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ON DEFORMATIONS OF F_n IN COMPACT LIE GROUPS*

BY

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ABSTRACT

We study some properties of the varieties of deformations of free groups in compact Lie groups. In particular, we prove a conjecture of Margulis and Soifer about the density of non-virtually free points in such variety, and a conjecture of Goldman on the ergodicity of the action of $\mathrm{Aut}(F_n)$ on such variety when $n \geq 3$.

Introduction

For $n > 1$, a generic n–tuple of elements in a connected compact non-abelian Lie group G generates a free group. G. A. Margulis and G. A. Soifer conjectured (cf. [So]) that every such tuple can be slightly deformed to one which generates a group which is not virtually free. In this note we prove this conjecture, and actually show that for $n \geq 3$ and for an arbitrary dense subgroup Γ, with some restriction on the minimal size of a generating set, the set of deformations of F_n whose image coincides with Γ is dense in the variety of all deformations. The idea is to move a given (almost arbitrary) n-tuple into any open subset of G^n by applying Nielsen transformations. Using the same idea we also prove a

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conjecture of W. M. Goldman [Go] on the ergodicity of the action of $\mathrm{Out}(F_n)$ on $\mathrm{Hom}(F_n, G)/G$ when $n \geq 3$. For $n = 2$, we prove the Margulis–Soifer conjecture by showing that any pair can be slightly deformed to one which generates an infinite group which has Serre's property (FA) and in particular is not virtually free.

1. The case $n > 2$

We will say that a group Γ is k-generated if it has a generating set of size $\leq k$.

THEOREM 1.1: *Let $n \geq 3$. Let G be a connected compact Lie group, and let $\Gamma \leq G$ be an $(n-1)$-generated dense subgroup. Then any n elements $s_1, \ldots, s_n \in G$ admit an arbitrarily small deformation $t_1, \ldots, t_n$ with $\Gamma = \langle t_1, \ldots, t_n \rangle$. In other words, the set*

$$\{f \in Hom(F_n, G) : f(F_n) = \Gamma\}$$

is dense in $Hom(F_n, G)$.

For example, Theorem 1.1 implies that if n is larger than the minimal size of a generating set of $\mathrm{SO}_5(\mathbb{Z}[1/5])$, then any n elements in $\mathrm{SO}_5(\mathbb{R})$ can be slightly deformed to a generating set of the group $\mathrm{SO}_5(\mathbb{Z}[1/5])$ (recall that $\mathrm{SO}_5(\mathbb{Z}[1/5])$ is dense in $\mathrm{SO}_5(\mathbb{R})$ and has Kazhdan property (T), cf. [M]). Similarly, any 5 elements in $\mathrm{SO}_3(\mathbb{R})$ can be deformed to a generating set of a surface group.

Let G be a connected compact Lie group. Since the topology of G is metrizable, we may assume that it is endowed with a metric d, and by averaging over all left and right translations with respect to the Haar measure, we may assume that d is left and right invariant. Let $\Gamma \leq G$ be an $(n-1)$-generated dense subgroup of G, let $s_1, \ldots, s_n$ be arbitrary n elements of G, and let $\epsilon > 0$. We will explain how to perturb the s_i to some t_i which satisfy $d(t_i, s_i) < \epsilon,\ i = 1, \ldots, n$ and $\langle t_1, \ldots, t_n \rangle = \Gamma$.

We will first treat the case where G is semisimple, since our proof in this case gives more than what is stated in Section 1.1.

1.1. THE SEMISIMPLE CASE. Let G be a connected compact semisimple Lie group. It is well-known that G admits a structure of an algebraic group over $\mathbb{R}$, and hence is equipped with a Zariski topology. Let $\mathfrak{g}$ denote the Lie algebra of G, write $\mathfrak{g} = \bigoplus_{i=1}^{p} \mathfrak{g}_i$ where $\mathfrak{g}_i$ are the simple factors of $\mathfrak{g}$, and set $\mathcal{A} = \bigoplus_{i=1}^{p} \mathrm{End}(\mathfrak{g}_i)$. Let G_i be the subgroup of G corresponding to $\mathfrak{g}_i$; recall that

G is an almost direct product of the G_i. Let $\overline{G}_i$ be the quotient of G by the product of the G_j, $j \neq i$ and let $\pi_i : G \to \overline{G}_i$ be the canonical projection. For each i, the restriction of the Adjoint representation to G_i is irreducible on $\mathfrak{g}_i$ and hence, by Burnside's lemma, $\mathrm{span}(\mathrm{Ad}(G)) = \mathcal{A}$. It is well-known that $\mathcal{A}$ is generated by two elements which can be taken from $\mathrm{Ad}(G)$, for instance, one can take two elements in $\mathrm{Ad}(G)$ which generate a dense subgroup (cf. [K]).

Lemma 1.2: *Let $m \in \mathbb{N}$ and let $g_1, \ldots, g_m$ be arbitrary m elements in G. The set*

$$\Omega(g_1, \ldots, g_m) := \{g \in G : Ad(g), Ad(g_1), \ldots, Ad(g_m) \ \ generate \ \ \mathcal{A}\}$$

is Zariski open in G.

Proof. If $\Omega(g_1, \ldots, g_m)$ is empty there is nothing to prove. Assume that $\Omega(g_1, \ldots, g_m) \neq \emptyset$ and let $g_0 \in \Omega(g_1, \ldots, g_m)$. Let $d = \dim(\mathcal{A})$. It follows from the definition of $\Omega(g_1, \ldots, g_m)$ that there are d words with $m+1$ letters $W_1, \ldots, W_d$ which, when evaluated at the point $(\mathrm{Ad}(g_0), \mathrm{Ad}(g_1), \ldots, \mathrm{Ad}(g_m))$, span $\mathcal{A}$. The set

$$\{g \in G : \mathrm{span}\{W_i\big(\mathrm{Ad}(g), \mathrm{Ad}(g_1), \ldots, \mathrm{Ad}(g_m)\big) : i = 1, \ldots, d\} = \mathcal{A}\}$$

is then a Zariski open subset of $\Omega(g_1, \ldots, g_m)$ which contains g_0. Since g_0 is an arbitrary element of $\Omega(g_1, \ldots, g_m)$ it follows that $\Omega(g_1, \ldots, g_m)$ is Zariski open. ∎

Suppose now that $g_1, \ldots, g_k \in G$ are such that $\mathrm{Ad}(g_1), \ldots, \mathrm{Ad}(g_k)$ generate $\mathcal{A}$, then for each $1 \leq i \leq k$ the set[1] $\Omega(g_1, \ldots, \hat{g}_i, \ldots, g_k)$ is Zariski open, and since it contains g_i it is non-empty and hence Zariski dense in G as G is Zariski connected. It follows that the set

$$\tilde{\Omega}(g_1, \ldots, g_k) := \bigcap_{i=1}^{k} \Omega(g_1, \ldots, \hat{g}_i \ldots, g_k)$$

is also Zariski open and non-empty, hence dense in G with respect to the housdorff topology. Moreover, we have

Lemma 1.3: *Let $g_1, \ldots, g_k \in G$ be k elements such that $Ad(g_1), \ldots, Ad(g_k)$ generate $\mathcal{A}$. The projection of $\langle g_1, \ldots, g_k \rangle$ to each simple factor $\overline{G}_i$, $i = 1, \ldots, p$*

[1] The hat above the i-th element means that we exclude it.

of G is either finite or dense. If for every $i \leq p$ this projection is infinite, then $\langle g_1, \ldots, g_k \rangle$ is dense in G.

Proof. Let C be the closure of $\langle g_1, \ldots, g_k \rangle$ and C° its identity connected component. Then C° is a connected compact Lie subgroup of G and $|C/C^\circ| < \infty$. Moreover, since the Lie algebra of C° is stable under the elements $\mathrm{Ad}(g_i), i = 1, \ldots, k$ which generate $\mathcal{A}$, C° is normal in G. Since for $i = 1, \ldots, p$, $\overline{G_i}$ is almost simple, it follows that the compact connected normal subgroup $\pi_i(C^\circ)$ is either trivial or equal to $\overline{G}_i$, which means that $\pi_i(\langle g_1, \ldots, g_k \rangle)$ is either finite or dense. Finally, if $\pi_i(\langle g_1, \ldots, g_k \rangle)$ is infinite for each $i \leq p$ then the normal subgroup C° projects onto every simple factor of G, and hence $C^\circ = G$, i.e. $\langle g_1, \ldots, g_k \rangle$ is dense. ∎

We will also use the following well-known lemma.

Lemma 1.4: *The set U of all pairs $(x_1, x_2) \in G \times G$ for which the group $\langle x_1, x_2 \rangle$ is dense in G, is open, dense and of full Haar measure in $G \times G$.*

Proof. The second and third properties follow from the fact that U contains the the following set which is clearly dense and has full Haar measure:

$$\{(x_1, x_2) \in G \times G : \mathrm{Ad}(x_1), \mathrm{Ad}(x_2) \text{ generate } \mathcal{A}, \\ \text{and } \pi_j(x_1) \text{ is non-torsion } \forall j \leq p\}.$$

It is well-known that U contains an open subset V near the identity of $G \times G$ (cf. [GZ] or [BG]). To see that U is open, note that if $(x_1, x_2) \in U$, then, as $\langle x_1, x_2 \rangle$ is dense, there are two words in two letters W_1, W_2 such that $\big(W_1(x_1, x_2), W_2(x_1, x_2)\big)$ belongs to V. It follows that if $U_i,\ i = 1, 2$ are sufficiently small neighbourhoods of x_i in G, then $((W_1(y_1, y_2), W_2(y_1, y_2)) \in V$ for any $(y_1, y_2) \in U_1 \times U_2$, which implies that $\langle y_1, y_2 \rangle$ is dense in G. Therefore $U_1 \times U_2 \subset U$, and since (x_1, x_2) is arbitrary, U is open. ∎

In order to prove Theorem 1.1 we will use repeatedly the so called product replacement moves which allows replacing one generating set $(\gamma_1, \ldots, \gamma_k)$ for the group Γ by another generating set of the same cardinality by multiplying one γ_i by some $\gamma_j^{\pm 1}$ where $j \neq i$ or, more generally, by an element of the group $\langle \gamma_j : j \neq i \rangle$. These operations are also called Nielsen transformations.

Let $(\gamma_1, \ldots, \gamma_{n-1})$ be a generating set for Γ. Applying Selberg's lemma to the projection of Γ to any simple factor $\overline{G}_i$ of G and taking the intersection, we see that Γ contains a subgroup of finite index Γ_0 such that the projection of any element of Γ_0 to any simple factor of G is either trivial or non-torsion. Since Γ is dense and G is connected, Γ_0 is also dense. Pick $\gamma_n \in \Gamma_0 \cap \tilde{\Omega}(\gamma_1, \ldots, \gamma_{n-1})$ with $\pi_j(\gamma_n) \neq 1$, for all $j \leq p$. Then from Lemma 1.3 and the assumption that Γ is dense we get that $\langle \gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_n \rangle$ is dense in G for every $1 \leq i \leq n$. Fix two open sets $U_i \subset B_\epsilon(s_i),\ i = 1, 2$ such that $U_1 \times U_2 \subset U$, and $U_1 \subset \tilde{\Omega}(\gamma_2, \ldots, \gamma_n)$, where $U \subset G \times G$ is the set defined in Lemma 1.4. Since $\langle \gamma_2, \ldots, \gamma_n \rangle$ is dense in G there is some $\hat{\gamma}_1 \in \langle \gamma_2, \ldots, \gamma_n \rangle$ for which $t_1 := \hat{\gamma}_1 \gamma_1 \in U_1$. Since t_1 belongs to $\tilde{\Omega}(\gamma_2, \ldots, \gamma_n)$, it follows from Lemma 1.3 and the property of γ_n that the group $\langle t_1, \gamma_3, \ldots, \gamma_n \rangle$ is dense in G. Chose $\hat{\gamma}_2 \in \langle t_1, \gamma_3, \ldots, \gamma_n \rangle$ such that $t_2 = \hat{\gamma}_2 \gamma_2$ lies in U_2. It follows that $\langle t_1, t_2 \rangle$ is dense in G and we can pick $\hat{\gamma}_i \in \langle t_1, t_2 \rangle,\ i = 3, \ldots, n$ so that $t_i := \hat{\gamma}_i \gamma_i \in B_\epsilon(s_i)$. This completes the proof of Theorem 1.1 in the semisimple case. ∎

Remarks 1.5: (1) The argument above, with some simple modifications, can be applied also to dense subgroups Γ of the form $\Gamma = \langle \Delta, \gamma \rangle$ where Δ is an $(n-1)$-generated dense subgroup of G, and $\gamma \in G \setminus \Delta$. It is therefore natural to ask wether any n-generated dense subgroup of G is of that form. This question makes sense for every $n \geq 3$, and the answer may depend on n and G. Nir Avni pointed out that for every G there is some integer $n(G)$ such that the answer is affirmative for all $n \geq n(G)$. It is of interest whether $n(G)$ is always 3 or some other constant independent of G. For $n \geq n(G)$ one can assume that Γ is n-generated in Theorem 1.1.

(2) It was shown in [GZ] that any k-generated dense subgroup Γ of a connected semisimple Lie group G admits a generating set of cardinality $(k+2)$ which lies arbitrarily close to the identity of G, and hence there is no uniform Kazhdan constant for generating sets of size $k+2$ for the natural representation of Γ on $L_2(G)$. It follows from Theorem 1.1 that when G is compact, one can replace $k+2$ by $k+1$. In view of the previous remark, for $k \geq n(G)$ one can even omit the "+1".

(3) The argument above can be also applied for some non-connected compact groups. M. Abert and L. Pyber have shown me that using results from [D] one can prove the following: Let G be a topologically k-generated pro-finite-soluble group and Γ a $(k+1)$-generated dense subgroup of G. Then for any $k+1$ elements $s_1, \ldots, s_{k+1}$ which topologically generate G and any open normal subgroup $N \lhd G$, one can choose $t_i \in s_i N,\ i = 1, \ldots, k+1$ such that $\Gamma = \langle t_1, \ldots, t_{k+1} \rangle$.

Moreover, any k-generated profinite group has a $k+1$-generated dense subgroup which is not virtually free.

(4) For non-compact Lie groups E. Ghys asked the following related question: Let H be a connected non-compact simple Lie group. Does every n-tuple $(s_1,\dots,s_n)$ which generates a dense subgroup of H admit an arbitrarily small deformation $(t_1,\dots,t_n)$ for which the group $\langle t_1,\dots,t_n\rangle$ is not free? We refer to [So] for other related problems.

1.2. The ergodicity of $\mathrm{Aut}(F_n)$ on $\mathrm{Hom}(F_n,G)$. By fixing a free generating set for the free group F_n we may identify the deformation variety $\mathrm{Hom}(F_n,G)$ with G^n. The automorphism group $\mathrm{Aut}(F_n)$ acts on $\mathrm{Hom}(F_n,G)$ by pre-compositions. The Nielsen operations on generating sets (the product replacement moves) when applied to the generators of the free group F_n correspond to elements of $\mathrm{Aut}(F_n)$. Let us denote by $L_{i,j}$ the operation of replacing the j-th generator by its product from the left with the i-th generator, e.g. $L_{2,1}(g_1,g_2,\dots,g_n)=(g_2g_1,g_2,\dots,g_n)$. Arguing as above one can show that the $\mathrm{Aut}(F_n)$-orbit of almost any point in G^n is dense. More precisely, let $Y\subset G^n$ be the set of n-tuples consisting of n elements, such that the Adjoint of any $n-1$ of them generate the algebra $\mathcal{A}$ and the projection of each to any simple factor of G is non-torsion, and let $Y_0=\bigcap_{\sigma\in\mathrm{Aut}(F_n)}\sigma(Y)$. Then Y_0 is $\mathrm{Aut}(F_n)$-invariant, has full measure and the orbit of any element in Y_0 is dense in G^n. Furthermore, we have

Theorem 1.6: *Let G be a compact connected semisimple Lie group and $n\geq 3$. The action of $Aut(F_n)$ on G^n is ergodic.*

Proof. Assume the contrary, and let $A\subset G^n=G\times G\times\cdots\times G$ be an $\mathrm{Aut}(F_n)$ almost invariant measurable subset which is neither null nor conull. Since $\mathrm{Aut}(F_n)$ is countable we may assume that A is invariant rather than almost invariant. Now since the action of G on itself by left translations is clearly ergodic, it follows from our assumption that A is not (almost) invariant under the action by left translations of one of the n factors, say the first, of G^n. Thus by Fubini's theorem, for a set of positive measure of $(g_2,\dots,g_n)\in G^{n-1}$ we have that $\{g\in G:(g,g_2,\dots,g_n)\in A\}$ is neither null nor conull in G. Let us fix a point $(g_2,\dots,g_n)$ in this subset such that the pair consisting of the first two components (g_2,g_3) belongs to the set of full measure $U\subset G\times G$ introduced in Lemma 1.4. Note that the orbits of the action of $\langle g_2,g_3\rangle$ by left translations on

G coincide with (the projection to the first factor of) the orbits of $\langle L_{2,1}, L_{3,1}\rangle$ on $\{(g, g_2, \ldots, g_n) : g \in G\}$. Set

$$A_1 := \{g \in G : (g, g_2, \ldots, g_n) \in A\}.$$

By our assumption, A_1 is neither null or conull. This however is a contradiction since the group $\langle g_2, g_3\rangle$ is dense in G and hence acts ergodically on G. ∎

Theorem 1.6 clearly implies, and is actually equivalent to (see [Go] Lemma 3.1):

THEOREM 1.7: *Let G be a compact connected semisimple Lie group and $n \geq 3$. Then the action of $Out(F_n)$ on $Hom(F_n, G)/G$ is ergodic.*

Theorem 1.7 was conjectured by W. M. Goldman in [Go], and was proved there under the assumption that $G = \mathrm{SU}(2)$. As explained in the last paragraph of [Go], the general case follows from the semisimple one, i.e., one can omit the assumption that the connected compact Lie group G is semisimple in Theorems 1.6 and 1.7. As pointed out in [Go] the assumption that $n \geq 3$ is necessary, since the function trace$([x, y])$ is $\mathrm{Aut}(F_2)$ invariant on G^2.

Applying the conclusions of Theorems 1.6 and 1.7 to G^2 instead of G we obtain furthermore:

THEOREM 1.8: *Let G be a compact connected Lie group and $n \geq 3$. Then the action of $Aut(F_n)$ on $Hom(F_n, G)$, and the action of $Out(F_n)$ of $Hom(F_n, G)/G$ are weakly mixing.*

Proof. Since $\mathrm{Hom}(F_n, G) \times \mathrm{Hom}(F_n, G)$ is canonically isomorphic to $\mathrm{Hom}(F_n, G \times G)$, and since $\mathrm{Hom}(F_n, G)/G \times \mathrm{Hom}(F_n, G)/G$ is canonically isomorphic to $\mathrm{Hom}(F_n, G\times G)/G\times G$, the assertions follows by applying Theorems 1.6 and 1.7 to the connected compact Lie group $G \times G$. ∎

Remark 1.9: An $(n \geq 2)$-tuple $(g_1, \ldots, g_n) \in G^n$ is said to have a spectral gap if the maximal eigenvalue of the corresponding Hecke operator on $L_2(G)$ is isolated in the spectrum, or equivalently, if the group $\langle g_1, \ldots, g_n\rangle$ acts strongly ergodically, by left multiplications, on G. D. Fisher [F] showed that Goldman's theorem about the ergodicity of the $\mathrm{Aut}(F_n)$ action on $\mathrm{SO}(3)^n$ implies that for $n \geq 3$ the set of n-tuples in $\mathrm{SO}(3)$ which posses a spectral gap is either null or conull. Indeed, this set is measurable and $\mathrm{Aut}(F_n)$-invariant. Theorem 1.6 implies that the same conclusion holds when $\mathrm{SO}(3)$ is replaced by any compact connected Lie group G. Furthermore, Theorem 1.1 implies that if G admits one $(n \geq 2)$-tuple with a spectral gap, then the set of $(n+1)$-tuples with spectral gap is dense in G^{n+1}.

1.3. THE PROOF OF THEOREM 1.1 IN THE CASE WHERE G IS NOT NECESSARILY SEMISIMPLE. Let us come back to the proof of Theorem 1.1 in the general case. Let G be an arbitrary connected compact Lie group. Then G is an almost direct product of G' and Z where G' is the commutator group of G and is semisimple, and Z is the center of G. We do not assume that G' and Z are non-trivial. Let $D \subset G \times G$ be the set of all pairs which generate a dense subgroup of G.

LEMMA 1.10: *D is dense in $G \times G$.*

Proof. Let $V \subset G \times G$ be the pre-image of the open dense subset of $G/Z \times G/Z$ that one gets from Lemma 1.4 applied to the semisimple group G/Z (if $G = Z$ take $V = G \times G$). If $(x_1, x_2) \in V$, then $\langle x_1, x_2 \rangle \cap G'$ is dense in G'.

The quotient G/G', being a connected compact abelian Lie group, is isomorphic to a product of circle groups, and it is well-known that in such a group the set of elements which generate a dense cyclic subgroup is dense. In fact, an element in $U(1)^m$ generates a dense cyclic subgroup iff its normalized coordinates are independent together with 1 over the rational. Let $\pi : G \to G/G'$ denote the canonical projection. The set

$$\{(x_1, x_2) \in V : \langle \pi(x_1) \rangle \text{ is dense in } G/G'\}$$

is clearly dense in $G \times G$ and contained in D. ∎

Fix a pair $(g_1, g_n) \in D \cap (B_{\frac{\epsilon}{2}}(s_1) \times B_{\frac{\epsilon}{2}}(s_n))$, and choose finitely many words $W_j,\ j = 1, \ldots, k$ in two letters, such that the set $\{W_j(g_1, g_n) : j = 1, \ldots, k\}$ form an $\epsilon/2$-net in G. Then fix $\epsilon_1 < \epsilon/2$ sufficiently small so that for any $(x_1, x_n) \in B_{\epsilon_1}(g_1) \times B_{\epsilon_1}(g_n)$ the set $\{W_j(x_1, x_n) : j = 1, \ldots, k\}$ is an ϵ-net in G. Let $\{\gamma_1, \ldots, \gamma_{n-1}\}$ be a generating set for the dense subgroup Γ. We will use the following

Claim 1.11: The set $X = \{x \in G : \langle \gamma_2, \ldots, \gamma_{n-1}, x \rangle \text{ is dense in } G\}$ is dense in G.

Indeed, X contains every x which satisfies the following conditions:

(1) $\mathrm{Ad}(\gamma_2), \ldots, \mathrm{Ad}(\gamma_{n-1}), \mathrm{Ad}(x)$ generate the algebra $\mathrm{span}(\mathrm{Ad}(G))$.

(2) The projection of x to every simple factor of G is non-torsion.
(3) The projection of x to G/G' generates a dense subgroup of G/G'.

Condition (1) defines an open dense subset of G, while the set of elements which satisfies (2) and (3) is clearly dense.

Fix $x_n \in X \cap B_{\epsilon_1}(g_n)$ and let W be a word in $n-1$ letters such that $W(\gamma_2, \ldots, \gamma_{n-1}, x_n) \in B_{\epsilon_1}(g_1)\gamma_1^{-1}$. Since Γ is dense in G we can pick $t_n \in \Gamma \cap B_{\epsilon_1}(g_n)$ sufficiently close to x_n so that $W(\gamma_2, \ldots, \gamma_{n-1}, t_n)$ still lies in $B_{\epsilon_1}(g_1)\gamma_1^{-1}$. Then $t_1 := W(\gamma_2, \ldots, \gamma_{n-1}, t_n)\gamma_1$ lies in $B_{\epsilon_1}(g_1)$. It follows that the set

$$\{W_j(t_1, t_n) : j = 1, \ldots, k\}$$

is an ϵ-net in G. Since the metric on G is invariant under right translations, there are $n-2$ (not necessarily distinct) elements of this ϵ-net: $W_{j_i}(t_1, t_n)$, $i = 2, \ldots, n-1$ such that the elements $t_i := W_{j_i}(t_1, t_n)\gamma_i$, $i = 2, \ldots, n-1$ belong to $B_\epsilon(s_i)$ respectively. Since $(t_1, \ldots, t_n)$ was obtained from $(\gamma_1, \ldots, \gamma_{n-1}, t_n)$ by product replacement moves, $(t_1, \ldots, t_n)$ generates Γ and the theorem is proved.

1.4. A CONCRETE EXAMPLE. We will now give a concrete example: Let G be a connected compact Lie group with center Z, and let $a, b, c \in G$ be arbitrary three elements. Slightly deforming a and b to a', b' we may assume that:

(1) a' is regular and non-torsion;
(2) the group $\langle a'Z, b'Z\rangle$ is dense in G/Z.

The centralizer $A = Z_G(a')$ of a' is a maximal torus in G. Since all maximal tori of G are conjugate and their union is equal to G, we can find $\gamma \in \langle a', b'\rangle$ such that $\gamma A \gamma^{-1}$ passes arbitrarily close to c. Then we can slightly deform c to some $c' \in \gamma A \gamma^{-1}$ such that c' and $\gamma a' \gamma^{-1}$ are independent over $\mathbb{Z}$, i.e., $\langle c', \gamma a' \gamma^{-1}\rangle \cong \mathbb{Z}^2$. The group $\langle a', b', c'\rangle$ is not virtually free since it contains a copy of $\mathbb{Z}^2$.

2. The case $n = 2$

The argument above, using the product replacement method, does not apply for $n = 2$. In this case we prove the following:

THEOREM 2.1: *Let G be a compact connected semisimple Lie group, and let $a, b \in G$. There is an arbitrarily small deformation a', b' of a, b such that $\langle a', b'\rangle$*

is dense[2] *in G and has Serre's property (FA). In particular $\langle a', b'\rangle$ is not virtually free.*

Recall that a group Γ has Serre's property (FA) if every action of Γ on a tree admits a global fixed point.

A finitely generated infinite virtually non-abelian-free group, being quasi isometric to F_2, has infinitely many ends, and hence by Stalling's theorem splits over a finite group, and therefore acts minimally on some tree with finite edge stabilizers. Additionally, it is well-known that an infinite virtually cyclic group admits a transitive action on the linear tree. Therefore, an infinite group with property (FA) is not virtually free.

Yves de Cornulier suggested to use the following result of Serre [Se]

LEMMA 2.2 (Serre [Se]): *Let $\Gamma = \langle a, b\rangle$ and suppose that a, b and ab are torsion. Then Γ has property (FA).*

For the sake of completeness let us give a proof for Lemma 2.2: Suppose that Γ acts on a tree T, and let T_a, T_b be the fixed subtrees of a, b, which are non-empty since a and b are torsion. Then $T_a \cap T_b$ is non-empty, for otherwise the segment I connecting T_a to T_b would be oriented with $(ab)\cdot I$ which in turn would imply that ab is hyperbolic, contrary to the assumption that it is torsion.

We will show that any two elements can be deformed to elements which satisfy the conditions of Lemma 2.2 and generate a dense subgroup. For the sake of simplicity let us assume that G is simple. The semisimple case follows easily by considering simultaneously all simple factors of G. By Lemma 1.4 the set of couples which generate a dense subgroup in G is open and dense in $G\times G$. Start with arbitrary $a, b \in G$. Slightly deforming them we may assume that:

(1) both a and b are torsion;
(2) both a and b are regular, in the sense that their centralizers are maximal tori in G;
(3) the group $\langle a, b\rangle$ is dense in G.

We will show that it is possible to deform a and b within their conjugacy class, i.e. find $g, h \in G$ close to the identity, such that $a^g b^h = (gag^{-1})(hbh^{-1})$ would be a torsion element.

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[2] By replacing the word "dense" with the word "infinite" in the conclusion of the theorem, one may assume that G is only non-abelian rather than semisimple.

Denote by $\mathcal{T}_a = Z_G(a)$ the centralizing torus of a, by $\mathcal{T}_b$ the one of b and by $\mathfrak{t}_a, \mathfrak{t}_b$ the corresponding abelian Lie subalgebras of $\mathfrak{g} = \mathrm{Lie}(G)$.

Claim 2.3: $\mathfrak{t}_a \cap \mathfrak{t}_b = \{0\}$.

Since $\mathfrak{t}_a \cap \mathfrak{t}_b$ is in the null space of both $\mathrm{Ad}(a)$ and $\mathrm{Ad}(b)$ while $\langle a, b\rangle$ is dense, the claim follows as the center of $\mathfrak{g}$ is trivial.

For $x \in G$, let $C(x) = \{gxg^{-1} : g \in G\}$ denote the conjugacy class of x. We will derive Theorem 2.1 from the following proposition, in which we do not require that G is compact:

Proposition 2.4: *Let G be a connected simple Lie group and $a, b \in G$ two regular elements in general position, i.e., $\mathfrak{t}_a \cap \mathfrak{t}_b = \{0\}$. The restriction of the product map $G \times G \to G$ to the conjugacy class $C(a) \times C(b)$ of (a, b) is open in a neighbourhood of (a, b).*

Proof. First note that $\mathfrak{t}_a = \ker(\mathrm{Ad}(a) - 1)$ and $\mathfrak{t}_b = \ker(\mathrm{Ad}(b) - 1)$. Now

$$(\mathrm{Ad}(x^{-1}) - 1)(\mathfrak{g}) = \big(\ker(\mathrm{Ad}(x) - 1)\big)^{\perp},$$

where the orthogonal complement is taken with respect to the Killing form, which is non-degenerate as G is simple. Since $\mathfrak{t}_a \cap \mathfrak{t}_b = \{0\}$ it follows that

$$(\mathrm{Ad}(a^{-1}) - 1)(\mathfrak{g}) + (\mathrm{Ad}(b^{-1}) - 1)(\mathfrak{g}) = \mathfrak{g}. \tag{1}$$

Since G admits a faithful linear representation, we my assume that it is linear. Let us compute the tangent space to the conjugacy class $C(a)$ at a, identified via the Lie algebra of left invariant vector fields as a subspace of the tangent space $\mathfrak{g} = T_1(G)$. For $X \in \mathfrak{g}$ we have

$$\frac{d}{dt}\Big|_{t=0}(\exp(tX)a\exp(-tX)) = Xa - aX,$$

and by multiplying from the left by a^{-1} we obtain:

$$L_{a^{-1}}T_a(C(a)) = (\mathrm{Ad}(a^{-1}) - 1)(\mathfrak{g}),$$

where $T_a(C(a))$ denotes the tangent space of $C(a)$ at a viewed as a subspace of $T_a(G)$. Similarly, $L_{b^{-1}}T_b(C(b)) = (\mathrm{Ad}(b^{-1}) - 1)(\mathfrak{g})$.

Finally, the differential of the product map $G \times G \to G$ at (a, b), again identified via a left translation as a subspace of $T_{(1,1)}(G \times G) = \mathfrak{g} \oplus \mathfrak{g}$, evaluated at (X, Y) is easily seen to be $\mathrm{Ad}(b^{-1})(X) + Y$. It follows that the image of the differential at (a, b) of the product map $C(a) \times C(b) \to G$ is

$$\mathrm{Ad}(b^{-1})(\mathrm{Ad}(a^{-1}) - 1)(\mathfrak{g}) + (\mathrm{Ad}(b^{-1}) - 1)(\mathfrak{g}).$$

Since the second summand is $\mathrm{Ad}(b)$-invariant, we derive from (1) that this differential is onto. Therefore, the proposition follows from the implicit function theorem. ∎

Now since the torsion elements are dense in G, we can pick, by Proposition 2.4 and Lemma 1.4, $g, h \in G$ arbitrarily close to 1 such that $\langle a^g, b^h \rangle$ is still dense and $a^g b^h$ is torsion. This completes the proof of Theorem 2.1. ∎

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