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Asymptotic Distribution of Eigenfrequencies for Damped Wave Equations

By

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Abstract

The eigenfrequencies associated to a damped wave equation, are known to belong to a band parallel to the real axis. We establish Weyl asymptotics for the distribution of the real parts of the eigenfrequencies, we show that up to a set of density 0, the eigenfrequencies are confined to a band determined by the Birkhoff limits of the damping coefficient. We also show that certain averages of the imaginary parts converge to the average of the damping coefficient.

Résumé

Il est bien connu que les fréquences propres associées à un d'Alembertien amorti sont confinées dans une bande parallèle à l'axe réel. Nous établissons une asymptotique de Weyl pour la distribution des parties réelles des fréquences propres, nous montrons que "presque toutes" les fréquences propres appartiennent à une bande déterminée par la limite de Birkhoff du coefficient d'amortissement. Nous montrons aussi que certaines moyennes des parties imaginaires convergent vers la moyenne du coefficient d'amortissement.

§0. Introduction

In control theory (see [L]) one is interested in the long time behaviour of solutions to the wave equation with a damping term

(0.1) $$(\partial_t^2 - \Delta + 2a(x)\partial_t)v(t,x) = 0,\ (t,x) \in \mathbf{R} \times M,$$

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for some compact Riemannian manifold M. Here Δ denotes the Laplace Beltrami operator, and a is some bounded real-valued function on M, that we shall assume to be C^∞ for simplicity. Because of the presence of a we will lose the unitary behaviour of the evolution generated by (0.1), and we may have exponential growth or decay of the solutions, when $|t| \to \infty$. At least in principle the growth or decay rates have some relation with the eigen-frequencies of the corresponding stationary problem: Putting $v(t,x) = e^{it\tau}u(x)$, $\tau \in \mathbf{C}$, we are lead to that problem:

$$(-\Delta - \tau^2 + 2ia(x)\tau)u(x) = 0. \tag{0.2}$$

We say that $\tau \in \mathbf{C}$ is an eigenfrequency or an eigen-value for (0.2), if there exist a corresponding non-vanishing distribution u (and actually smooth function, by elliptic regularity), which solves the equation. It is easy to see that the eigenvalues are confined to a band parallel to the real axis. More precisely, if τ is an eigenvalue, then we have

$$\inf a \leq \operatorname{Im}\tau \leq \sup a, \text{ when } \operatorname{Re}\tau \neq 0, \tag{0.3}$$

$$2\min(\inf a, 0) \leq \operatorname{Im}\tau \leq 2\max(\sup a, 0), \text{ when } \operatorname{Re}\tau = 0. \tag{0.4}$$

Using Fredholm theory, we see that the set of eigenvalues is discrete.

G.Lebeau [L] has obtained several results which relate the stationary problem (0.2) and the evolution problem (0.1). P. Freitas [F] has obtained various estimates for the eigenfrequencies of (0.2).

There are three equivalent ways of defining the multiplicity of the eigenvalues. The first one consists in transforming (0.2) into an ordinary eigenvalue problem

$$(\mathcal{P} - \tau)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = 0, \tag{0.5}$$

where

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ -\Delta & 2ia(x) \end{pmatrix} : H^1 \times H^0 \to H^1 \times H^0, \tag{0.6}$$

is elliptic in the Agmon-Douglis-Nirenberg sense, with domain $H^2 \times H^1$. The relation between (0.2) and (0.5) is given by $u_0 = u$, $u_1 = \tau u$. Then the eigenvalues of (0.2) are precisely the eigenvalues of $\mathcal{P}$ and the multiplicity of an eigenvalue τ_0 is then defined to be the rank of the spectral projection $\Pi_{\tau_0} = \frac{1}{2\pi i}\int_\gamma (\tau - \mathcal{P})^{-1} d\tau$, where γ is a sufficiently small circle centered at τ_0.

The second way is to consider a perturbation

$$\widetilde{P}(\tau) = P(\tau) + K(\tau), \tag{0.7}$$

where $P(\tau) = (-\Delta - \tau^2 + 2ia(x)\tau)$ and $K(\tau)$ is of finite rank and depends holomorphically on $\tau \in \operatorname{neigh}(\tau_0, \mathbf{C})$ (i.e. some neighborhood of τ_0 in $\mathbf{C}$), with the property that $\widetilde{P}(\tau_0) : H^2(M) \to H^0(M)$ is bijective. The existence of such a K follows from Fredholm theory. Then the multiplicity of τ_0 is defined as the multiplicity of τ_0 as a zero of the Fredholm determinant $\det(\widetilde{P}(\tau)^{-1}P(\tau))$. This determinant is well defined since $\widetilde{P}(\tau)^{-1}P(\tau) - 1 : L^2 \to L^2$ is of trace class.

The third way is to define the multiplicity of τ_0 as the trace

$$\operatorname{tr} \frac{1}{2\pi i} \int_\gamma P(\tau)^{-1} \partial_\tau P(\tau) d\tau,$$

with γ as above. In the appendix at the end of this introduction, we show that the three notions coincide.

In the following, we shall always count the eigenvalues with their multiplicity. We are interested in the asymptotic distribution of large eigenvalues. Since (0.2) is invariant under the map $(\tau, u) \mapsto (-\overline{\tau}, \overline{u})$, the eigenvalues are situated symmetrically around the imaginary axis, so without loss of generality, we may restrict the attention to the region $\operatorname{Re}\tau \geq 0$.

For $T > 0$, we put

$$\langle a \rangle_T = \frac{1}{2T} \int_{-T}^{T} a \circ \exp(tH_p) dt, \text{ on } p^{-1}(1), \tag{0.8}$$

where $p = \xi^2$ denotes the principal symbol of $-\Delta$ defined on T^*M and H_p is the corresponding Hamilton field. Recall that $\exp tH_p : p^{-1}(1) \to p^{-1}(1)$ can be identified with the geodesic flow on the sphere bundle of M. It is easy to show (see [L] or [S], appendix A) that

$$\begin{aligned} A_+ :&= \inf_{T>0} \sup_{p^{-1}(1)} \langle a \rangle_T = \lim_{T\to\infty} \sup_{p^{-1}(1)} \langle a \rangle_T \\ A_- :&= \sup_{T>0} \inf_{p^{-1}(1)} \langle a \rangle_T = \lim_{T\to\infty} \inf_{p^{-1}(1)} \langle a \rangle_T. \end{aligned} \tag{0.9}$$

G. Lebeau [L] established the following theorem (cf. Rauch-Taylor [RT]).

Theorem 0.0. *For every $\epsilon > 0$, there are at most finitely many eigenvalues outside $\mathbf{R} + i]A_- - \epsilon, A_+ + \epsilon[$.*

Actually in [L], the result is stated only for the eigenvalues in $\mathbf{R} + i]-\infty, A_- - \epsilon]$, and with the assumption that $a \geq 0$ (which corresponds to actual damping). On the other hand Lebeau allows M to have a boundary, and he then takes Dirichlet boundary conditions. It would be interesting to see if the results below extend to the case when M has a boundary.

We are interested in the distribution of eigenvalues inside the band in the above theorem. The next result says that we have Weyl asymptotics with (in general) optimal remainder estimate for the distribution of the real parts. See note added in proof.

Theorem 0.1. *The number of eigenvalues τ with $0 \leq \operatorname{Re} \tau \leq \lambda$ is equal to*

$$\left(\frac{\lambda}{2\pi}\right)^n \left(\iint_{p^{-1}([0,1])} dx d\xi + \mathcal{O}(\lambda^{-1})\right),$$

when $\lambda \to \infty$.

It follows that the number of eigenvalues with $\lambda \leq \operatorname{Re} \tau \leq \lambda + 1$ is $\mathcal{O}(\lambda^{n-1})$, when $\lambda \to +\infty$. In view of the Birkhoff ergodic theorem, the limit

$$\langle a \rangle_\infty := \lim_{T \to \infty} \langle a \rangle_T \tag{0.10}$$

exists on $p^{-1}(1)$ almost everywhere with respect to the flow invariant Liouville measure. The essential supremum and infimum of $\langle a \rangle_\infty$ satisfy

$$A_- \leq \operatorname{ess\,inf} \langle a \rangle_\infty \leq \operatorname{ess\,sup} \langle a \rangle_\infty \leq A_+. \tag{0.11}$$

When the geodesic flow is ergodic, we have equality in the middle, and we may have strict inequality to the left and to the right. In the non-ergodic case, we can find a for which we have strict inequality in the middle. The next result implies that for every $\epsilon > 0$, most of the eigenvalues belong to the band

$$\operatorname{ess\,inf} \langle a \rangle_\infty - \epsilon < \operatorname{Im} \tau < \operatorname{ess\,sup} \langle a \rangle_\infty + \epsilon.$$

Theorem 0.2. *For every $\epsilon > 0$, the number of eigenvalues in $[\lambda, \lambda + 1] + i(\mathbf{R} \backslash]\operatorname{ess\,inf} \langle a \rangle_\infty - \epsilon, \operatorname{ess\,sup} \langle a \rangle_\infty + \epsilon[)$ is $o(\lambda^{n-1})$, when $\lambda \to \infty$.*

Our last result concerns the average distribution of the imaginary parts of the eigenvalues.

Theorem 0.3. *Fix some $C_0 > 1$, and let $\lambda_2 > \lambda_1 \gg 1$ with $\lambda_2/\lambda_1 \leq C_0$, $\lambda_2 - \lambda_1 \geq \log \lambda_1$. Let $N(\lambda_1, \lambda_2)$ denote the number of eigenvalues in $[\lambda_1, \lambda_2] +$*

$i\mathbf{R}$. *Then*

(0.12)
$$\frac{1}{N(\lambda_1,\lambda_2)}\sum_{\tau\in\sigma(P)\cap([\lambda_1,\lambda_2]+i\mathbf{R})}\operatorname{Im}\tau=\frac{1}{\operatorname{vol}(M)}\int_M a(x)dx+\mathcal{O}(1)\frac{\log\lambda_1}{\lambda_2-\lambda_1}.$$

Here dx denotes the Riemann volume element, $\operatorname{vol}(M)=\int_M dx$, *and $\sigma(P)$ denotes the set of eigenvalues.*

The author's interest in the problems of this paper comes from earlier works on resonances, and more precisely certain situations where some part of the resonances are captured in some band like domain, isolated from the other resonances. This happens in the case of the exterior problem for strictly convex obstacles as was established by Zworski and the author in [SZ]. Theorem 0.1 can be viewed as an analogue of the main result of that paper in a technically easier situation, and we have used some ideas of the proof in [SZ] (and the strategy of [S2]). Similarly, Theorem 0.2, can be viewed as an analogue of a result of [S] for resonances in the case of strictly convex obstacles with analytic boundary. In that work we approached the resonances only from one side, and the problem of getting upper bounds for the density of resonances in a marginal region of the first band of resonances opposite to the real axis, is still open, but perhaps attainable. As far as the author knows, there is no analogue to Theorem 0.3 in the case of strictly convex obstacles.

The plan of the paper is the following. In Section 1, we make a simple reduction to a semi-classical framework, in which we work in the remainder of the paper, and which permits us to establish the results in a more general form. In Section 2, we show how one can average the lower order part of the operator along the trajectories of the principal symbol, by means of simple conjugation with pseudodifferential operators (pseudors from now on), and we explain how to obtain Theorem 0.0 in this way. In Section 3 we discuss certain perturbations of the operator (very similar to those used by J.F.Bony [B]) which are used in Section 4 to prove Theorem 0.2. In Section 5, we make different perturbations of the operator and create gaps in the spectrum. We also estimate the corresponding difference of certain trace integrals along contours in the complex spectral plane. In Section 6, we study the trace integrals for the perturbed operator, and combining this with the results of the preceding section, we obtain semi-classical analogues of Theorem 0.1 and 0.3, from which we also get the versions stated above. At this stage, we also use ideas from [S2], [SZ].

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Appendix

Let $\tau = \tau_0$ be an eigenvalue of (0.2), and write

$$P(\tau) = (-\Delta - \tau^2 + 2ia(x)\tau). \tag{A.1}$$

We recall the three different definitions of multiplicity and show that they are equivalent.

I) Let $K(\tau)$ be a finite rank operator depending holomorphically on $\tau \in \mathrm{neigh}(\tau_0, \mathbf{C})$, and assume that $\widetilde{P}(\tau_0) = H^2(M) \to H^0(M)$ is bijective, where $\widetilde{P}(\tau) = P(\tau) + K(\tau)$. Then we define $m_I(\tau_0) \in \mathbf{N}$ to be the order of vanishing of $\widetilde{D}(\tau) := \det(\widetilde{P}(\tau)^{-1}P(\tau))$ at $\tau = \tau_0$.

II) Define $m_{II}(\tau_0) := \mathrm{tr}\, \frac{1}{2\pi i}\int_\gamma P(\tau)^{-1}\partial_\tau P(\tau) d\tau$, where γ is a sufficiently small circle centered at τ_0. We shall see that the integral is a trace class operator.

III) Define $m_{III}(\tau_0)$ to be the rank of the spectral projection $\Pi_\gamma = (2\pi i)^{-1}\int_\gamma (\tau - \mathcal{P})^{-1} d\tau$, with γ as above, and $\mathcal{P}$ defined in (0.6).

Proposition A.1. *We have $m_I(\tau_0) = m_{II}(\tau_0) = m_{III}(\tau_0)$.*

Proof. We first show the equality of m_I and m_{II}. We have

$$\begin{aligned} m_I(\tau_0) &= \frac{1}{2\pi i}\int_\gamma \widetilde{D}(\tau)^{-1}\partial_\tau \widetilde{D}(\tau) d\tau \\ &= \frac{1}{2\pi i}\mathrm{tr} \int_\gamma (\widetilde{P}(\tau)^{-1}P(\tau))^{-1}\partial_\tau(\widetilde{P}(\tau)^{-1}P(\tau)) d\tau \\ &= \frac{1}{2\pi i}\mathrm{tr}\, (\int P(\tau)^{-1}\partial_\tau P(\tau) d\tau - \int P(\tau)^{-1}(\partial_\tau \widetilde{P}(\tau))\widetilde{P}(\tau)^{-1}P(\tau) d\tau). \end{aligned}$$

It suffices to show that the last integral is of trace class and has trace 0. Since $P(\tau) = \widetilde{P}(\tau) - K(\tau)$, it is equal to

$$\int P(\tau)^{-1}(\partial_\tau \widetilde{P}(\tau))(1 - \widetilde{P}(\tau)^{-1}K(\tau)) d\tau. \tag{A.2}$$

The contribution from $\widetilde{P}(\tau)^{-1}K(\tau)$ is of trace class already before integration, so (A.2) is of trace class precisely when

$$\int (1-\widetilde{P}(\tau)^{-1}K(\tau))P(\tau)^{-1}\partial_\tau \widetilde{P}(\tau)d\tau \tag{A.3}$$

is, and when so, the two integrals have the same trace. But (A.3) is equal to

$$\int_\gamma \widetilde{P}(\tau)^{-1}P(\tau)P(\tau)^{-1}\partial_\tau \widetilde{P}(\tau)d\tau = \int_\gamma \widetilde{P}(\tau)^{-1}\partial_\tau \widetilde{P}(\tau)d\tau = 0, \tag{A.4}$$

where the last equality follows from the fact that $\widetilde{P}(\tau)^{-1}$ is holomorphic inside γ.

It only remains to prove that $m_{II}(\tau_0) = m_{III}(\tau_0)$. A straight forward computation shows that

$$(\tau - \mathcal{P})^{-1} = \begin{pmatrix} P(\tau)^{-1}(2ia-\tau) & -P(\tau)^{-1} \\ P(\tau)^{-1}(2ia\tau - \tau^2) - 1 & -P(\tau)^{-1}\tau \end{pmatrix}. \tag{A.5}$$

We know that $\int_\gamma (\tau - \mathcal{P})^{-1}d\tau$ is of trace class, by spectral Fredholm theory. From (A.5), we get

$$\begin{aligned} m_{III}(\tau_0) &= \operatorname{tr} \frac{1}{2\pi i}\int_\gamma (\tau-\mathcal{P})^{-1}d\tau \\ &= \operatorname{tr}\left(\frac{1}{2\pi i}\int_\gamma P(\tau)^{-1}(2ia-\tau)d\tau - \frac{1}{2\pi i}\int_\gamma P(\tau)^{-1}\tau d\tau\right) \\ &= \operatorname{tr}\left(\frac{1}{2\pi i}\int_\gamma P(\tau)^{-1}\partial_\tau P(\tau)d\tau\right) = m_{II}(\tau_0). \end{aligned}$$

□

§1. Semi-Classical Reduction

Let M be a compact smooth Riemannian manifold of dimension n. Let $a \in C^\infty(M;\mathbf{R})$ and consider

$$(-\Delta - \tau^2 + 2ia(x)\tau)v = 0,\ v \not\equiv 0. \tag{1.1}$$

We recall that the eigenvalues to the problem (1.1) form a discrete set which is invariant under reflexion in the imaginary axis and contained in some band parallel to the real axis. We will only be interested in the eigenvalues τ with large modulus, and by reflexion symmetry, we may restrict the attention to such values with $\operatorname{Re}\tau \gg 1$.

Write $\tau = \lambda/h$ with $|\lambda| \sim 1$, $0 < h \ll 1$, $|\arg \lambda| < \pi/4$, so that

$$(-h^2\Delta - \lambda^2 + 2ia(x)\lambda h)v = 0. \tag{1.2}$$

put $z = \lambda^2$, $\lambda = \sqrt{z}$, so that $|z| \sim 1$, $|\arg z| < \pi/2$ and

$$(\mathcal{P} - z)v = 0, \tag{1.3}$$

$$\mathcal{P} = P + ihQ(z),\ P = -h^2\Delta,\ Q(z) = 2a(x)\sqrt{z}. \tag{1.4}$$

We notice that $Q(z)$ is self adjoint for $z > 0$.

In the following it will be convenient to consider the problem in a more general frame work of pseudors. For $m \in \mathbf{R}$, let $S(\langle\xi\rangle^m) = S^m_{1,0}(\mathbf{R}^{2n}_{x,\xi})$ be the space of $a \in C^\infty(\mathbf{R}^{2n}_{x,\xi})$, such that

$$\partial_x^\alpha \partial_\xi^\beta a = \mathcal{O}(\langle\xi\rangle^{m-|\beta|}),\ \forall \alpha, \beta \in \mathbf{N},\ \langle\xi\rangle = (1+\xi^2)^{1/2}, \tag{1.5}$$

see [H]. This definition extends naturally to symbols defined on T^*M. If a depends on a semi-classical parameter $h \in]0, h_0]$ and possibly on other parameters as well, we require (1.5) to hold uniformly with respect to these parameters. For h dependent symbols, we say that $a \in S_{\mathrm{cl}}(\langle\xi\rangle^m)$ if there exists $a_0 \in S(\langle\xi\rangle^m)$ independent of h such that $a - a_0 \in hS(\langle\xi\rangle^{m-1})$, and we call a_0 the leading symbol or principal symbol of a (and of the corresponding h pseudor, to be defined). If $a = a(x,\xi;h) \in S(\langle\xi\rangle^m)$ on $\mathbf{R}^{2n}$, we let $\mathrm{Op}\,(a) = \mathrm{Op}_h(a) = a(x, hD_x; h)$ be the classical h quantization of a (see (A.3) in the appendix of section 6, for the standard formula). If $a_j \in S(\langle\xi\rangle^{m_j})$, $j = 1, 2$, we have $\mathrm{Op}_h(a_1)\mathrm{Op}_h(a_2) = \mathrm{Op}_h(a)$, where

$$S(\langle\xi\rangle^{m_1+m_2}) \ni a \equiv a_1 a_2 \bmod hS(\langle\xi\rangle^{m_1+m_2-1}).$$

In particular, if $a_j \in S_{\mathrm{cl}}(\langle\xi\rangle^{m_j})$, then $a \in S_{\mathrm{cl}}(\langle\xi\rangle^{m_1+m_2})$ and we have $a_0 = a_{1,0}a_{2,0}$ for the principal symbols. This very standard calculus extends to the case of compact manifolds in the usual way.

In the following, we consider $\mathcal{P} = P + ihQ(z)$, where $P \in \mathrm{Op}_h(S_{\mathrm{cl}}(\langle\xi\rangle^2))$ is formally self-adjoint with (real) principal symbol $p(x,\xi)$, satisfying $dp \neq 0$ on $p^{-1}([\alpha,\beta])$, for some $0 < \alpha < 1 < \beta < +\infty$ and with $p \sim \langle\xi\rangle^2$ for large ξ. (Then P becomes essentially self-adjoint.) Further, we assume that $Q = Q(z) \in \mathrm{Op}_h(S_{\mathrm{cl}}(\langle\xi\rangle))$ depends holomorphically on $z \in \Omega := e^{i]-\theta_0,\theta_0[}]\alpha,\beta[$, for some $\theta_0 \in]0,\pi/4[$. We assume that Q is formally self-adjoint for real z and let $q(z) = q(x,\xi,z)$ denote the principal symbol of $Q(z)$.

§2. Averaging

Lemma 2.1. *We assume that v is a non-trivial solution of* (1.3) *for some $z \in \Omega$. Then*

$$h \inf_{p^{-1}(\operatorname{Re} z)} q(\operatorname{Re} z) - \mathcal{O}(h^2) \leq \operatorname{Im} z \leq h \sup_{p^{-1}(\operatorname{Re} z)} q(\operatorname{Re} z) + \mathcal{O}(h^2). \tag{2.1}$$

Proof. Because of the ellipticity of the operator in (1.3) for large ξ, we see that

$$\|Av\| \leq \mathcal{O}(1)\|v\|, \tag{2.2}$$

if $A \in \operatorname{Op}(S(\langle\xi\rangle^2))$. In particular,

$$Pv,\ Q(z)v = \mathcal{O}(1) \text{ in } L^2, \tag{2.3}$$

and considering $0 = \operatorname{Im}((\mathcal{P} - z)v|v)$, we see that

$$\operatorname{Im} z = \mathcal{O}(h). \tag{2.4}$$

It follows that $(P - \operatorname{Re} z)v = \mathcal{O}(h)$ in L^2.

Choose $\widetilde{q} = \widetilde{q}_{\operatorname{Re} z} \in S(1)$, such that $\widetilde{q}(x,\xi) = q(x,\xi,\operatorname{Re} z)$ on $p^{-1}(\operatorname{Re} z)$, and

$$\inf_{p^{-1}(\operatorname{Re} z)} q(\operatorname{Re} z) \leq \inf \widetilde{q} \leq \sup \widetilde{q} \leq \sup_{p^{-1}(\operatorname{Re} z)} q(\operatorname{Re} z). \tag{2.5}$$

We have

$$q(\operatorname{Re} z) = \widetilde{q}_{\operatorname{Re} z} + k(p - \operatorname{Re} z),$$

with $k \in S(\langle\xi\rangle^{-1})$. If we let $\widetilde{Q}$ and K be h pseudors with $\widetilde{q}$ and k as leading symbols, then

$$\begin{aligned}
0 &= \operatorname{Im}((P + ihQ(z) - z)v|v) = h(Q(\operatorname{Re} z)v|v) + (\mathcal{O}(h^2) - \operatorname{Im} z)\|v\|^2 \\
&= h(\widetilde{Q}v|v) + h(K(P - \operatorname{Re} z)v|v) - (\operatorname{Im} z)\|v\|^2 + \mathcal{O}(h^2)\|v\|^2 \\
&= (h(\widetilde{Q}v|v) - (\operatorname{Im} z)\|v\|^2) + \mathcal{O}(h^2)\|v\|^2.
\end{aligned}$$

The semi-classical version of the sharp Gårding inequality shows that

$$(\inf \widetilde{q} - \mathcal{O}(h))\|v\|^2 \leq (\widetilde{Q}v|v) \leq (\sup \widetilde{q} + \mathcal{O}(h))\|v\|^2,$$

and combining this with the preceding identity and (2.5), we get the desired conclusion. □

We now try to improve (2.1) by conjugating $\mathcal{P} = P + ihQ(z)$ by an elliptic selfadjoint pseudor $A \in \mathrm{Op}(S_{\mathrm{cl}}(1))$ with leading symbol $a = e^g$. We have

$$A^{-1}PA = P + A^{-1}[P, A] = P - ihB, \tag{2.6}$$

where B is an h-pseudor in $\mathrm{Op}(S_{\mathrm{cl}}(\langle\xi\rangle))$ with leading symbol

$$b = a^{-1}\{p, a\} = a^{-1}H_p(a) = H_p(g). \tag{2.7}$$

We get

$$A^{-1}(P + ihQ(z))A = P + ih\mathrm{Op}(q(\mathrm{Re}\, z) - H_p(g)) + h^2R(z), \tag{2.8}$$

with $R(z) \in \mathrm{Op}(S(\langle\xi\rangle))$. The idea is now to choose $g = g_{\mathrm{Re}\, z}$ so that $\sup_{p^{-1}(\mathrm{Re}\, z)}(q(\mathrm{Re}\, z) - H_p(g))$ becomes smaller or so that $\inf_{p^{-1}(\mathrm{Re}\, z)}(q(\mathrm{Re}\, z) - H_p(g))$ becomes larger. Let us first notice that we cannot hope to change very much the long time averages of $q = q(\mathrm{Re}\, z)$ along the H_p trajectories, since

$$\begin{aligned}\langle H_p g\rangle_T := \frac{1}{2T}\int_{-T}^{T} H_p(g)\circ\exp(tH_p)dt &= \frac{1}{2T}\int_{-T}^{T}\frac{d}{dt}(g\circ\exp tH_p)dt\\ &= \frac{1}{2T}(g\circ\exp(TH_p) - g\circ\exp(-TH_p)) = \mathcal{O}\left(\frac{1}{T}\right),\end{aligned}$$

whenever g is fixed. On the other hand, we can replace $q(\mathrm{Re}\, z)$ on $p^{-1}(\mathrm{Re}\, z)$ by its average

$$\langle q\rangle_T = \frac{1}{2T}\int_{-T}^{T} q\circ\exp(tH_p)dt.$$

To see that, we first work on the real axis and try to solve

$$\frac{d}{dt}u(t) = \frac{1}{2T}1_{[-T,T]} * v - v = \left(\frac{1}{2T}1_{[-T,T]} - \delta_0\right) * v, \tag{2.9}$$

and we first solve

$$\frac{df_T}{dt} = \frac{1}{2T}1_{[-T,T]} - \delta_0, \tag{2.10}$$

by means of $f_T(t) = f(t/T)$, and $f(t) = 1_{[-1,0]}(t)(t+1)/2 + 1_{[0,1]}(t)(t-1)/2$. Along an integral curve $\rho(t) = \exp(tH_p)(\rho(0))$ in $p^{-1}(\mathrm{Re}\, z)$, we choose $g_T(\rho(t)) = -f_T * (q\circ\rho)$, so that

$$-\frac{d}{dt}g_T(\rho(t)) = \left(\frac{1}{2T}1_{[-T,T]} - \delta_0\right) * (q\circ\rho).$$

In other terms, with $q = q(\operatorname{Re} z)$, we get on $p^{-1}(\operatorname{Re} z)$:

$$-g_T = \int f_T(s) q \circ \exp((t-s)H_p) ds, \tag{2.11}$$

and

$$H_p(g_T) = q - \langle q \rangle_T. \tag{2.12}$$

Choose $g_T \in S(1)$ (depending also on $\operatorname{Re} z$) satisfying (2.11) on $p^{-1}(\operatorname{Re} z)$. With $A = A_T$ chosen correspondingly, (2.8) gives

$$A_T^{-1}(P + ihQ(z))A_T = P + ih\operatorname{Op}(q_T) + h^2 R_T(z), \tag{2.13}$$

with $R_T \in \operatorname{Op}(S(\langle\xi\rangle))$, and with $q_T \in S(\langle\xi\rangle)$ equal to $\langle q \rangle_T$ on $p^{-1}(\operatorname{Re} z)$. If v is a nontrivial solution of (1.3), we get

$$(A_T^{-1}(P + ihQ)A_T - z)A_T^{-1}v = 0, \tag{2.14}$$

and from the lemma (as well as its proof which takes care of the contribution from $h^2 R_T(z)$) we get

(2.15)
$$h \inf_{p^{-1}(\operatorname{Re} z)} \langle q(\operatorname{Re} z)\rangle_T - \mathcal{O}_T(h^2) \le \operatorname{Im} z \le h \sup_{p^{-1}(\operatorname{Re} z)} \langle q(\operatorname{Re} z)\rangle_T + \mathcal{O}_T(h^2),$$

for non-trivial solutions of (1.3) when $z = \mathcal{O}(h)$.

In Appendix A of [S], it was established that

$$\sup_{T\ge 0} \inf_{p^{-1}(\operatorname{Re} z)} \langle q \rangle_T = \lim_{T\to\infty} \inf_{p^{-1}(\operatorname{Re} z)} \langle q \rangle_T, \tag{2.16}$$

$$\inf_{T\ge 0} \sup_{p^{-1}(\operatorname{Re} z)} \langle q \rangle_T = \lim_{T\to\infty} \sup_{p^{-1}(\operatorname{Re} z)} \langle q \rangle_T. \tag{2.17}$$

The argument there also shows that the limits are locally uniform in $\operatorname{Re} z$. Moreover, if we introduce the a.e. limit given by Birkhoff's ergodic theorem:

$$\langle q \rangle_\infty = \lim_{T\to\infty} \langle q \rangle_T, \tag{2.18}$$

then

(2.19)
$$\lim_{T\to\infty} \inf_{p^{-1}(\operatorname{Re} z)} \langle q \rangle_T \le \operatorname*{inf\,ess}_{p^{-1}(\operatorname{Re} z)} \langle q \rangle_\infty \le \operatorname*{sup\,ess}_{p^{-1}(\operatorname{Re} z)} \langle q \rangle_\infty \le \lim_{T\to\infty} \sup_{p^{-1}(\operatorname{Re} z)} \langle q \rangle_T.$$

It is easy to find examples where the first and the last of these inequalities are strict, we may for instance consider a two-dimensional torus and assume that the support of a is contained in a strip parallel to one of the axes.

From (2.15–17), we get for eigenvalues in (1.3):

(2.20)
$$h(\lim_{T\to\infty}\inf_{p^{-1}(\mathrm{Re}\,z)}\langle q(\mathrm{Re}\,z)\rangle_T - o(1)) \leq \mathrm{Im}\,z \leq h(\lim_{T\to\infty}\sup_{p^{-1}(\mathrm{Re}\,z)}\langle q(\mathrm{Re}\,z)\rangle_T + o(1)),$$

locally uniformly in $\mathrm{Re}\,z$. This result implies Theorem 0.0.

We are interested in bounds on the density of eigenvalues in the marginal regions

$$\mathrm{Im}\,z \leq h(\inf_{p^{-1}(\mathrm{Re}\,z)}\mathrm{ess}\,\langle q\rangle_\infty - \epsilon_0),\ \mathrm{Im}\,z \geq h(\sup_{p^{-1}(\mathrm{Re}\,z)}\mathrm{ess}\,\langle q\rangle_\infty + \epsilon_0),$$

where $\epsilon_0 > 0$ is any fixed number.

§3. Perturbations with Controlled Trace Norm

In the following two sections, we let z vary in a disc of radius $\mathcal{O}(h)$ around some real value, that we take $= 1$ for simplicity. Thus we will work with $z = 1 + \zeta$, $\zeta = \mathcal{O}(h)$. We consider the operator (2.13), that we write

$$\mathcal{P}_T = P + ihQ_T + h^2R_T(z), Q_T = Q_T(1). \tag{3.1}$$

Here R_T is slightly modified compared to the R_T in (2.13) but has the same properties. For simplicity we sometimes drop the subscript T, also for the principal symbol of $Q = Q_T$, that we denote by $q = q_T$. We recall that

$$H_pq = \mathcal{O}(\frac{1}{T}),\ \mathrm{on} p^{-1}(1). \tag{3.2}$$

It follows that

$$\|[P,Q]u\| \leq (\mathcal{O}(\frac{h}{T}) + \mathcal{O}_T(h^2))\|u\| + \mathcal{O}_T(h)\|(P-1)u\|. \tag{3.3}$$

In the new notations, (2.15) becomes

$$h\inf_{p^{-1}(1)} q - \mathcal{O}_T(h^2) \leq \mathrm{Im}\,\zeta \leq h\sup_{p^{-1}(1)} q + \mathcal{O}_T(h^2), \tag{3.4}$$

for non-trivial solutions of

$$(\mathcal{P} - z)u = 0,\ z = 1 + \zeta, \tag{3.5}$$

with $\zeta = \mathcal{O}(h)$.

We now want to make a small perturbation $\widetilde{\mathcal{P}}$ of $\mathcal{P}$, so that the upper bound in (3.4) is improved for solutions of $(\widetilde{\mathcal{P}} - z)u = 0$, and we want a corresponding control over the resolvent of $\widetilde{\mathcal{P}}$. We also want a good control over the norm and the trace class norm of $\widetilde{\mathcal{P}}-\mathcal{P}$. For that, assume that we have constructed a pseudor $\widetilde{Q}\in \mathrm{Op}(S(\langle\xi\rangle))$ with leading symbol $\widetilde{q}$ such that $\widetilde{q}\le q$ on $p^{-1}(1)$ and

$$H_p\widetilde{q}=\mathcal{O}\left(\frac{1}{T}\right) \text{ on } p^{-1}(1). \tag{3.6}$$

For instance, we can take $\widetilde{q}=a(q)$, on $p^{-1}(1)$, where a is real and smooth, $a(E)\le E$, $|a'|\le 1$.

Let $0\le f\in\mathcal{S}(\mathbf{R})$, with $\widehat{f}\in C_0^\infty$, where $\widehat{f}(t)=\int e^{-\imath tE}f(E)dE$ is the Fourier transform. We will assume either that

$$\operatorname{supp}\widehat{f}\subset]-\frac{1}{2}T_{\min}(1),\frac{1}{2}T_{\min}(1)[, \tag{3.7}$$

where $T_{\min}(1)$ is the smallest possible period >0 of a closed H_p trajectory in $p^{-1}(1)$, or that

(3.8) the union of all closed H_p trajectories in $p^{-1}(1)$ is of measure 0.

Put

$$\widetilde{\mathcal{P}}=P+ih\widehat{Q}+h^2R_T(z), \tag{3.9}$$

with

$$\widehat{Q}=Q+f\left(\frac{P-1}{h}\right)(\widetilde{Q}-Q)f\left(\frac{P-1}{h}\right). \tag{3.10}$$

Notice that

$$\|(\widetilde{Q}-Q)u\|\le(\sup_{p^{-1}(1)}(q-\widetilde{q})+\mathcal{O}_T(h))\|u\|+\mathcal{O}_T(1)\|(P-1)u\|,$$

and that $\|(P-1)f(\frac{P-1}{h})\|=\mathcal{O}(h)$. It follows that

$$\begin{aligned}\left\|f\left(\frac{P-1}{h}\right)(\widetilde{Q}-Q)f\left(\frac{P-1}{h}\right)\right\|&\\ \le\|f\|_\infty^2\sup_{p^{-1}(1)}(q-\widetilde{q})&+\mathcal{O}_{f,T}(h),\ h\to 0.\end{aligned} \tag{3.11}$$

For the trace class norm, we notice that by the sharp Gårding inequality, $f(\frac{P-1}{h})(Q-\widetilde{Q}+Ch)f(\frac{P-1}{h})\ge 0$, if $C=C_{f,T}>0$ is sufficiently large. Hence,

$$
\begin{aligned}
(3.12)\quad & \left\| f\left(\frac{P-1}{h}\right)(Q-\widetilde{Q})f\left(\frac{P-1}{h}\right)\right\|_{\mathrm{tr}} \\
& \le \left\| f\left(\frac{P-1}{h}\right)(Q-\widetilde{Q}+Ch)f\left(\frac{P-1}{h}\right)\right\|_{\mathrm{tr}} + Ch\left\| f\left(\frac{P-1}{h}\right)^2\right\|_{\mathrm{tr}} \\
& \le \mathrm{tr}\left(f\left(\frac{P-1}{h}\right)(Q-\widetilde{Q}+Ch)f\left(\frac{P-1}{h}\right)\right) + \mathcal{O}_{f,T}(h^{2-n}) \\
& \le \mathrm{tr}\left(f\left(\frac{P-1}{h}\right)(Q-\widetilde{Q})f\left(\frac{P-1}{h}\right)\right) + \mathcal{O}_{f,T}(h^{2-n}).
\end{aligned}
$$

Here

$$
\begin{aligned}
\mathrm{tr}\left(f\left(\frac{P-1}{h}\right)(Q-\widetilde{Q})f\left(\frac{P-1}{h}\right)\right) &= \mathrm{tr}\, f\left(\frac{P-1}{h}\right)^2 (Q-\widetilde{Q}) \\
&= \mathrm{tr}\, \frac{1}{2\pi}\int \widehat{f^2}(t) e^{it\frac{P-1}{h}}(Q-\widetilde{Q})dt,
\end{aligned}
$$

which under one of the assumptions (3.7), (3.8) is equal to

$$
(3.13)\qquad C_n h^{1-n}\left(\int_{p^{-1}(1)} (q-\widetilde{q})L_0(d\rho)\widehat{f^2}(0) + o_{f,T}(1)\right),\ h\to 0,
$$

where $C_n > 0$ only depends on the dimension n of M and L_0 is the Liouville measure on $p^{-1}(1)$. (See for instance [DS] for this classical fact.) Further, $\widehat{f^2}(0) = \|f\|^2_{L^2}$, so combining this with (3.12), we get

$$
\begin{aligned}
(3.14)\quad & \left\| f\left(\frac{P-1}{h}\right)(Q-\widetilde{Q})f\left(\frac{P-1}{h}\right)\right\|_{\mathrm{tr}} \\
& \le C_n h^{1-n}\int_{p^{-1}(1)} (q-\widetilde{q})L_0(d\rho)\|f\|^2_{L^2} + o_{f,T}(1)h^{1-n}.
\end{aligned}
$$

(3.11) and (3.14) provide estimates for the operator and trace norms of

$$
\widetilde{\mathcal{P}} - \mathcal{P} = ihf\left(\frac{P-1}{h}\right)(\widetilde{Q}-Q)f\left(\frac{P-1}{h}\right).
$$

We now study the invertibility of $z-\widetilde{\mathcal{P}}$. If A, B are bounded self-adjoint operators, we have

$$
\|(A+iB)u\|^2 = \|Au\|^2 + \|Bu\|^2 + i([A,B]u|u),
$$

and applying this to (3.9), we get

$$
\begin{aligned}
&(3.15)\\
&2\|(\widetilde{\mathcal{P}}-z)u\|^2 \geq \|(P+ih\widehat{Q}-z)u\|^2 - \mathcal{O}_T(h^4)(\|(P-1)u\|^2+\|u\|^2)\\
&\geq \|(P-\operatorname{Re} z)u\|^2 + h^2\left\|\left(\frac{\operatorname{Im} z}{h}-\widehat{Q}\right)u\right\|^2 + ih([P,\widehat{Q}]u|u)\\
&\quad -\mathcal{O}_T(h^4)(\|(P-1)u\|^2+\|u\|^2)\\
&= \|(P-\operatorname{Re} z)u\|^2 + h^2\left\|\left(\frac{\operatorname{Im} z}{h}-\widehat{Q}\right)u\right\|^2\\
&\quad +(\mathcal{O}(1)\frac{h^2}{T}(1+\|f\|_{L^\infty}^2)+\mathcal{O}_{f,T}(h^3))\|u\|^2 - \mathcal{O}_T(h^2)\|(P-\operatorname{Re} z)u\|^2.
\end{aligned}
$$

Here we also used that $z-1=\mathcal{O}(h)$. This implies (for h small enough depending on T) that

$$
(3.16)\quad \|(P-\operatorname{Re} z)u\| \leq \sqrt{3}\|(\widetilde{\mathcal{P}}-z)u\| + \left(\mathcal{O}_f(1)\frac{h}{\sqrt{T}}+\mathcal{O}_{f,T}(1)h^{3/2}\right)\|u\|.
$$

On the other hand, we have

$$
\begin{aligned}
(3.17)\quad &\operatorname{Im}\left(\frac{1}{h}(z-\widetilde{\mathcal{P}})u|u\right)\\
&= \left(\left(\frac{\operatorname{Im} z}{h}-\widehat{Q}\right)u|u\right) + \mathcal{O}_T(h)(\|u\|+\|(\operatorname{Re} z-P)u\|)\|u\|\\
&= \left(\left(\frac{\operatorname{Im} z}{h}-Q+f\left(\frac{P-1}{h}\right)(Q-\widetilde{Q})f\left(\frac{P-1}{h}\right)\right)u|u\right)\\
&\quad +\mathcal{O}_T(h)(\|u\|+\|(\operatorname{Re} z-P)u\|)\|u\|.
\end{aligned}
$$

Here we would like to eliminate $f(\frac{P-1}{h})$ and for that purpose we factorize

$$
(3.18)\qquad f(\lambda)=f(\mu)+g_\mu(\lambda)(\lambda-\mu),
$$

so that $g_\mu(\lambda)=\int_0^1 f'(\mu+t(\lambda-\mu))dt$ and $|g_\mu(\lambda)|\leq \min(\|f'\|_{L^\infty}, 2\|f\|_{L^\infty}/|\lambda|)$. With $g(\lambda)=g_{\frac{\operatorname{Re} z-1}{h}}(\lambda)=g_{\frac{\operatorname{Re}\zeta}{h}}(\lambda)$, we get

$$
\begin{aligned}
(3.19)\quad &\left|\left(f\left(\frac{P-1}{h}\right)(Q-\widetilde{Q})f\left(\frac{P-1}{h}\right)u|u\right) - f\left(\frac{\operatorname{Re}\zeta}{h}\right)^2((Q-\widetilde{Q})u|u)\right|\\
&= \left|\left((Q-\widetilde{Q})f\left(\frac{P-1}{h}\right)u|f\left(\frac{P-1}{h}\right)u\right) - f\left(\frac{\operatorname{Re}\zeta}{h}\right)^2((Q-\widetilde{Q})u|u)\right|\\
&\leq \left|\left((Q-\widetilde{Q})g\left(\frac{P-1}{h}\right)\frac{P-\operatorname{Re} z}{h}u|f\left(\frac{P-1}{h}\right)u\right)\right|\\
&\quad + \left|\left((Q-\widetilde{Q})f\left(\frac{\operatorname{Re}\zeta}{h}\right)u|g\left(\frac{P-1}{h}\right)\frac{P-\operatorname{Re} z}{h}u\right)\right|\\
&\leq (2\sup_{p^{-1}(1)}(q-\widetilde{q})\|f\|_\infty\|f'\|_\infty+\mathcal{O}_{f,T}(h))\|u\|\left\|\frac{P-\operatorname{Re} z}{h}u\right\|.
\end{aligned}
$$

Use this in (3.17)

$$
\begin{aligned}
&\operatorname{Im}\left(\frac{1}{h}(z-\widetilde{\mathcal{P}})u|u\right) \geq \\
&\left(\left(\frac{\operatorname{Im}\zeta}{h} - Q + f\left(\frac{\operatorname{Re}\zeta}{h}\right)^2 (Q-\widetilde{Q})\right)u|u\right) \\
&-2(\sup_{p^{-1}(1)}(q-\widetilde{q})\|f\|_\infty\|f'\|_\infty + \mathcal{O}_{f,T}(h))\|u\|\|\frac{1}{h}(P-\operatorname{Re} z)u\| - \mathcal{O}_{f,T}(h)\|u\|^2.
\end{aligned}
\tag{3.20}
$$

Let $\alpha(E) > 0$, be a continuous function defined on a bounded interval J containing 0 and restrict z by assuming that

$$
\frac{\operatorname{Im}\zeta}{h} - q + f\left(\frac{\operatorname{Re}\zeta}{h}\right)^2 (q-\widetilde{q}) \geq \alpha\left(\frac{\operatorname{Re}\zeta}{h}\right), \text{on } p^{-1}(\operatorname{Re} z), \frac{\operatorname{Re}\zeta}{h} \in J.
\tag{3.21}
$$

(Notice that from (3.21), we get the same lower bound on $p^{-1}(1)$, provided that α is replaced by $\alpha - \mathcal{O}(h)$.) Then for the same z and for $(x,\xi) \in T^*M$:

$$
\frac{\operatorname{Im}\zeta}{h} - q + f\left(\frac{\operatorname{Re}\zeta}{h}\right)^2 (q-\widetilde{q}) = \alpha\left(\frac{\operatorname{Re}\zeta}{h}\right) + \beta_{\frac{\operatorname{Re}\zeta}{h}}(x,\xi) + \gamma_{\frac{\operatorname{Re}\zeta}{h}}(x,\xi)(p-\operatorname{Re} z),
\tag{3.22}
$$

where $S(1) \ni \beta_{\frac{\operatorname{Re}\zeta}{h}}(x,\xi) \geq 0$, $\gamma_{\frac{\operatorname{Re}\zeta}{h}} \in S(\langle\xi\rangle^{-1})$, and we deduce from the sharp Gårding inequality that

$$
\begin{aligned}
&\left(\left(\frac{\operatorname{Im}\zeta}{h} - Q + f\left(\frac{\operatorname{Re}\zeta}{h}\right)^2 (Q-\widetilde{Q})\right)u|u\right) \geq \\
&\qquad \left(\alpha\left(\frac{\operatorname{Re}\zeta}{h}\right) - \mathcal{O}_{f,T}(h)\right)\|u\|^2 - \mathcal{O}_{f,T}(1)\|u\|\|(P-\operatorname{Re} z)u\|.
\end{aligned}
\tag{3.23}
$$

Using this in (3.20), we get

$$
\begin{aligned}
\operatorname{Im}\left(\frac{1}{h}(z-\widetilde{\mathcal{P}})u|u\right) &\geq \left(\alpha\left(\frac{\operatorname{Re}\zeta}{h}\right) - \mathcal{O}_{f,T}(h)\right)\|u\|^2 \\
&- (2\sup_{p^{-1}(1)}(q-\widetilde{q})\|f'\|_\infty\|f\|_\infty + \mathcal{O}_{f,T}(1)h)\|u\|\|\frac{1}{h}(P-\operatorname{Re} z)u\|,
\end{aligned}
\tag{3.24}
$$

and using (3.16), we get

$$
\begin{aligned}
(3.25)\quad \operatorname{Im}\left(\frac{1}{h}(z-\widetilde{\mathcal{P}})u|u\right) &\geq \left(\alpha\left(\frac{\operatorname{Re}\zeta}{h}\right)-\mathcal{O}_{f,T}(h)\right)\|u\|^2 \\
&-\sqrt{3}(2\sup_{p^{-1}(1)}(q-\widetilde{q})\|f'\|_\infty\|f\|_\infty+\mathcal{O}_{f,T}(1)h)\|u\|\|\frac{1}{h}(z-\widetilde{\mathcal{P}})u\| \\
&-\left(\mathcal{O}_f(1)\frac{1}{\sqrt{T}}+\mathcal{O}_{f,T}(1)h^{\frac{1}{2}}\right)\|u\|^2.
\end{aligned}
$$

Using also that $\operatorname{Im}(\frac{z-\widetilde{\mathcal{P}}}{h}u|u)\leq\|\frac{1}{h}(z-\widetilde{\mathcal{P}})u\|\|u\|$, we get

$$
\begin{aligned}
(3.26)\quad &\left(\alpha\left(\frac{\operatorname{Re}\zeta}{h}\right)-(\mathcal{O}_f(1)\frac{1}{\sqrt{T}}+\mathcal{O}_{f,T}(1)h^{\frac{1}{2}})\right)\|u\| \\
&\leq(1+2\sqrt{3}\sup_{p^{-1}(1)}(q-\widetilde{q})\|f'\|_\infty\|f\|_\infty+\mathcal{O}_{f,T}(1)h)\|\frac{1}{h}(z-\widetilde{\mathcal{P}})u\|.
\end{aligned}
$$

Since q and $\widetilde{q}$ remain bounded on $p^{-1}(1)$, when $T\to\infty$, we see that for every $\epsilon\in]0,1[$, we can first choose T large enough, then h small enough depending on ϵ,T,α, and get

$$
\begin{aligned}
(3.27)\quad &(1-\epsilon)\alpha\left(\frac{\operatorname{Re}\zeta}{h}\right)\|u\| \\
&\leq(1+2\sqrt{3}\sup_{p^{-1}(1)}(q-\widetilde{q})\|f'\|_\infty\|f\|_\infty+\mathcal{O}_{f,T}(1)h)\|\frac{1}{h}(z-\widetilde{\mathcal{P}})u\|.
\end{aligned}
$$

Summing up the discussion so far, we have

Proposition 3.1. *Let $P\in\operatorname{Op}S_{\mathrm{cl}}(\langle\xi\rangle^2)$ be formally selfadjoint with real principal symbol p, and assume that $p(x,\xi)\sim\langle\xi\rangle^2$ for large ξ and that dp does not vanish on $p^{-1}(1)$. Let $Q=Q(z)\in\operatorname{Op}(S_{\mathrm{cl}}(\langle\xi\rangle))$ with principal symbol $q(z)$ depend holomorphically on $z\in\Omega:=e^{i]-\theta_0,\theta_0[}]\alpha,\beta[$, where $0<\theta_0<\pi/4, 0<\alpha<1<\beta$, and be formally self-adjoint when z is real. Let*

$$\mathcal{P}_T=P+ihQ_T+h^2R_T(z),\ Q_T=Q_T(1),\ z=1+\zeta,\ \zeta=\mathcal{O}(h),$$

be the operator (3.1), so that $Q_T\in\operatorname{Op}(S_{\mathrm{cl}}(\langle\xi\rangle))$ has leading symbol q_T with $q_T=\langle q\rangle_T$ on $p^{-1}(1)$, and $R_T(z)\in\operatorname{Op}(S(h^2\langle\xi\rangle))$. Let $\widetilde{Q}_T\in\operatorname{Op}(S_{\mathrm{cl}}(\langle\xi\rangle))$ have leading symbol $\widetilde{q}_T$ with $\widetilde{q}_T=a\circ q_T$ on $p^{-1}(1)$, where $C^\infty\ni a(t)\leq t, |a'(t)|\leq 1$, and put

$$
(3.28)\quad \widetilde{\mathcal{P}}_T=P+ih\widehat{Q}_T+h^2R_T(z),\ \widehat{Q}_T=Q_T(1)+f\left(\frac{P}{h}\right)(\widetilde{Q}_T-Q_T)f\left(\frac{P}{h}\right),
$$

where $0\leq f\in\mathcal{S}(\mathbf{R})$, $\widehat{f}\in C_0^\infty$.

Then

$$\|\widetilde{\mathcal{P}}_T - \mathcal{P}_T\| \le h(\|f\|_\infty^2 \sup_{p^{-1}(1)} (q_T - \widetilde{q}_T) + \mathcal{O}_{f,T}(h)). \tag{3.29}$$

If we assume either (3.7) *or* (3.8), *then*

$$\|\widetilde{\mathcal{P}}_T - \mathcal{P}_T\|_{\mathrm{tr}} \le C_n h^{1-n} \int_{p^{-1}(1)} (q_T - \widetilde{q}_T) L_0(d\rho)\|f\|_{L^2}^2 + o_{f,T}(1) h^{1-n}. \tag{3.30}$$

If we drop (3.7), (3.8), *but restrict z further by assuming that for some continuous function $\alpha(E) > 0$, defined on some bounded interval J containing* 0, *we have for T large enough*:

$$\frac{\operatorname{Im}\zeta}{h} - q_T + f\left(\frac{\operatorname{Re}\zeta}{h}\right)^2 (q_T - \widetilde{q}_T) \ge \alpha\left(\frac{\operatorname{Re}\zeta}{h}\right), \text{ on } p^{-1}(1), \frac{\operatorname{Re}\zeta}{h} \in J, \tag{3.31}$$

then for every $\epsilon > 0$, $T \ge T(\epsilon) > 0$ and $h \le h(\epsilon, T) > 0$, $(z - \widetilde{\mathcal{P}}_T)^{-1}$ exists, and we have

$$\left\| \left(\frac{1}{h}(z - \widetilde{\mathcal{P}}_T)\right)^{-1} \right\| \le \frac{1 + 2\sqrt{3}\sup_{p^{-1}(1)}(q_T - \widetilde{q}_T)\|f'\|_\infty \|f\|_\infty}{(1-\epsilon)\alpha(\frac{\operatorname{Re}\zeta}{h})}. \tag{3.32}$$

Similar perturbations based on $f(\frac{P-1}{h})$ are used by J.F.Bony [B].

§4. Upper Bounds on the Density of Eigenvalues

We consider the same situation as in Proposition 3.1 and we choose $T > 0$ sufficiently large, $h > 0$ sufficiently small depending on T and possibly other parameters as well. Let $\omega = \omega(E)$, $w = w(E)$ be continuous functions on $\mathbf{R}$, independent of T, such that

$$w(E) > \omega(E) > \frac{1}{C} + \sup_{p^{-1}(1)}(q_T - f(E)^2(q_T - \widetilde{q}_T)),\ E \in \mathbf{R}, \tag{4.1}$$

$$\omega(E) > \sup_{p^{-1}(1)}(q_T) + \frac{1}{C}, \text{ when } |E| \ge C, \tag{4.2}$$

for some fixed constant $C \ge 1$. With $D(z_0, r) = \{z \in \mathbf{C}; |z - z_0| < r\}$, put

$$\Omega = \left\{\zeta \in D(0, 2Ch);\ \omega\left(\frac{\operatorname{Re}\zeta}{h}\right) < \frac{\operatorname{Im}\zeta}{h}\right\}, \tag{4.3}$$

$$W = \left\{\zeta \in D(0, 2Ch);\ w\left(\frac{\operatorname{Re}\zeta}{h}\right) < \frac{\operatorname{Im}\zeta}{h}\right\}. \tag{4.4}$$

Notice that $\Omega = h\Omega_1$, $W = hW_1$, where W_1, Ω_1 are independent of h.

For $0 < \epsilon < 2\epsilon_0$, let $\Omega_{+,\epsilon}$, $W_{+,\epsilon}$ denote the intersections of Ω and W respectively with $\{\zeta \in \mathbf{C};\ \mathrm{Im}\,\zeta/h > \sup_{p^{-1}(1)}(q_T) + \epsilon\}$. For $\zeta = z - 1 \in \Omega_{+,\epsilon_0}$ and T sufficiently large, we have

$$\|h(z - \widetilde{\mathcal{P}}_T)^{-1}\|, \|h(z - \mathcal{P}_T)^{-1}\| \leq \frac{2}{\epsilon_0}. \tag{4.5}$$

For $\widetilde{\mathcal{P}}_T$ this follows from Proposition 3.1 and for $\mathcal{P}_T$ a simplified version of the proof gives the same fact (or else we can put $f = 0$ in that proposition).

Let $\alpha(E) \geq$ Const. > 0 be a continuous function, independent of T, such that (cf. (4.1))

$$\omega(E) \geq \alpha(E) + \sup_{p^{-1}(1)} (q_T - f(E)^2(q_T - \widetilde{q}_T)),\ E \in \mathbf{R}. \tag{4.6}$$

Then by Proposition 3.1, we have for $\zeta = z - 1 \in \Omega$:

$$\|h(z - \widetilde{\mathcal{P}}_T)^{-1}\| \leq \frac{2(1 + \sqrt{3}\sup_{p^{-1}(1)}(q_T - \widetilde{q}_T)\|f'\|_\infty \|f\|_\infty)}{\inf \alpha(E)} =: D_T, \tag{4.7}$$

when h is small enough (depending on T).

Also recall from Proposition 3.1 that

$$\left\|\frac{1}{h}(\mathcal{P}_T - \widetilde{\mathcal{P}}_T)\right\| \leq \|f\|_\infty^2 \sup_{p^{-1}(1)} (q_T - \widetilde{q}_T) + \mathcal{O}_{f,T}(h) =: A_T, \tag{4.8}$$

$$\left\|\frac{1}{h}(\mathcal{P}_T - \widetilde{\mathcal{P}}_T)\right\|_{\mathrm{tr}} \leq C_n h^{1-n} \left(\|f\|_{L^2}^2 \int_{p^{-1}(1)} (q_T - \widetilde{q}_T) L_0(d\rho) + o_{f.T}(1) \right) =: B_T h^{1-n}. \tag{4.9}$$

For $z - 1 \in \Omega$, we write

$$(z - \mathcal{P}_T) = (z - \widetilde{\mathcal{P}}_T)(1 - K(z)), K(z) = (z - \widetilde{\mathcal{P}}_T)^{-1}(\mathcal{P}_T - \widetilde{\mathcal{P}}_T). \tag{4.10}$$

Here $K(z)$ is of trace class with

$$\|K\|_{\mathrm{tr}} \leq D_T B_T h^{1-n}. \tag{4.11}$$

It follows that z is an eigenvalue of $\mathcal{P}_T(z)$ precisely when

$$\mathcal{D}(z) := \det(1 - K(z)) \tag{4.12}$$

vanishes. Let us define the multiplicity of such an eigenvalue as the corresponding multiplicity of the zero of $\mathcal{D}$ (following one of the equivalent definitions discussed in Section 0). This multiplicity is independent of the choice of $\widetilde{\mathcal{P}}_T$, as well as of T. (Recall from the dicussion in Section 2, that z is an eigenvalue of $\mathcal{P}_T(z)$ iff it is an eigenvalue of $\mathcal{P}(z) = P + ihQ(z)$.)

From (4.11), and a general estimate on Fredholm determinants (see [GK]), we get the upper bound

$$|\mathcal{D}(z)| \leq \exp \|K\|_{\mathrm{tr}} \leq \exp(B_T D_T h^{1-n}). \tag{4.13}$$

On the other hand, for $z - 1 \in \Omega_{+,\epsilon_0}$, we have

$$(1 - K(z))^{-1} = (z - \mathcal{P}_T)^{-1}(z - \widetilde{\mathcal{P}}_T) = 1 + (z - \mathcal{P}_T)^{-1}(\mathcal{P}_T - \widetilde{\mathcal{P}}_T),$$

so

$$\|(1 - K(z))^{-1}\| \leq 1 + \frac{2A_T}{\epsilon_0}. \tag{4.14}$$

Write

$$(1 - K)^{-1} = 1 + K(1 - K)^{-1},$$

and observe that

$$\|K(1 - K)^{-1}\|_{\mathrm{tr}} \leq \|K\|_{\mathrm{tr}}\|(1 - K)^{-1}\| \leq D_T B_T h^{1-n}\left(1 + \frac{2A_T}{\epsilon_0}\right).$$

Consequently, for $z - 1 \in \Omega_{+,\epsilon_0}$:

$$\begin{aligned} |\mathcal{D}(z)|^{-1} &= |\det(1 - K)^{-1}| \\ &= |\det(1 + K(1 - K)^{-1})| \\ &\leq \exp\left(D_T\left(1 + \frac{2A_T}{\epsilon_0}\right) B_T h^{1-n}\right). \end{aligned} \tag{4.15}$$

So we have

$$\log|\mathcal{D}(z)| \leq D_T B_T h^{1-n}, \ z - 1 \in \Omega, \tag{4.16}$$

$$\log|\mathcal{D}(z)| \geq -D_T\left(1 + \frac{2A_T}{\epsilon_0}\right) B_T h^{1-n}, \ z - 1 \in \Omega_{+,\epsilon_0}. \tag{4.17}$$

Recall that $\log|\mathcal{D}(z)|$ is subharmonic and that $\Delta_z \log|\mathcal{D}(z)| = 2\pi \sum \delta(z - z_j)$, where z_j are the eigenvalues counted with their multiplicity and δ denotes the Dirac measure. It follows either by Jensen's inequality, or by working more directly with the Green and Poisson kernels for Ω, that if $N = N(W)$ is the number of zeros of $\mathcal{D}(z)$ in $1 + W$, then

$$N(W) \leq C(\Omega_1, W_1) D_T\left(1 + \frac{A_T}{\epsilon_0}\right) B_T h^{1-n}. \tag{4.18}$$

Here B_T is the most interesting constant (cf. (4.9)).

Recall that on $p^{-1}(1)$, we have

$$q_T = \langle q \rangle_T,\ \widetilde{q}_T = a(\langle q \rangle_T),\ q = q(1), \tag{4.19}$$

where $a(E) \leq E$, $|a'(E)| \leq 1$, $a \in C^\infty$. Also recall that $\langle q \rangle_\infty = \lim_{T\to\infty} \langle q \rangle_T$ is the a.e. limit in Birkhoff's ergodic theorem, and that

$$\operatorname{ess\,sup} \langle q \rangle_\infty \leq \lim_{T\to\infty} \sup \langle q \rangle_T. \tag{4.20}$$

Assume that we have strict inequality in (4.20) and choose constants α, β with

$$\operatorname{ess\,sup} \langle q \rangle_\infty < \alpha < \beta < \lim_{T\to\infty} \sup \langle q \rangle_T. \tag{4.21}$$

Let $a = a_{\alpha,\beta}$ be increasing with

$$a_{\alpha,\beta}(E) = E \text{ for } E \leq \alpha,\ a_{\alpha.\beta}(E) \leq \beta, \tag{4.22}$$

Choose f in (4.1) with

$$f(E) \geq 1,\ |E| \leq C. \tag{4.23}$$

Then for $|E| \leq C$, we have on $p^{-1}(1)$

$$q_T - f(E)^2(q_T - \widetilde{q}_T) \leq q_T - (q_T - \widetilde{q}_T) = \widetilde{q}_T = a(\langle q \rangle_T) \leq \beta, \tag{4.24}$$

while for $|E| > C$, we have

$$q_T - f(E)^2(q_T - \widetilde{q}_T) \leq \langle q \rangle_T. \tag{4.25}$$

Choose $w(E), \omega(E)$ continuous on $\mathbf{R}$, such that

$$w(E) > \omega(E) > \epsilon_1 + \beta 1_{[-C,C]}(E) + (\lim_{T\to\infty} \sup \langle q \rangle_T) 1_{\mathbf{R}\setminus[-C,C]}(E), \tag{4.26}$$

for some fixed but arbitrarily small $\epsilon_1 > 0$. Then (4.24,25) imply that (4.1,2) hold if T is large enough, and we can find a corresponding function $\alpha(E)$ in (4.6).

The constants D_T, A_T are bounded by some T independent constant. As for B_T, we notice that

$$\int_{p^{-1}(1)} (\langle q \rangle_T - a(\langle q \rangle_T)) L_0(d\rho) \to \int_{p^{-1}(1)} (\langle q \rangle_\infty - a(\langle q \rangle_\infty)) L_0(d\rho) = 0,$$

by the dominated convergence theorem, since $a(\langle q\rangle_\infty)=\langle q\rangle_\infty$ a.e. We conclude from (4.18) that

$$N(W)=o(1)h^{1-n}. \tag{4.27}$$

If $\widetilde{\beta}\in]\mathrm{ess\,sup}\,\langle q\rangle_\infty, \lim_{T\to\infty}\sup\langle q\rangle_T[$, then we can choose α,β in (4.21) with $\beta<\widetilde{\beta}$ and for every $0<\widetilde{C}<C$, we can choose $w(E)$ and ϵ_1 in (4.26), such that $w(E)<\widetilde{\beta}1_{[-\widetilde{C},\widetilde{C}]}(E)$. We have then showed that the number of eigenvalues $z=1+\zeta$ of P with $|\mathrm{Re}\,\zeta/h|<\widetilde{C}$, $\mathrm{Im}\,z/h>\widetilde{\beta}$, (and $\zeta=\mathcal{O}(h)$) is $o(1)h^{1-n}$.

The analogous result holds for the number of eigenvalues with

$$\left|\frac{\mathrm{Re}\,\zeta}{h}\right|<\widetilde{C},\ \frac{\mathrm{Im}\,\zeta}{h}<\widehat{\beta}<\mathrm{ess\,inf}\,\langle q\rangle_\infty,$$

and we get the semiclassical version of the main theorem. It suffices to apply the reduction in Section 1, to get Theorem 0.2.

§5. Comparison with an Operator with Gaps in the Spectrum

As in Section 1, we consider

$$(\mathcal{P}-z)v=0, \tag{5.1}$$

$$\mathcal{P}=P+ihQ(z),\ P=-h^2\Delta, Q(z)=2a(x)\sqrt{z}. \tag{5.2}$$

We notice that $Q(z)$ is selfadjoint for $z>0$.

In the following, we shall only use that $Q\in\mathrm{Op}\,(S_{\mathrm{cl}}(\langle\xi\rangle))$ depends holomophically on $z\in\Omega$, defined after (1.4), and that $P\in\mathrm{Op}\,(S_{\mathrm{cl}}(\langle\xi\rangle^2))$ is formally selfadjoint and has the properties that $dp(x,\xi)\neq 0$, on $p^{-1}([\alpha,\beta])$ and $p(x,\xi)\sim\langle\xi\rangle^2$ for large ξ. Notice that P is essentially selfadjoint with domain $H^2(M)$.

Fix $C_0>1$ and let $\alpha+\frac{1}{C_0}\le E_1<E_2\le\beta-\frac{1}{C_0}$ satisfy $E_2-E_1\ge 4h$. So E_j may depend on h. It will be convenient to introduce

$$E_0=\frac{E_1+E_2}{2},\ r_0=\frac{E_2-E_1}{2}.$$

Lemma 5.1. *For every $C>0$, there exists a self-adjoint operator $\widetilde{P}$ with the same domain as P, such that*

$$(E_j+[-Ch,Ch])\cap\sigma(\widetilde{P})=\emptyset,\ j=1,2, \tag{5.3}$$

$$\|P-\widetilde{P}\|\le Ch, \tag{5.4}$$

$$\|P-\widetilde{P}\|_{\mathrm{tr}}\le\widetilde{C}(C)h^{2-n}. \tag{5.5}$$

Proof. This is a direct consequence of the fact that the number of eigenvalues of P in $E_j + [-Ch, Ch]$ is $\mathcal{O}(1)h^{1-n}$. □

Write

$$\widetilde{P} = P + h\delta P,\ \widetilde{\mathcal{P}} = \widetilde{P} + ihQ(z), \tag{5.6}$$

so that

$$\|\delta P\| \leq C,\ \|\delta P\|_{\mathrm{tr}} \leq \widetilde{C}(C)h^{1-n}. \tag{5.7}$$

If we choose C large enough, we can arrange so that $(z - \widetilde{\mathcal{P}})^{-1}$ exists and satisfies

$$\|(z - \widetilde{\mathcal{P}})^{-1}\| \leq \frac{\mathcal{O}(1)}{h + |\mathrm{Im}\, z|}, \tag{5.8}$$

for z in the region

$$D(E_0; r_0 - 2h, r_0 + 2h) \cup \{z \in D(E_0; r_0 + 2h);\ |\mathrm{Im}\, z| \geq Ch\}. \tag{5.9}$$

Here we denote by $D(z_0; r, R)$ the open annulus $\{z \in \mathbf{C};\ r < |z - z_0| < R\}$. For simplicity we shall assume that $D(E_0; r_0 + 2h)$ is contained in $\Omega = e^{i]-\theta_0,\theta_0[}]\alpha, \beta[$. In the following, everything works without this extra assumption, if we replace certain sets in the complex plane by their images under the map $\mathbf{C} \ni E \mapsto \mathrm{Re}\, E + i\kappa \mathrm{Im}\, E$, for some fixed $\kappa > 0$ which is small enough.

Write

$$z - \mathcal{P} = (z - \widetilde{\mathcal{P}})(1 + h(z - \widetilde{\mathcal{P}})^{-1}\delta P), \tag{5.10}$$

and put

$$D(z) = \det(1 + h(z - \widetilde{\mathcal{P}})^{-1}\delta P)$$

for z in the domain (5.9). Notice that

$$\|h(z - \widetilde{\mathcal{P}})^{-1}\delta P\|_{\mathrm{tr}} \leq \frac{C_1 h^{2-n}}{h + |\mathrm{Im}\, z|}, \tag{5.11}$$

for z in (5.9). It follows that

$$|D(z)| \leq \exp(C_1 h^{2-n}/(h + |\mathrm{Im}\, z|)) \tag{5.12}$$

in the same region. If we restrict z to the subset of points of (5.9) with $|\mathrm{Im}\, z| > Ch$, we may assume that

$$\|(z - \mathcal{P})^{-1}\| \leq \frac{C_0}{h + |\mathrm{Im}\, z|}, \tag{5.13}$$

and from (5.10), we get

$$(5.14)\qquad (1+h(z-\widetilde{\mathcal{P}})^{-1}\delta P)^{-1}=(z-\mathcal{P})^{-1}(z-\widetilde{\mathcal{P}})=1-h(z-\mathcal{P})^{-1}\delta P,$$

with

$$\|h(z-\mathcal{P})^{-1}\delta P\|_{\mathrm{tr}}\le C_1h^{2-n}/(h+|\operatorname{Im}z|).$$

The determinant of (5.14) is equal to $1/D(z)$, which leads to

$$(5.15)\qquad |D(z)|\ge \exp(-C_1h^{2-n}/(h+|\operatorname{Im}z|)).$$

Let z_j, $j=1,..,N$ be the eigenvalues of $\mathcal{P}$ in

$$(5.16)\qquad D(E_0,r_0-h,r_0+h).$$

We know from section 4 that $N=\mathcal{O}(1)h^{1-n}$, and we have just seen that we also have $|\operatorname{Im}z_j|\le Ch$. (Actually, the bound on N could easily be rederived here following the usual method.) Of course, we count the z_j with their multiplicities as zeros of $D(z)$. For each z_j, let $\widetilde{b}_{z_j}(z)$ be the corresponding Blaschke factor on $D(E_0,r_0+3h/2)$, defined by

$$(5.17)\qquad \widetilde{b}_{z_j}(z)=b_{\frac{z_j-E_0}{r_0+3h/2}}\left(\frac{z-E_0}{r_0+3h/2}\right).$$

See the appendix of this section. We then know that

$$(5.18)\qquad |\widetilde{b}_{z_j}(z)|\sim\frac{|z-z_j|}{h},\ |z-z_j|\le\mathcal{O}(1)h,$$

$$(5.19)\qquad |\widetilde{b}_{z_j}(z)|=1+\mathcal{O}(1)\frac{h}{h+|z-z_j|},\ |z-z_j|\ge\mathcal{O}(1)h.$$

Let

$$(5.20)\qquad D_b(z)=\prod_{j=1}^{N}\widetilde{b}_{z_j}(z),$$

be the corresponding Blaschke product. Then, if we restrict the attention to the set $D(E_0,r_0+h)$, we get

$$(5.21)\qquad \exp\left(-\mathcal{O}(1)\frac{h^{2-n}}{h+|\operatorname{Im}z|}\right)\le|D_b(z)|,\ |\operatorname{Im}z|>Ch,$$

$$(5.22)\qquad |D_b(z)|\le 1,\ \text{in general}.$$

Now write

$$D(z) = G(z)D_b(z),\ z \in D(E_0, r_0 + h), \tag{5.23}$$

so that $G(z) \neq 0$ in the region analogous to (5.9):

$$D(E_0, r_0 - h, r_0 + h) \cup \{z \in D(E_0, r_0 + h); |\mathrm{Im}\, z| \geq Ch\}. \tag{5.24}$$

Combining (5.21,22,15,12) we get

$$|\log|G(z)|| \leq \mathcal{O}(1)\frac{h^{2-n}}{h + |\mathrm{Im}\, z|} \tag{5.25}$$

in $\{z \in D(E_0, r_0 + h);\ |\mathrm{Im}\, z| > Ch\}$.

As in [S2], we see that if $-h \leq a < b \leq h$ with $b - a \geq h/\mathcal{O}(1)$, then there exists $c \in]a, b[$, such that (5.21) holds everywhere on the circle $\partial D(E_0, r_0 + c)$. It follows from this and (5.12), that $\log|G(z)| \leq \mathcal{O}(1)h^{2-n}/(h + |\mathrm{Im}\, z|)$ on the same circle. Since $\log|G(z)|$ is harmonic, we can use the maximum principle to conclude that

$$\log|G(z)| \leq \mathcal{O}(1)\frac{h^{2-n}}{h + |\mathrm{Im}\, z|}, z \in D(E_0, r_0 - 3h/4, r_0 + 3h/4). \tag{5.26}$$

Combining (5.25,26) with Harnack's inequality (as in [S2]), we get

(5.27)

$$|\log|G(z)|| \leq \mathcal{O}(1)\frac{h^{2-n}}{h + |\mathrm{Im}\, z|},$$
$$z \in D\left(E_0, r_0 - \frac{2}{3}h, r_0 + \frac{2}{3}h\right) \cup \left\{z \in D\left(E_0, r_0 + \frac{2}{3}h\right);\ |\mathrm{Im}\, z| > Ch\right\}.$$

Since $\log|G(z)|$ is harmonic, it follows that

(5.28)

$$|\nabla_z \log|G(z)|| \leq \mathcal{O}(1)\frac{h^{2-n}}{(h + |\mathrm{Im} z|)\min(h + |\mathrm{Im}\, z|, r_0 + \frac{2}{3}h - |z|)},$$
$$z \in D\left(E_0, r_0 - \frac{7}{12}h, r_0 + \frac{7}{12}h\right) \cup \left\{z \in D\left(E_0, r_0 + \frac{7}{12}h\right);\ |\mathrm{Im}\, z| > Ch\right\},$$

after an arbitrarily small increase of the constant C in (5.27). Since $\log|G(z)| =$ Re $\log G(z)$, where $\log G(z)$ is a multivalued holomorphic function (well defined modulo $2\pi i\mathbf{Z}$), it follows from (5.28) and the Cauchy-Riemann equations that

$$\left|\frac{d}{dz}\log G(z)\right| \leq \mathcal{O}(1)\frac{h^{2-n}}{(h + |\mathrm{Im}\, z|)\min(h + |\mathrm{Im}\, z|, r_0 + \frac{2}{3}h - |z|)} \tag{5.29}$$

in the same domain as in (5.28).

Let $f(z)$ be holomorphic in $D(E_0, r_0+h)$ and let γ be the oriented boundary of the hexagon with corners in $E_0 \pm r(h) \pm iCh$, $E_0 \pm ir_0/2$, with $r = r(h) \in]r_0 - \frac{7}{12}h, r_0 + \frac{7}{12}h[$. We assume that no z_j is on γ and let $\operatorname{int}\gamma$ be the open hexagon just defined.

Now pass to integrals and observe first that if we have a relation $A(t) = B(t)C(t)$ between bounded invertible operators between Hilbert spaces which are C^1 functions of t on some interval, and if $\frac{dC}{dt}$ is of trace class, then so is $\frac{dA}{dt}A^{-1} - \frac{dB}{dt}B^{-1}$, and

$$\operatorname{tr}\left(\frac{dA}{dt}A^{-1} - \frac{dB}{dt}B^{-1}\right) = \operatorname{tr}\frac{dC}{dt}C^{-1} = \operatorname{tr}C^{-1}\frac{dC}{dt}. \tag{5.30}$$

This applies to holomorphic functions and using (5.10), we get

$$\begin{aligned}(5.31)\ \operatorname{tr}\Big(&\frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\mathcal{P})(z-\mathcal{P})^{-1}dz \\ &\qquad -\frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\widetilde{\mathcal{P}})(z-\widetilde{\mathcal{P}})^{-1}dz\Big) \\ &= \frac{1}{2\pi i}\int_\gamma f(z)\operatorname{tr}((1-\partial_z\mathcal{P})(z-\mathcal{P})^{-1} - (1-\partial_z\widetilde{\mathcal{P}})(z-\widetilde{\mathcal{P}})^{-1})dz \\ &= \frac{1}{2\pi i}\int_\gamma f(z)\operatorname{tr}\frac{d}{dz}((1+h(z-\widetilde{\mathcal{P}})^{-1}\delta P))(1+h(z-\widetilde{\mathcal{P}})^{-1}\delta P)^{-1}dz \\ &= \frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}\log D(z)dz.\end{aligned}$$

Here $\log D(z)$ is a multivalued function, but its derivative is single valued. Notice also that

$$\operatorname{tr}\left(\frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\mathcal{P})(z-\mathcal{P})^{-1}dz\right) = \sum_{\mu\in\sigma(\mathcal{P})\cap\operatorname{int}(\gamma)} f(\mu), \tag{5.32}$$

and similarly for $\widetilde{\mathcal{P}}$, which is the motivation for the considerations of this section.

From (5.23) we get

$$\frac{d}{dz}\log D(z) = \frac{d}{dz}\log G(z) + \frac{d}{dz}\log D_b(z), \tag{5.33}$$

which leads to a corresponding decomposition of the last integral in (5.31). One of the terms is

$$\frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}\log D_b(z)dz = \sum_{j;\, z_j\in\operatorname{int}\gamma} f(z_j), \tag{5.34}$$

since

$$\frac{d}{dz}\log D_b(z) = \sum_{j=1}^{N} \frac{d}{dz}\log \widetilde{b}_{z_j}(z) \tag{5.35}$$

and $\frac{d}{dz}\log \widetilde{b}_{z_j}(z)$ is holomorphic in $D(E_0, r_0 + h)$ except for a simple pole at $z = z_j$ with singularity $(z - z_j)^{-1}$ there.

Let us first assume that

$$f(z) = \mathcal{O}(1) \text{ on } \gamma. \tag{5.36}$$

Then since $N = \mathcal{O}(1)h^{1-n}$, we get from (5.34):

$$\left|\frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}\log D_b(z)dz\right| = \mathcal{O}(1)h^{1-n}. \tag{5.37}$$

On the other hand, using (5.29), we get

$$\left|\frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}\log G(z)dz\right| \leq \mathcal{O}(1)h^{2-n}\int_0^1 \frac{1}{(h+t)^2}dt = \mathcal{O}(1)h^{1-n}. \tag{5.38}$$

In conclusion, we get under the assumption (5.36):

$$\begin{aligned} &\operatorname{tr}\Big(\frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\widetilde{\mathcal{P}})(z-\widetilde{\mathcal{P}})^{-1}dz \\ &-\frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\mathcal{P})(z-\mathcal{P})^{-1}dz\Big) = \mathcal{O}(1)h^{1-n}. \end{aligned} \tag{5.39}$$

Taking $f = 1$, we get the following result which implies Theorem 0.1:

Theorem 5.2. *Let $C > 0$ be sufficiently large. The number of eigenvalues z of $\mathcal{P}$ with $E_1 \leq \operatorname{Re} z \leq E_2$, $|\operatorname{Im}| \leq Ch$ is equal to $(2\pi h)^{-n}(\iint_{E_1\leq p(x,\xi)\leq E_2} dxd\xi + \mathcal{O}(h))$.*

Proof. We take $f = 1$ in (5.39), (5.32) and get

$$\#\sigma(\mathcal{P}) \cap \operatorname{int}\gamma = \#\sigma(\widetilde{\mathcal{P}}) \cap \operatorname{int}\gamma + \mathcal{O}(1)h^{1-n}.$$

Define $\widetilde{\mathcal{P}}_t = t\widetilde{\mathcal{P}} + (1-t)\widetilde{P}$ by (6.1) below so that $\widetilde{\mathcal{P}}_1 = \widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{P}}_0 = \widetilde{P}$. It is clear from the definition (6.1) that the number of eigenvalues of $\widetilde{\mathcal{P}}_t$ in $\operatorname{int}\gamma$ is independent of t. From this and the fact that the total number of eigenvalues of $\mathcal{P}$ in an h neighborhood of E_1 or E_2 is $\mathcal{O}(h^{1-n})$, it follows that the number of eigenvalues of $\mathcal{P}$ in the region in the statement of the theorem is equal to

$\mathcal{O}(h^{1-n})$ plus the number of eigenvalues of $\widetilde{P}$ in the interval $[E_1, E_2]$. We can choose $\widetilde{P}$ in Lemma 5.1, so that the latter number is equal to $\mathcal{O}(h^{1-n})$ plus the number of eigenvalues of P in the same interval, and by well-known results on the counting function for eigenvalues of h pseudors, the latter number is given by the expression in the theorem. □

We now change our assumptions on f:

$$f(z) \in \mathbf{R} \text{ when } z \text{ is real and } f'(z) = \mathcal{O}(1) \text{ in } D(E_0, r_0 + h). \tag{5.40}$$

We want to estimate the imaginary part of (5.31). We have $\operatorname{Im} f(z_j) = \mathcal{O}(1) \operatorname{Im} z_j = \mathcal{O}(h)$ and it follows from (5.34) that

$$\operatorname{Im} \frac{1}{2\pi i} \int_\gamma f(z) \frac{d}{dz} \log D_b(z) dz = \mathcal{O}(h^{2-n}). \tag{5.41}$$

For the other contribution to (5.31), we make an integration by parts:

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma f(z) \frac{d}{dz} \log(G(z)) dz &= \\ \frac{1}{2\pi i} [f(z) \log(G(z))]_{E_0+r+i0}^{E_0+r-i0} &- \frac{1}{2\pi i} \int_\gamma f'(z) \log(G(z)) dz, \end{aligned} \tag{5.42}$$

where $E_0 + r$ is the point of intersection of γ with $[E_0, +\infty[$ and where we choose a branch of $\log G$ after placing a cut along $[E_0, +\infty[$. Since $f(E_0 + r)$ is real and $\log(G(E_0 + r - i0) - \log(G(E_0 + r + i0))$ is imaginary, we see that the first term of the RHS of (5.42) is real, so

$$\operatorname{Im} \frac{1}{2\pi i} \int_\gamma f(z) \frac{d}{dz} \log(G(z)) dz = -\operatorname{Im} \frac{1}{2\pi i} \int_\gamma f'(z) \log(G(z)) dz. \tag{5.43}$$

Here we recall (5.27):

$$|\operatorname{Re} \log G(z)| = |\log |G(z)|| \leq \mathcal{O}(1) \frac{h^{2-n}}{h + |\operatorname{Im} z|}. \tag{5.44}$$

In order to get an estimate for Im $\log G(z)$, we integrate (5.29). For $z \in \gamma$ with $\operatorname{Im} z \geq 0$, we integrate from $E_0 + ir_0/2$ and get

$$\begin{aligned} |\operatorname{Im} \log G(z) - \operatorname{Im} \log G(E_0 + ir_0/2)| &\leq \mathcal{O}(1) \int_{|\operatorname{Im} z|}^{+\infty} \frac{h^{2-n}}{(h+t)^2} dt \\ &= \mathcal{O}(1) \left[-\frac{h^{2-n}}{(h+t)} \right]_{|\operatorname{Im} z|}^{\infty} = \mathcal{O}(1) \frac{h^{2-n}}{h + |\operatorname{Im} z|}. \end{aligned} \tag{5.45}$$

For $\mathrm{Im}\, z < 0$, we integrate from $E_0 - ir_0/2$ instead. If $C_\pm = \mathrm{Im}\, \log G(E_0 \pm ir_0/2) \in \mathbf{R}$, we get from (5.44), (5.45) and its analogue, that

$$\log G(z) = iC_\pm + \mathcal{O}(1)\frac{h^{2-n}}{(h + |\mathrm{Im} z|)},\ z \in \gamma,\ \pm\mathrm{Im}\, z > 0. \tag{5.46}$$

Let $\gamma_\pm$ be the part of γ in $\pm\mathrm{Im}\, z > 0$. Then

$$\frac{1}{2\pi i}\int_{\gamma_+} f'(z) iC_+ dz = \frac{C_+}{2\pi}(f(E_0 - r) - f(E_0 + r)) \in \mathbf{R},$$
$$\frac{1}{2\pi i}\int_{\gamma_-} f'(z) iC_- dz = \frac{C_-}{2\pi}(f(E_0 + r) - f(E_0 - r)) \in \mathbf{R},$$

so the term $iC_\pm$ in (5.46) gives no contribution to (5.43). The contribution from the remainder in (5.46) to (5.43) is

$$\mathcal{O}(1)\int_0^1 \frac{h^{2-n}}{(h+t)}dt = \mathcal{O}(1)h^{2-n}[\log(h+s)]_0^1 = \mathcal{O}(1)h^{2-n}\log\frac{1}{h}.$$

It follows that

$$\mathrm{Im}\,\frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}\log(G(z))dz = \mathcal{O}(1)h^{2-n}\log\frac{1}{h}. \tag{5.47}$$

Combining this with (5.41), we get the conclusion that under the assumption (5.40):

(5.48)
$$\mathrm{Im\ tr}\left(\frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\widetilde{\mathcal{P}})(z-\widetilde{\mathcal{P}})^{-1}dz - \frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\mathcal{P})(z-\mathcal{P})^{-1}dz\right)$$
$$= \mathcal{O}(1)h^{2-n}\log\frac{1}{h}.$$

Appendix. Blaschke Factors for the Unit Disc

Let $w \in D(0,1)$ and put

$$b_w(z) = \frac{z-w}{\overline{w}z - 1},\ z \in \overline{D(0,1)}, \tag{A.1}$$

so that $b_w(z)$ vanishes precisely at $z = w$ and satisfies $|b_w(z)| = 1$, for $|z| = 1$. Let $|w| = 1 - \epsilon$, where $\epsilon > 0$ is small. We recall the order of magnitude of $b_w(z)$, for $|z - w| \gg \epsilon$, $|z - w| \sim \epsilon$, $|z - w| \ll \epsilon$. For simplicity, we may assume that $w = 1 - \epsilon$.

Case 1. $|z-w| \geq C\epsilon$, $C \gg 1$. Then

$$\begin{aligned}\frac{z-w}{\overline{w}z-1} &= \frac{z-(1-\epsilon)}{(1-\epsilon)(z-\frac{1}{1-\epsilon})} = \frac{1}{1-\epsilon}\left(1+\frac{\frac{1}{1-\epsilon}-(1-\epsilon)}{z-\frac{1}{1-\epsilon}}\right) \\ &= \frac{1}{1-\epsilon}\left(1-\frac{1-(1-\epsilon)^2}{(1-\epsilon)(z-\frac{1}{1-\epsilon})}\right) = \frac{1}{1-\epsilon}\left(1+\frac{2\epsilon+\mathcal{O}(\epsilon^2)}{z-\frac{1}{1-\epsilon}}\right).\end{aligned}$$

In the region under consideration, $|z-\frac{1}{1-\epsilon}| \sim |z-(1-\epsilon)| \leq \mathcal{O}(1)$, so we get

$$|b_w(z)| = 1+\frac{\mathcal{O}(\epsilon)}{|z-w|}.$$

Case 2. $\epsilon/C \leq |z-w| \leq C\epsilon$. Here $b_w(z) \sim 1$.

Case 3. $|z-w| \leq \epsilon/C$: Here $|z-1/\overline{w}| \approx |(1-\epsilon)-\frac{1}{1-\epsilon}| \approx 2\epsilon$ and $|z-w|/|z-1/\overline{w}| \approx |z-w|/2\epsilon$, so

$$|b_w(z)| = \frac{|z-w|}{|\overline{w}(z-\frac{1}{\overline{w}})|} \approx \frac{|z-w|}{2\epsilon}.$$

§6. Trace Integrals for $\widetilde{\mathcal{P}}$

Recall that $\widetilde{\mathcal{P}} = \widetilde{P}+ihQ(z)$ by (5.6), and put

$$\widetilde{\mathcal{P}}_t = \widetilde{P}+ihtQ(z), \ 0 \leq t \leq 1. \tag{6.1}$$

Under the assumption (5.40), we are interested in

$$\operatorname{tr}\frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\widetilde{\mathcal{P}})(z-\widetilde{\mathcal{P}})^{-1}dz. \tag{6.2}$$

Put

$$I(t) = \operatorname{tr}\frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\widetilde{\mathcal{P}}_t)(z-\widetilde{\mathcal{P}}_t)^{-1}dz, \tag{6.3}$$

so that $I(1)$ is the expression (6.2). Notice that

$$I(t) = \sum_{z\in\sigma(\widetilde{\mathcal{P}}_t)\cap\operatorname{int}(\gamma)} f(z) \tag{6.4}$$

and hence that $\operatorname{Im} I(0) = 0$, since $\sigma(\widetilde{\mathcal{P}}_0) = \sigma(\widetilde{P})$ is real. Let us first assume that $Q(z)$ is of trace class. Then

$$\begin{aligned}
\partial_t I(t) &= \partial_t \frac{1}{2\pi i} \int_\gamma f(z) \partial_z \log\det((z-\widetilde{\mathcal{P}}_0)^{-1}(z-\widetilde{\mathcal{P}}_t))dz \\
&= \frac{1}{2\pi i} \int_\gamma f(z) \partial_z \partial_t \log\det((z-\widetilde{\mathcal{P}}_0)^{-1}(z-\widetilde{\mathcal{P}}_t))dz \\
&= \frac{1}{2\pi i} \int_\gamma f'(z) \operatorname{tr}((z-\widetilde{\mathcal{P}}_t)^{-1}\partial_t \mathcal{P}_t)dz \\
&= \frac{1}{2\pi i} \int_\gamma f'(z) \operatorname{tr}((z-\widetilde{\mathcal{P}}_t)^{-1} ihQ(z))dz \\
&= \operatorname{tr} \frac{h}{2\pi} \int_\gamma f'(z)(z-\widetilde{\mathcal{P}}_t)^{-1} Q(z) dz.
\end{aligned}$$

Notice that the last expression can also be written

$$(-1)^N \operatorname{tr} \frac{h}{2\pi} \int_\gamma C^{(N)}(z) \frac{d^N}{dz^N}(z-\widetilde{\mathcal{P}}_t)^{-1} dz,$$

where $C^{(N)}$ is an N fold primitive of $f'(z)Q(z)$. Using this trick, we can drop the assumption that $Q(z)$ is of trace class and choose a sequence $Q_j(z)$ of trace class operators, such that $(z-\widetilde{\mathcal{P}}_t)^{-1}Q_j(z) \to (z-\widetilde{\mathcal{P}}_t)^{-1}Q(z)$ in operator norm. We get

$$\partial_t I(t) = \operatorname{tr} \frac{h}{2\pi} \int_\gamma f'(z)(z-\widetilde{\mathcal{P}}_t)^{-1} Q(z) dz. \tag{6.5}$$

We shall compare $(z-\widetilde{\mathcal{P}}_t)^{-1}$ for $z \in \gamma$ with an approximate inverse of $z - \mathcal{P}_t$. For this we concentrate on the only difficult region $z \approx E_j$, $p(x,\xi) \approx E_j$, $j = 1, 2$, say with $j = 1$. If $(x_0, \xi_0) \in p^{-1}(E_1)$, let κ : neigh$((0,0), \mathbf{R}^{2n}) \to$ neigh$((x_0,\xi_0), T^*M)$ be a canonical transformation with $\kappa^{-1}(p^{-1}(E_1)) \subset \{(x,\xi);\, \xi_n = 0\}$. Let U be a corresponding Fourier integral operator, noncharacteristic and microlocally unitary at $((x_0,\xi_0);(0,0))$. Then $U^*(\mathcal{P}_t - z)U$ can be viewed as an operator with symbol in Σ^1 (see the appendix of this section) with $\widetilde{h} = h + |z - E_1|$ which is elliptic in that class, near (0,0) when $|z - E_1| \gg h$, and outside an h-neighborhood of $\xi_n = 0$ near (0,0), when $|z - E_1| = \mathcal{O}(h)$. We can then apply Proposition A.3 and find $q \in \Sigma^{-1}$ such that

$$U^*(z - \mathcal{P}_t)Uq(x, hD_x; h) = 1 + r_1(x, hD_x, z; h), \tag{6.6}$$

where $r_1 \in \cap_{N=0}^{\infty} h^N \Sigma^{-N}$ in a fixed neighborhood of $(0,0)$. Further,

$$q(x,\xi,z;h) \equiv (z - p\circ\kappa)^{-1}\left(1-\chi\left(\frac{|z-p\circ\kappa(x,\xi)|}{h}\right)\right) \operatorname{mod} h\Sigma^{-2},$$

in a neighborhood of $(0,0)$, where $\chi \in C_0^\infty(\mathbf{R})$ is equal to 1 near 0.

We can find finitely many points $(x_j,\xi_j) \in p^{-1}(E_1)$, $j = 1,..,N$, $U = U_j$, $q = q_j$ as above and $\chi_j \in C_0^\infty(T^*M)$ with support in a neighborhood of (x_j,ξ_j), as well as an operator

$$\mathcal{E}_t^r = \mathcal{E}_{0,t}\chi_0 + \sum_{j=1}^{N} U_j q_j(x, hD_x, z; h) U_j^* \chi_j, \tag{6.7}$$

with the following properties:

$$1 = \sum_0^N \chi_j, \tag{6.8}$$

where $\operatorname{supp}\chi_0 \cap p^{-1}(E_1) = \emptyset$ and in (6.7) we let χ_j denote corresponding h-pseudors. $\mathcal{E}_{0,t} \in \operatorname{Op}(S(\langle\xi\rangle^{-2}))$ with leading symbol $(z-p)^{-1}$, near the support of χ_0. $q_j \in \Sigma^{-1}$ with $\widetilde{h} = h+|z-E_1|$, and $q_j \equiv (z-p\circ\kappa_j)^{-1}(1-\chi(h^{-1}|z-p\circ\kappa_j|))$ $\operatorname{mod} h\Sigma^{-2}$ in a neighborhood of $\operatorname{supp}(\chi_j \circ \kappa_j)$. Further,

$$(z - \mathcal{P}_t)\mathcal{E}_t^r = 1 - K_r, \tag{6.9}$$

where

$$\|K_r\| = \mathcal{O}(1)\left(\frac{h}{h+|z-E_1|}\right)^N, \text{ for all } N > 0, \tag{6.10}$$

$$\|K_r\|_{\mathrm{tr}} = \mathcal{O}(1)h^{1-n}\left(\frac{h}{h+|z-E_1|}\right)^N, \text{ for all } N > 0. \tag{6.11}$$

Similarly with

$$\mathcal{E}_t^\ell = \chi_0\mathcal{E}_{0,t} + \sum_{j=1}^{N} \chi_j U_j q_j U_j^*, \tag{6.12}$$

we get

$$\mathcal{E}_t^\ell(z - \mathcal{P}_t) = 1 - K_\ell, \tag{6.13}$$

where K_ℓ satisfy (6.10,11).

We also have with $\mathcal{E} = \mathcal{E}_t^\ell, \mathcal{E}_t^r$,

(6.14)

$$\mathcal{E},\, \mathcal{E}Q,\, Q\mathcal{E},\, (z-\widetilde{\mathcal{P}}_t)^{-1},\, (z-\widetilde{\mathcal{P}}_t)^{-1}Q,\, Q(z-\widetilde{\mathcal{P}}_t)^{-1} = \frac{\mathcal{O}(1)}{h+|\operatorname{Im} z|} \text{ in norm,}$$

for $z \in \gamma$, and from this and (5.4), (5.5), we get

$$(z-\widetilde{\mathcal{P}}_t)\mathcal{E}_t^r = 1-\widetilde{K}_r,\ \mathcal{E}_t^\ell(z-\widetilde{\mathcal{P}}_t) = 1-\widetilde{K}_\ell, \tag{6.15}$$

where $\widetilde{K} = \widetilde{K}_r$, $\widetilde{K}_\ell$ satisfy

$$\|\widetilde{K}\| = \mathcal{O}(1)\frac{h}{h+|\operatorname{Im} z|}, \tag{6.16}$$

$$\|\widetilde{K}\|_{\mathrm{tr}} = \mathcal{O}(1)h^{1-n}\frac{h}{h+|\operatorname{Im} z|}. \tag{6.17}$$

These constructions and estimates extend to all $z \in \gamma$ and not just to the ones that are close to E_1 or E_2.

Write

$$(z-\widetilde{\mathcal{P}}_t)^{-1} = \mathcal{E}_t^r + (z-\widetilde{\mathcal{P}}_t)^{-1}\widetilde{K}_r = \mathcal{E}_t^\ell + \widetilde{K}_\ell(z-\widetilde{\mathcal{P}}_t)^{-1}, \tag{6.18}$$

and conclude that

(6.19)

$$(z-\widetilde{\mathcal{P}}_t)^{-1} - \mathcal{E}_t^\ell, ((z-\widetilde{\mathcal{P}}_t)^{-1} - \mathcal{E}_t^\ell)Q = \begin{cases} \dfrac{\mathcal{O}(1)h}{(h+|\operatorname{Im} z|)^2} & \text{in norm,} \\ \dfrac{\mathcal{O}(1)h^{2-n}}{(h+|\operatorname{Im} z|)^2} & \text{in trace norm.} \end{cases}$$

We recall that $f'(z) = \mathcal{O}(1)$ by (5.40) and approximate $\partial_t I(t)$ in (6.5) by

$$J(t) := \operatorname{tr} \frac{h}{2\pi}\int_\gamma f'(z)\mathcal{E}_t^\ell Q dz. \tag{6.20}$$

Using (6.19), we get

$$\partial_t I(t) - J(t) = \mathcal{O}(1)h^{3-n}\int_\gamma \frac{1}{(h+|\operatorname{Im} z|)^2}dz = \mathcal{O}(1)h^{2-n}. \tag{6.21}$$

It remains to study $J(t)$. When $d(z) := \operatorname{dist}(z, \{E_1, E_2\}) \geq 1/\mathcal{O}(1)$ then $\mathcal{E}_t^\ell \in \operatorname{Op}(S_{\mathrm{cl}}(\langle\xi\rangle^{-2}))$ with leading symbol $(z-p(x,\xi))^{-1}$. When $d(z)$ is small, we can represent $\mathcal{E}_t^\ell$ as a finite sum, where one of the terms is in $\operatorname{Op}(S_{\mathrm{cl}}(\langle\xi\rangle^{-2}))$, supported away from $p^{-1}(E_1)$ or $p^{-1}(E_2)$, and the other terms are conjugates by means of a Fourier integral operator of second microlocal inverses of the corresponding (inverse) conjugation of $P-z$, in such a way that $p^{-1}(E_1)$ or $p^{-1}(E_2)$ is transformed into $\xi_n = 0$ by the corresponding canonical transformation. Moreover, in the region $|\xi| \gg 1$, the full symbol of $\mathcal{E}_t^\ell$ is holomorphic with respect to z in a neighborhood of the closure of the interior of γ, modulo a term in $\cap h^N S(\langle\xi\rangle^{-2-N})$. When computing the traces of the various terms, we have to integrate the corresponding symbols over T^*M (for one of the terms) and over $\mathbf{R}^{2n}$ for the others. The integrals over $\mathbf{R}^{2n}$ can be transformed into integrals over T^*M by means of the canonical transformation. It follows that

$$J(t) = h^{1-n}\frac{1}{2\pi}\int_\gamma dz \frac{1}{(2\pi)^n}\iint_{T^*M} f'(z)a(x,\xi,z;h)dxd\xi,$$

with

$$a(x,\xi,z;h) = \frac{q(x,\xi,z)(1-\chi(\frac{|z-p(x,\xi)|}{h}))}{(z-p(x,\xi))} + r(x,\xi,z;h).$$

Here $\chi \in C_0^\infty(\mathbf{R})$ is equal to 1 near 0, while

$$r(x,\xi,z;h) = \begin{cases} \mathcal{O}\left(\dfrac{h}{(h+|z-p(x,\xi)|)^2}\right), |\xi| \leq \mathcal{O}(1), \\ \mathcal{O}\left(\dfrac{h}{\langle\xi\rangle^3}\right), |\xi| \gg 1. \end{cases}$$

Moreover, modulo a term which is $\mathcal{O}(h^N\langle\xi\rangle^{-2-N})$ for every $N \geq 0$, we know that $r(x,\xi,z;h)$ extends to a holomorphic function in $z \in \text{neigh}\,(\overline{\text{int}\,(\gamma)})$, when $|\xi| \gg 1$.

The contribution from $\dfrac{q(x,\xi,z)}{z-p(x,\xi)}$ to $J(t)$ becomes

$$J_1(t) = ih^{1-n}\frac{1}{(2\pi)^n}\iint_{T^*M} f'(p(x,\xi))q(x,\xi,p(x,\xi))1_{[E_1,E_2]}(p(x,\xi))dxd\xi.$$

The contribution from $-\dfrac{\chi(\frac{|z-p(x,\xi)|}{h})q}{z-p(x,\xi)}$ to $J(t)$ is

$$\begin{aligned} &\mathcal{O}(1)h^{1-n}\int_{\{z\in\gamma;\, d(z)\leq\mathcal{O}(h)\}}\iint_{d(p(x,\xi))\leq\mathcal{O}(h)} \frac{1}{|z-p(x,\xi)|}dxd\xi|dz| \\ &= \mathcal{O}(1)h^{1-n}\int_{\{z\in\gamma; d(z)\leq\mathcal{O}(h)\}} -\log(d(z))|dz| \\ &= \mathcal{O}(1)h^{1-n}\int_0^h -\log(t)dt = \mathcal{O}\left(h^{2-n}\log\frac{1}{h}\right). \end{aligned}$$

When estimating the contribution from $r(x,\xi z;h)$, we first integrate w.r.t. z and get

$$h^{1-n}\iint_{T^*M} s(x,\xi;h)dxd\xi,$$

where $s(x,\xi;h)=\mathcal{O}(h^N\langle\xi\rangle^{-2-N})$ for $|\xi|\geq C\gg 1$ and for all $N\geq 0$, while for $|\xi|\leq C$:

$$\begin{aligned}s(x,\xi;h)&=\mathcal{O}(1)\int_\gamma \frac{h}{(h+|z-p(x,\xi)|)^2}|dz|\\&=\mathcal{O}(1)\int_0^1 \frac{h}{(h+d(p)+t)^2}dt=\mathcal{O}(1)\frac{h}{h+d(p)}.\end{aligned}$$

So the contribution from r to $J(t)$ is

$$\mathcal{O}(h^N)+\mathcal{O}(1)h^{2-n}\iint_{|\xi|\leq C}\frac{1}{h+d(p(x,\xi))}dxd\xi=\mathcal{O}\left(h^{2-n}\log\frac{1}{h}\right).$$

Summing up our computations and estimates, we get

(6.22)
$$J(t)=\frac{ih}{(2\pi h)^n}\iint_{E_1\leq p(x,\xi)\leq E_2} f'(p(x,\xi))q(x,\xi,p(x,\xi))dxd\xi+\mathcal{O}\left(h^{2-n}\log\frac{1}{h}\right).$$

Combining this with (6.21), the fact that $\operatorname{Im} I(0)=0$ and (5.48), (5.32), we get:

Theorem 6.1. *For f satisfying* (5.40), *and for $C>0$ sufficiently large, we have*

(6.23)
$$\operatorname{Im}\sum_{\substack{z\in\sigma(\mathcal{P})\\ \operatorname{Re} z\in[E_1,E_2]\\ |\operatorname{Im} z|\leq Ch}} f(z)=\frac{h}{(2\pi h)^n}\iint_{E_1\leq p\leq E_2} f'(p)q(x,\xi,p)dxd\xi+\mathcal{O}(1)h^{2-n}\log\frac{1}{h}.$$

Using Cauchy's inequality, we see that the LHS of (6.23) is equal to

$$\sum_{z\in\sigma(\mathcal{P})\cap([E_1,E_2]+i[-Ch,Ch])} f'(\operatorname{Re} z)\operatorname{Im} z+\mathcal{O}(1)h^{2-n}\log\frac{1}{h}. \tag{6.24}$$

The integral in the RHS of (6.23) is equal to

$$\int_{E_1}^{E_2} f'(E)\Big(\int_{p^{-1}(E)} q(x,\xi,E)L_E(d(x,\xi))\Big)dE,$$

where L_E denotes the Liouville measure on $p^{-1}(E)$. If we view $f'(E)$ as test functions, we may say that the average distribution of the imaginary parts of the eigenvalues of $\mathcal{P}$ is given by the density

$$\frac{h\int_{p^{-1}(E)} q(x,\xi,E)L_E(d(x,\xi))}{\int_{p^{-1}(E)} L_E(d(x,\xi))}. \tag{6.25}$$

We finally derive Theorem 0.3 from Theorem 6.1. The relation between the eigenvalues z of $\mathcal{P}$ and the eigenvalues τ of the original operator P in the introduction is given by $z=(h\tau)^2$, and we recall that $q=2a(x)\sqrt{z}$. Here $\mathrm{Im}\,\tau = \mathcal{O}(1)$, $h\sim(\mathrm{Re}\,\tau)^{-1}$, so $\mathrm{Re}\,z=(h\mathrm{Re}\,\tau)^2+\mathcal{O}(h^2)$, $\mathrm{Im}\,z = 2h(\mathrm{Re}\,\tau)h(\mathrm{Im}\,\tau)$, and (6.23,24) lead to

$$\begin{aligned}2h \sum_{\substack{\tau\in\sigma(P)\\ \mathrm{Re}\,\tau\in[\frac{\sqrt{E_1}}{h},\frac{\sqrt{E_2}}{h}]}} & f'((h\mathrm{Re}\,\tau)^2)(h\mathrm{Re}\,\tau)\mathrm{Im}\,\tau \\ &= \frac{h}{(2\pi h)^n}\iint_{E_1\le p\le E_2} f'(p)\sqrt{p}2a(x)dxd\xi + \mathcal{O}(1)h^{2-n}\log\frac{1}{h}.\end{aligned} \tag{6.26}$$

Putting $g(p)=f'(p)\sqrt{p}$, we get

$$\begin{aligned}\sum_{\substack{\tau\in\sigma(P)\\ \mathrm{Re}\,\tau\in[\frac{\sqrt{E_1}}{h},\frac{\sqrt{E_2}}{h}]}} & g(h\mathrm{Re}\,\tau)\mathrm{Im}\,\tau \\ &= \frac{1}{(2\pi h)^n}\left(\iint_{E_1\le p\le E_2} g(p(x,\xi))a(x)dxd\xi + \mathcal{O}(1)h\log\frac{1}{h}\right).\end{aligned} \tag{6.27}$$

Now choose $f(p)=2\sqrt{p}$, so that $g(p)=1$, let λ_1,λ_2 be as in Theorem 0.3, choose $h=1/\lambda_1$, $E_1=1$, $E_2=(\lambda_2/\lambda_1)^2$, to get

$$\sum_{\substack{\tau\in\sigma(P)\\ \mathrm{Re}\,\tau\in[\lambda_1,\lambda_2]}} \mathrm{Im}\,\tau = \left(\frac{\lambda_1}{2\pi}\right)^n\left(\iint_{1\le p\le(\frac{\lambda_2}{\lambda_1})^2} a(x)dxd\xi + \mathcal{O}(1)\frac{\log\lambda_1}{\lambda_1}\right). \tag{6.28}$$

On the other hand, we know that $N(\lambda_1,\lambda_2)$ (defined in Theorem 0.3) obeys

$$N(\lambda_1,\lambda_2) = \left(\frac{\lambda_1}{2\pi}\right)^n\left(\iint_{1\le p\le(\frac{\lambda_2}{\lambda_1})^2} dxd\xi + \mathcal{O}(1)\frac{1}{\lambda_1}\right), \tag{6.29}$$

and (0.12) follows from (6.28,29).

Appendix

We review here some second microlocal calculus with respect to $\xi_n = 0$ (cf. [SZ]). If $m \in \mathbf{R}$, we let Σ^m denote the space of functions $a = a(x,\xi;h,\widetilde{h})$, defined for $(x,\xi) \in \mathbf{R}^{2n}$, $0 < h \leq \widetilde{h} \leq h_0$, or possibly for $(h,\widetilde{h})$ in some smaller set, with $a(\cdot;h,\widetilde{h}) \in C_0^\infty(\mathbf{R}^{2n})$, such that

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta}(\widetilde{h} + |\xi_n|)^{m-\beta_n}, \ \alpha,\beta \in \mathbf{N}^n. \tag{A.1}$$

$$\operatorname{supp} a(\cdot;h,\widetilde{h}) \subset K \subset\subset \mathbf{R}^{2n} \text{ for some } K \text{ independent of } h,\widetilde{h}. \tag{A.2}$$

In order to avoid some probably purely technical difficulties, we shall work with the "classical" h-quantization

(A.3)
$$\operatorname{Op}_h(a)u(x) = a(x,hD_x;h,\widetilde{h})u = \frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot\theta} a(x,\theta;h,\widetilde{h})u(y)dyd\theta.$$

For $a_1, a_2 \in C_0^\infty(\mathbf{R}^{2n})$, we recall that

$$\operatorname{Op}_h(a_1)\operatorname{Op}_h(a_2) = \operatorname{Op}_h(a_1 \# a_2), \tag{A.4}$$

where

$$a_1\#a_2(x,\xi;h,\widetilde{h}) = e^{-i\frac{x\cdot\xi}{h}} a_1(x,hD_x;h,\widetilde{h})(e^{\frac{i(\ )\xi}{h}} a_2(\cdot,\xi;h,\widetilde{h})). \tag{A.5}$$

Notice that if $\operatorname{supp} a_2 \subset \mathbf{R}^n \times K_2$, then $\operatorname{supp}(a_1\#a_2) \subset \mathbf{R}^n \times K_2$, and if $\operatorname{supp} a_1 \subset K_1 \times \mathbf{R}^n$, then $\operatorname{supp}(a_1\#a_2) \subset K_1 \times \mathbf{R}^n$. In particular,

$$\operatorname{supp} a_1\#a_2 \subset \pi_x(\operatorname{supp} a_1) \times \pi_\xi(\operatorname{supp} a_2), \tag{A.6}$$

where π_x and π_ξ denote the projections $(x,\xi) \mapsto x$ and $(x,\xi) \mapsto \xi$ respectively.

Proposition A.1. *If $a_j \in \Sigma^{m_j}$, $j = 1,2$, then $a_1\#a_2 \in \Sigma^{m_1+m_2}$. Moreover,*

$$a_1\#a_2 \sim \sum_{\alpha\in\mathbf{N}^n} \frac{h^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha a_1)(x,\xi;h,\widetilde{h})(D_x^\alpha a_2)(x,\xi;h,\widetilde{h}), \tag{A.7}$$

in the sense that for every $N \in \{1,2,\dots\}$,

$$a_1\#a_2 - \sum_{|\alpha|<N} \dots \in h^N \Sigma^{m_1+m_2-N}. \tag{A.8}$$

Notice that $\widetilde{h}^N \Sigma^{m_1+m_2-N} \subset \Sigma^{m_1+m_2}$, when $N \geq 0$, so we can view (A.7) as an asymptotic expansion in powers of $h/\widetilde{h}$.

Proof. Write $x = (x', x_n)$ and similarly for ξ, and notice that our symbols are completely standard in (x', ξ'), i.e. they belong to the symbol space $S^0_{0,0}$ in these variables, for each fixed (x_n, ξ_n). We can have them become quite standard also in (x_n, ξ_n) by means of a change of variables in ξ_n: Let $a \in \Sigma^m$, and put

$$\widetilde{a}(x, \xi; h, \widetilde{h}) = a(x, \xi', \widetilde{h}\xi_n; h, \widetilde{h}). \tag{A.9}$$

Then

$$|\partial_x^\alpha \partial_\xi^\beta \widetilde{a}| \leq C_{\alpha,\beta}(\widetilde{h} + \widetilde{h}|\xi_n|)^{m-\beta_n} \widetilde{h}^{\beta_n} = C_{\alpha,\beta} \widetilde{h}^m (1 + |\xi_n|)^{m-\beta_n}, \tag{A.10}$$

$$\operatorname{supp}(\widetilde{a}) \subset \{(x, \xi) \in \mathbf{R}^{2n}; (x, \xi', \widetilde{h}\xi_n) \in K\}, \tag{A.11}$$

where K is compact and independent of $h, \widetilde{h}$. Conversely, if $\widetilde{a}$ satisfies (A.10,11), and we define a by (A.9), then $a \in \Sigma^m$. Now we notice that $\widetilde{a}$ is a standard symbol of type 0,0 in (x', ξ') and of type 1,0 in (x_n, ξ_n): $\widetilde{a} \in \widetilde{h}^m S^0_{0,0} \otimes S^m_{1,0}(\mathbf{R}^{2n})$ (See [H].) Moreover, we have trivially

$$a(x, hD_x; h, \widetilde{h}) = \widetilde{a}\left(x, hD_{x'}, \frac{h}{\widetilde{h}} D_{x_n}; h, \widetilde{h}\right), \tag{A.12}$$

and this operator is then a standard h-pseudor with symbol of type 0,0 in (x', ξ') and a standard $h/\widetilde{h}$ pseudor with symbol of type (1,0) in (x_n, ξ_n). The asymptotic expansion (A.7) when expressed in terms of the corresponding symbols $\widetilde{a}_j$ is then the obvious combination of the corresponding composition formulas for the two groups of variables. Since the symbols have their support in $|\xi_n| \leq \mathcal{O}(1)$, we notice that a gain of a factor $h/(\widetilde{h} + |\xi_n|)$ is always weaker than a gain of a factor h. □

The following result is a consequence of wellknown criteria for a pseudor to be L^2 bounded or to be of trace class (see for instance [DS]).

Proposition A.2. *If $a \in \Sigma^0$, then $a(x, hD; h, \widetilde{h}) = \mathcal{O}(1) : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$. If $a \in h^m \Sigma^{-m}$, $m > 1$, then $a(x, hD; h, \widetilde{h}) : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is a trace class operator of trace class norm $\leq \mathcal{O}(1) h^{1-n} (h/\widetilde{h})^{m-1}$.*

As for the invertibility of elliptic operators, we have

Proposition A.3. *Let $a \in \Sigma^m$, and let $\Omega_1 \subset\subset \Omega_2 \subset\subset \mathbf{R}^{2n}$ be open sets independent of h. Assume that*

$$|a(x, \xi; h, \widetilde{h})| \geq \frac{1}{C}(\widetilde{h} + |\xi_n|)^m, (x, \xi) \in \Omega_2,$$

or more generally that there exists $b_0 \in \Sigma^{-m}$, such that

$$ab_0 = 1 + r,$$

where r is of class $\cap_{N\in\mathbf{N}} h^N \Sigma^{-N}$ in some neighborhood of the closure of $\overline{\Omega}_1$. Then there exists $b \in \Sigma^{-m}$, such that

$$a\#b = 1 + r_1,\ b\#a = 1 + r_2,$$

where r_1, r_2 are of class $\cap_{N=0}^{\infty} h^N \Sigma^{-N}$ in Ω_1.

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