty\} \subset L^\infty(S^1)$$

We need to find some supplementary space to H which is bigger than the space of constant functions. The following Lemma is the core fact.

Lemma 3. *For all $h \in H$, we have* $\mathrm{Infess}_A(h) \leq 0$.

Let us prove that thing. We write

$$h = \sum_{k=1}^{N} \left((f_k)_{t_k} - f_k \right),$$

with $f_k \in L^\infty(S^1)$ and $t_1, \ldots, t_k \in S^1$. We then pick $x \in A$ such that for all

$$t \in \mathbb{N}t_1 + \mathbb{N}t_2 + \ldots + \mathbb{N}t_N,$$

we have $x + t \in A$. Such an x is legit by the Baire category theorem. Next we consider

$$T(x) := \sum_{\alpha \in \{1,\ldots,M\}^N} h(x + \alpha_1 t_1 + \ldots + \alpha_N t_n).$$

If Infess(h) $= \epsilon > 0$ then we get

$$T(x) \geq M^N \epsilon.$$

On the other hand, a telescoping series argument shows that

$$|T(x)| \leq \sum_{k=1}^{N} \left| \sum_{\alpha} \left(f_k(x + t_k + \alpha.t) - f_k(x + \alpha.t) \right| \right| \leq 2N \max_k \|f_k\|_{L^\infty} M^{N-1}.$$

We get a contradiction by choosing M large enough. $\square$.

Going back to the main proof, we consider the sum

$$V := H \oplus \mathbb{R}\mathbf{1} \oplus \mathbb{R}\chi_{A^c}.$$

This sum is indeed direct by the preceding Lemma. We define ν uniquely on V by setting

$$\nu(h + \alpha + \beta\chi_{A^c}) := \alpha.$$

Because we have (Lemma again)

$$|\nu(h + \alpha + \beta\chi_{A^c})| = |\alpha| \leq \mathrm{Supess}_A|h + \alpha + \beta\chi_{A^c}| \leq \|h + \alpha + \beta\chi_{A^c}\|_{L^\infty},$$

we can use Hahn-Banach to extend the functional ν to $L^\infty(S^1)$ in such a way that

$$|\nu(f)| \leq \|f\|_{L^\infty}.$$

Because we have obviously $\nu(\mathbf{1}) = 1$, it implies that ν is positive (exercise!). Obviously $\nu(H) = 0$ so ν is an invariant mean and

$$\nu(A^c) = 0 \neq \lambda(A^c) = 1 - \lambda(A) > 0,$$

so ν is not Lebesgue measure. $\square$

It turns out that this construction of "exotic" measures is related to the ideas encountered in the theory of *amenable groups* (which is a notion relevant for topological groups with discrete topology). We will see later on that the only invariant mean is on S^n for $n \geq 2$ is Lebesgue measure. Banach-Tarski paradox tells us something weaker: an invariant mean has to be absolutely continuous with respect to Lebesgue, see Lubotzky [6], chapter 2. The goal of this section is to show the following fact which connects clearly Ruziewicz problem to existence of Hecke operators with spectral gaps.

Proposition 4. *Assume that* $S = \{\gamma_1, \gamma_2, \ldots, \gamma_p, \gamma_1^{-1}, \ldots, \gamma_p^{-1}\} \subset SO(n+1)$ *and that the associated Hecke operator*

$$T_S := \frac{1}{|S|}\sum_{g\in S} \delta_g : L^2(SO(n+1)) \to L^2(SO(n+1))$$

has a spectral gap. Then Lebesgue measure on S^n *is the* <u>only</u> *invariant mean.*

Proof. The easiest proof proceeds through rather delicate abstract arguments of functional analysis which we outline here. Since we have a diffeomorphism $S^n \simeq SO(n+1)/SO(n)$, the spectral gap property on $L^2(SO(n+1))$ implies a spectral gap on $L^2(S^n)$ and we will still denote Hecke operators on the Sphere by

$$T_S(f)(x) := \frac{1}{|S|}\sum_{g\in S} f(g.x).$$

Observe that given an invariant mean $\nu \in L^\infty(S^n)$, we do have $\nu \in (L^1(S^n))^{**}$ the topological bidual of L^1. By weak-$*$ density of L^1 into its bidual (and convexity arguments) [1] we can find a sequence [2] (h_k) in $L^1(S^n)$ such that

[1] See Rudin's functional analysis, chapter 3, Theorem 3.13.

[2] Here we are cheating a lot: the weak-$*$ topology is not metrizable so ν may not be a weak-$*$ limit of a sequence of L^1-elements. Hence, the correct proof needs to rely on the existence of a converging *net* $(h_\alpha)_\alpha$ or on Bourbaki's delightful notion of *filter*. Since this does not change the main argument, we avoid this technicality here.

- $\lim_{k\to\infty} h_k = \nu$ (in weak $*$ sense).
- For all k, $\int h_k d\lambda = 1$.
- For all $\gamma \in S$, $\lim_{k\to\infty} \|h_k \circ \gamma - h_k\|_{L^1} = 0$.
- For all k, $h_k \geq 0$.

We now set $f_k := \sqrt{h_k}$. Obviously $f_k \in L^2$ and $\|f_k\|_{L^2} = 1$. Since we have [3]

$$\|f_k \circ \gamma - f_k\|_{L^2}^2 \leq \|h_k \circ \gamma - h_k\|_{L^1},$$

we also obtain that for all $\gamma \in S$, $f_k \circ \gamma - f_k \to 0$ in L^2 as $k \to +\infty$. We now fix $\epsilon > 0$. By the spectral gap property, one can find N such that for all k,

$$\|T_s^N(f_k) - \mathbf{1}\|_{L^2} \leq \epsilon.$$

On the other hand for all k large enough we have

$$\|T_S^N(f_k) - f_k\|_{L^2} \leq \epsilon.$$

Hence we have

$$\lim_{k\to\infty} \|f_k - \mathbf{1}\|_{L^2} = 0.$$

On the other hand, by Cauchy-Schwarz

$$\|h_k - \mathbf{1}\|_{L^1} \leq \|f_k + \mathbf{1}\|_{L^2}\|f_k - \mathbf{1}\|_{L^2} \leq 2\|f_k - \mathbf{1}\|_{L^2}.$$

We conclude therefore that for all $\varphi \in L^\infty$,

$$\nu(\varphi) = \lim_{k\to\infty} \int h_k \varphi d\lambda = \lambda(\varphi).$$

The proof is done. □

The *crucial fact* we used here is that, in operator norm,

$$\lim_{N\to+\infty} \|T_S^N|_{L_0^2}\| = 0,$$

which is much weaker than the spectral gap property. An alternative way to obtain uniqueness of invariant means is to rely on the so-called Kazhdan property (T).

Definition 5. *A locally compact group G has Kazhdan property (T) if there exists $\epsilon > 0$ and a non-empty compact subset $K \subset G$ such that for all irreducible unitary representation $\rho : G \to V_\rho$, we have that for all $v \in V_\rho$ with $\|v\| = 1$,*

$$\|\rho(k)v - v\| \geq \epsilon,$$

for some $k \in K$.

The main observation is the following.

Proposition 6. *Assume that one can find a finitely generated, Zariski dense subgroup $\Gamma \subset SO(n+1)$ with property (T), then the only invariant mean on S^n is Lebesgue measure.*

Proof. We give only the outline. Let S be a finite system of generators for Γ. First we start as in the previous proof, until we come up with a sequence $f_k \in L^2$ with $\|f_k\|_{L^2} = 1$ and such that for all $\gamma \in S$,

$$f_k \circ \gamma - f_k \to_{L^2} 0.$$

[3] Which follows from the fact that for $a, b \geq 0$, $(a-b)^2 \leq |a^2 - b^2|$.

Using Peter-Weyl theorem and Plancherel formula, we can restrict ourself to the case when f_k is a sequence of coefficients of a fixed irreducible unitary non-trivial representation ρ of $SO(n+1)$. This means that

$$f_k(x) = \mathrm{Tr}(\rho(x)A_k),$$

for some matrices $A_k \in \mathrm{End}(V_\rho)$. Computing $\|f_k \circ \gamma - f_k\|_{L^2}$, we obtain that the Hilbert-Schmidt norms

$$\|\rho(\gamma)A_k - A_k\|_{HS}$$

must tend to 0 as $k \to +\infty$, for all $\gamma \in S$. Because Γ is *Zariski dense* in $SO(n+1)$ any irreducible representation of $SO(n+1)$ restricted to Γ is also irreducible. We also point out that Kazdhan holds with S in place of K. It is now clear that

$$\inf_k \|A_k\|_{HS} = 0,$$

otherwise we would get a contradiction with Kazhdan property which says that there exists $\epsilon > 0$ such that for all k there exists $\gamma_k \in S$ with

$$\|\rho(\gamma_k)A_k - A_k\|_{HS} \geq \epsilon.$$

Therefore, by passing to a subsequence (or a subnet), we get that

$$\lim_{k\to+\infty} \|f_k - \mathbf{1}\|_{L^2} = 0.$$

The end of the proof is the same. □

By considering arithmetic lattices in $SO(n-2,2) \times SO(n)$, Sullivan [8] was able to show property (T) for certain finitely generated subgroups of $SO(n)$ for all $n \geq 5$ and thus solving Ruziewicz problem for S^n with $n \geq 4$. The case of S^2 and S^3 was solved by Drinfeld [4] using sophisticated arguments based on Jacquet-Langlands correspondence.

3. Examples of operators with spectral gap for $SU(2)$

Our main goal in this section is to show how to construct Hecke operators with spectral gap on $SU(2)$ (and therefore $SO(3)$), following the elementary approach of Gamburd-Jakobson-Sarnak [5].

3.1. **A refresher on** $SU(2)$**.**

Below we recall some facts on the representation theory of $SU(2)$. Recall that $SU(2)$ is just the group of matrices

$$g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \ \alpha, \beta \in \mathbb{C} \text{ s.t. } |\alpha|^2 + |\beta|^2 = 1.$$

Let $\mathcal{P}_m$ denote the complex vector space of Homegeneous polynomials of degree m with two variables, i.e. the vector space of dimension $m+1$ spanned by

$$f_j(u,v) := u^j v^{m-j}, \ j = 0, \dots, m$$

The representation $\rho_m : SU(2) \to \mathcal{P}_m$ is defined by

$$\rho_m(g)f(u,v) := f(au + cv, bu + dv),$$

if we have

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

These representation are irreducible and exhaust (up to equivalence) all the irreducible representations of $SU(2)$. Given $m \geq 0$, one defines the character of ρ_m by

$$\chi_m(g) := \mathrm{Tr}(\rho_m(g)).$$

Characters are central functions i.e. for all $x, g \in SU(2)$ we have

$$\chi_m(gxg^{-1}) = \chi_m(x).$$

By a classical linear algebra result (actually Cartan's theorem on maximal torii), each element $g \in SU(2)$ is conjugate to a diagonal element

$$g(r) := \begin{pmatrix} e^{ir} & 0 \\ 0 & e^{-ir} \end{pmatrix}.$$

The character formula says that for each $g(r)$ we have

$$\chi_m(g(r)) = \frac{\sin((m+1)r)}{\sin(r)},$$

with the obvious convention that

$$\chi_m(g(k\pi)) = (-1)^{mk}(m+1).$$

3.2. **Hecke operators.** Given a symmetric set $S = \{\gamma_1, \gamma_2, \ldots, \gamma_k, \gamma_1^{-1}, \ldots, \gamma_k^{-1}\}$, we associate the Hecke operator

$$T_m : \mathcal{P}_m \to \mathcal{P}_m$$

defined by

$$T_m(f) := \sum_{\gamma \in S} \rho_m(\gamma) f.$$

Endowing $\mathcal{P}_m$ with an inner product which make each $\rho_m(g)$ *unitary*, we observe that T_m is *self-adjoint*. What we would like is to exhibit some sets S such that we have, uniformly for all m large enough,

$$\|T_m\|_{\mathcal{P}_m} \leq 2k - \epsilon$$

for some $\epsilon > 0$.

We start with a combinatorial observation. Let Γ be the group generated by S and we assume in the latter that Γ is *free*. A word in $\gamma_\alpha \in \Gamma$ given by

$$\gamma_\alpha := \gamma_{\alpha_1}\gamma_{\alpha_2}\cdots\gamma_{\alpha_n}$$

is called *reduced* iff $\gamma_{\alpha_{j+1}} \neq \gamma_{\alpha_j}^{-1}$ for all $j = 1, \ldots, n-1$. The length n is denoted by $|\alpha|$. Set $p = 2k - 1$ and let $U_n(X)$ be the *Tchebychev polynomial of the second kind* [4] such that

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)},$$

for all $\theta \in \mathbb{R}$. A key observation is the following.

$$p^{n/2} U_n\left(\frac{T_m}{2\sqrt{p}}\right) = \sum_{\substack{0 \leq k \leq n \\ k \equiv n [2]}} \sum_{\substack{|\alpha| = k \\ \alpha \text{ reduced}}} \rho_m(\gamma_\alpha).$$

[4] We have $U_0(X) = 1$, $U_1(X) = 2X$, and for all $n \geq 2$,

$$U_{n+1}(X) = 2XU_n(X) - U_{n-1}(X).$$

The proof is elementary by induction on n, exploiting the inductive formula for $U_n(X)$. We will write the eigenvalues of T_m as $\lambda_j = 2\sqrt{p}\cos(\theta_{j,m})$, $j = 0, \dots, m$, having in mind that

$$\begin{cases} \theta_{j,m} \in [0,\pi] & \text{if} \quad |\lambda_j| \leq 2\sqrt{p} \\ \theta_{j,m} = i\xi_{j,m} \in i\mathbb{R}^+ & \text{if} \quad \lambda_j > 2\sqrt{p} \\ \theta_{j,m} = \pi + i\xi_{j,m} \in \pi + i\mathbb{R}^+ & \text{if} \quad \lambda_j < -2\sqrt{p} \end{cases}.$$

All eigenvalues with $|\lambda_j| \leq 2\sqrt{p}$ will correspond to the *bulk of the spectrum* whereas the others will be called exceptional. The starting point of the strategy of Gamburd-Jakobson-Sarnak is to compute the traces (using the character formula):

$$p^{n/2}\sum_{j=0}^{m}\frac{\sin((n+1)\theta_{j,m})}{\sin(\theta_{j,m})} = \sum_{|\alpha|\leq n}{}'\frac{\sin((m+1)r_\alpha)}{\sin(r_\alpha)},$$

where the right sum is understood over all reduced words with length $\equiv n$ [2]. Assuming n even, we have

$$\sum_{\xi_{j,m}\ \text{except.}}\frac{\sinh((n+1)\xi_{j,m})}{\sinh(\xi_{j,m})} + O(nm) = \frac{1}{p^{n/2}}\sum_{|\alpha|\leq n}{}'\frac{\sin((m+1)r_\alpha)}{\sin(r_\alpha)}.$$

Let us stare at this formula for a while. As n becomes large, the contribution of exceptional eigenvalues is magnified since we have an exponential growth (in n) coming from $\sinh((n+1)\xi)$. On the other hand, the bulk of the spectrum will contribute only to polynomial growth in n. Let us prove an actual statement out of this observation.

Definition 7. *Let Γ be the free group generated by S. We say that Γ satisfies condition (A) iff there exists $D > 0$ such that for all reduced word $\gamma_\alpha \in \Gamma \setminus \{Id\}$ we have*

$$\gamma_\alpha \sim \pm\begin{pmatrix} e^{ir_\alpha} & 0 \\ 0 & e^{-ir_\alpha}\end{pmatrix},$$

with $r_\alpha \in [D^{-|\alpha|}, \pi - D^{-|\alpha|}]$.

This is a diophantine-like condition: we ask that non-trivial elements in Γ cannot approach too fast $\pm Id$. We will see that this condition is satisfied for groups with algebraic generators.

Proposition 8. *Assume that Γ satisfies (A), then for all $r > 2\sqrt{p}$, there exists $\epsilon(r) > 0$ such that as $m \to +\infty$.*

$$N(r) := \#\{\lambda_{j,m}\ :\ p+1 \geq |\lambda_{j,m}| \geq r\} = O(m^{1-\epsilon}).$$

Proof. We set $n = [\beta \log m]$ where $\beta > 0$ will be adjusted later on. Using the trace formula we have the bound

$$CN(r)e^{n\widetilde{r}} \leq O(m\log(m)) + O\left(m^{-\alpha\log(p)/2}\sum_{|\alpha|\leq n}{}'\frac{\sin((m+1)r_\alpha)}{\sin(r_\alpha)}\right),$$

where $\widetilde{r} > 0$ is such that

$$2\sqrt{p}\cos(i\widetilde{r}) = r.$$

We therefore have

$$N(r) = O\left(\log(m)m^{1-\beta\widetilde{r}}\right) + O\left(m^{-\beta\log(p)/2}\sum_{|\alpha|\leq n}{}'\frac{\sin((m+1)r_\alpha)}{\sin(r_\alpha)}\right).$$

Using condition (A), we write [5] (we must isolate the identity term from the other ones)

$$\left|\sum_{|\alpha|\leq n}{}' \frac{\sin((m+1)r_\alpha)}{\sin(r_\alpha)}\right| \leq m+1+O\left(np^nD^n\right) = O\left(m+\log(m)m^{+\beta(\log(p)+\log(D))}\right).$$

It is now clear that by taking $\beta > 0$ small enough so that

$$\beta(\log(p)/2+\log(D)) < 1$$

we have obtained

$$N(r) = O(m^{1-\epsilon}),$$

for some $\epsilon(r, D, p) > 0$. We have not attempted to optimize the exponents, but they can be made explicit for sure. □

The above proposition clearly shows that for the large m limit, which corresponds to a semi-classical limit in geometric quantization, the majority of the eigenvalues will concentrate in the bulk $[-2\sqrt{p}, 2\sqrt{p}]$. Proving a spectral gap now requires that we are able to catch some cancellations in the sums

$$\sum_{|\alpha|\leq n}{}' \frac{\sin((m+1)r_\alpha)}{\sin(r_\alpha)},$$

for $n \asymp \log(m)$.

To this end, we will "average over the representation parameter m" in the following way. Let $\varphi \in C_0^\infty((1/2, 3/2)$ be a positive test function such that $\varphi(1) = 1$. We have the following fact.

Lemma 9. *Let $0 < r < \pi$, $m_0 \geq 1$ and $0 < \delta < 1$. For all $A \in \mathbb{N}_0$, we have*

$$\sum_{m\in\mathbb{Z}} \varphi\left(\frac{m+1}{m_0+1}\right)\frac{\sin((m+1)r)}{\sin(r)} = \begin{cases} O(m_0^2) \text{ if } 0 < r < m_0^{\delta-1} \\ O_A\left(m_0^{-A}(\sin(r))^{-1}\right) \text{ if } r \in (m_0^{\delta-1}, \pi) \end{cases}$$

Proof. The first case is a crude estimate. For the second claim, use Poisson summation formula and then integrate by parts as much times as needed. □

When applying Lemma 9 to the trace identity, we will gain cancellations for all words α with r_α large enough. Thus the main obstacle is now to estimate how many "bad words" α with $|\alpha| \leq n$ contribute to the small angles r_α. Let us show which adhoc estimate implies a spectral gap.

Definition 10. *Let Γ be the free group generated by S. We say that Γ satisfies condition (B) iff for all $0 < \delta < 1$, there exists $C > 0$ such that for all m_0 and n large we have*

$$\#\left\{|\alpha| \leq n \ : \ r_\alpha \in (0, m_0^{\delta-1}]\right\} \leq C\frac{p^n}{m_0^{3-3\delta}}.$$

[5] We simply bound each character contribution by

$$\left|\frac{\sin(m+1)\theta)}{\sin(\theta)}\right| \leq \min\left\{m+1, \frac{1}{|\sin(\theta)|}\right\}.$$

This definition is plausible as long as one is aware of the following fact. Let $f \in C^\infty(\mathbf{G})$ be a central function i.e. a function that lives on the maximal torus

$$\left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},\ \theta \in [0,\pi] \right\}.$$

Then f can be expanded in Fourier series (with uniform convergence) as

$$f(\theta) = \sum_{m=0}^{\infty} a(m)\chi_m(\theta),$$

with (by a well known formula for Haar measure induced on the maximal torus)

$$a(m) = \int_{\mathbf{G}} f(g)\chi_m(g)dg = \frac{2}{\pi}\int_0^\pi f(\theta)\chi_m(\theta)\sin^2(\theta)d\theta.$$

Set $C_n := \sum'_{|\alpha|\leq n} 1$. Then the trace identities (for m fixed and $n \to \infty$) say that for all $m \geq 1$,

$$\lim_{n\to\infty} \frac{1}{C_n} \sum_{|\alpha|\leq n}{}' \chi_m(r_\alpha) = 0$$

This fact combined with Fourier expansion actually shows that the angles r_α become equidistributed in the large n limit according to Haar measure:

$$\lim_{n\to\infty} \frac{1}{C_n} \sum_{|\alpha|\leq n}{}' f(r_\alpha) = a(1) = \frac{2}{\pi}\int_0^\pi f(\theta)\sin^2(\theta)d\theta.$$

If $f(\theta) = \mathbf{1}_{[0,\epsilon]}$, we obtain that as $n \to +\infty$,

$$\lim_{n\to\infty} \frac{1}{C_n} \sum_{r_\alpha\in[0,\epsilon]}{}' = O(\epsilon^3),$$

which explains the exponent 3 in the definition of condition (B). Therefore (B) is a strengthening of this equidistribution property, where one is able to keep track of small intervals. It is expected to hold generically. We can finally show the following:

Theorem 11. *Assume that both (A) and (B) hold, then the Hecke operator has a spectral gap.*

Proof. Let $\xi_{m_0}^{max} \geq 0$ be such that

$$2\sqrt{p}\cosh(\xi_{m_0}^{max}) = \max\{p+1 \geq |\lambda_{j,m_0}| \ : |\lambda_{j,m_0}| \geq 2\sqrt{p}\}.$$

We have a spectral gap whenever uniformly as $m_0 \to \infty$ we have

$$2\sqrt{p}\cosh(\xi_{m_0}^{max}) < 2k = p+1,$$

i.e.

$$\xi_{m_0}^{max} < \frac{\log(p)}{2}.$$

We will choose $n = [\gamma\log(m_0)]$ where $\gamma > 0$ will be specified later on. Combining Lemma 9 plus both (A) and (B), we get

$$\frac{\sinh((n+1)\xi_{m_0}^{max})}{\sinh(\xi_{m_0}^{max})} = O(nm_0^2) + O(m_0^{-\infty}) + O(p^{n/2}m_0^{-1+3\delta}),$$

which shows that we must have for all m_0 large,

$$\gamma\xi_{m_0}^{max} \leq \max\{2+\epsilon, \frac{\gamma}{2}\log(p) - 1 + \epsilon\},$$

therefore choosing any

$$\gamma > \frac{4}{\log(p)}$$

will do the job. One can even obtain some explicit estimates for the spectral gap (see [5]) which show that

$$\max_j |\lambda_{j,m}| \leq \sqrt{p}(p^{1/3} + p^{-1/3}) + \epsilon. \ \square$$

Now it remains to exhibit some group generators which satisfy both (A) and (B). This is where number theory is involved.

Let $H_{\mathbb{Z}} := \{x_0 + ix_1 + jx_2 + kx_3 \ : \ x_0, x_1, x_2, x_3 \in \mathbb{Z}\}$ be the set of *integral quaternions* with the well known relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

Given $g = x_0 + ix_1 + jx_2 + kx_3 \in H_{\mathbb{Z}}$, we denote by

$$N(g) := g\overline{g} = x_0^2 + x_1^2 + x_2^3 + x_3^2.$$

There a multiplicative homomorphism (exercise!) $\mathcal{I} : H_{\mathbb{Z}} \to SU_2(\mathbb{C})$ defined by

$$\mathcal{I}(g) := \frac{1}{\sqrt{N(g)}} \begin{pmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{pmatrix}.$$

Let $q \geq 3$ be a prime number. Then

$$\#\{g \in H_{\mathbb{Z}} \ : \ N(g) = q\} = 8q + 8.$$

It is possible to pick a set $\widetilde{g_1}, \ldots, \widetilde{g_k} \in \{g \in H_{\mathbb{Z}} \ : \ N(g) = q\}$ such that

$$g_1 := \mathcal{I}(\widetilde{g_1}), \ldots, g_k := \mathcal{I}(\widetilde{g_k})$$

and their inverses generate a *free group* Γ. Given a reduced element

$$g_\alpha = g_{\alpha_1} \ldots g_{\alpha_n} = \mathcal{I}(\widetilde{g_{\alpha_1}} \ldots \widetilde{g_{\alpha_n}}) \in \Gamma,$$

we have

$$2\cos(r_\alpha) = \mathrm{Tr}(g_\alpha) = \frac{2x_0^\alpha}{q^{n/2}},$$

so that

$$\sin(r_\alpha) = \frac{\sqrt{q^n - (x_0^\alpha)^2}}{q^{n/2}}.$$

Therefore (A) is clearly satisfied since if $g_\alpha \neq Id$ we must have $(x_0^\alpha)^2 \neq q^n$, and thus

$$\sin(r_\alpha) \geq q^{-n/2}.$$

To check condition (B) one has to count solutions of the diophantine equation

$$x_0^2 + x_1^2 + x_2^3 + x_3^2 = q^n, \ (x_0)^2 \neq q^n,$$

with the additional constraint that

$$\frac{\sqrt{q^n - (x_0^\alpha)^2}}{q^{n/2}} \leq N^{-1+\delta},$$

for N large. This lead us to bound

$$\sum_{q^{n/2}\sqrt{1-N^{-2+2\delta}}\leq x_0<q^{n/2}} r_3(q^n-x_0^2),$$

where $r_3(a)$ is the number of representations of a as a sum of three squares. Since

$$r_3(a)=O(a^{1/2+\epsilon})$$

this estimate ultimately leads to

$$\#\left\{|\alpha|\leq n\ :\ r_\alpha\in(0,N^{\delta-1}]\right\}\leq C\frac{q^{n+\epsilon}}{N^{3-3\delta}},$$

and thus (B) is (almost) satisfied, provided we take $2k-1=p$ close enough to q, (which is possible to do).

4. More recent results and open questions

In the paper [3], Bourgain and Gamburd manage to prove the following statement:

Theorem 12. *Assume that $\Gamma=\langle g_1,\dots,g_k,g_1^{-1},\dots,g_k^{-1}\rangle$ as above satisfies (A), then the operator*

$$T=\sum_j(\delta_{g_j}+\delta_{g_j^{-1}}):L^2(\mathbf{G})\to L^2(\mathbf{G})$$

has a spectral gap.

According to the previous §, it is enough to show that (B) always holds if (A) is assumed, which is basically what they achieved here. Their proof relies heavily on additive combinatorics. Since (A) is satisfied when the generators have algebraic entries, we therefore now that there is a spectral gap for a "dense set" of generators. But it is unknown if (A) is generic in measure ! Bourgain and Gamburd then generalized their result to $SU(d)$ for all d (see [2]).

In a recent paper, Yves Benoist and Nicolas de Saxcé obtained the following statement.

Theorem 13. *Let $\mathbf{G}$ be a connected simple compact Lie group. Let μ be a symmetric Borel probability measure on $\mathbf{G}$. We say that μ is almost diophantine iff there exist $C_1,C_2>0$ such that for all proper closed subgroups H of $\mathbf{G}$, we have for all $n\geq 1$,*

$$\mu^n(\{g\in\mathbf{G}\ :\ d(g,H)\leq e^{-C_1n}\})\leq e^{-C_2n}.$$

Then $T_\mu:L^2(\mathbf{G})\to L^2(\mathbf{G})$ has a spectral gap off μ is almost diophantine.

This smart statement generalizes the result of Bourgain-Gamburd and characterizes operators with spectral gaps. The main issue here is to be able to check the almost diophantine property: it is known to hold so far for generators with algebraic entries.

Below we list open questions that seem to be relevant.

- Is the almost diophantine property generic in measure ?
- Can one produce examples of spectral gaps for non algebraic generators ?
- In all of the operators studied above, symmetry of the measure (self-adjointness) has been used in a critical way. Can one produce examples of spectral gaps for semi-groups or non self-adjoint Hecke operators ?
- It can be shown, using a result of Kesten, that the continuous spectrum, in the above examples, is given by $[-2\sqrt{p},+2\sqrt{p}]$. Can one say something smart about the peripheral spectrum before the spectral gap ?

- In the $SU_2(\mathbb{C})$ case, the large m limit coincides with a semi-classical limit in geometric quantization. Can one give a general semi-classical interpretation of this spectral gap in term of trapped set and quantum resonances ? Can we view Hecke operators as FIO with a simple canonical relation and use it efficiently ?

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Laboratoire de Mathématiques d'Avignon,, Campus Jean-Henri Fabre, 301 rue Baruch de Spinoza,, 84916 Avignon Cedex 9.

E-mail address: frederic.naud@univ-avignon.fr