On the Gibbs states of the non-critical Potts model on $\mathbb{Z}^2$

Joint work with H. Duminil-Copin, D. Ioffe, and Y. Velenik.

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Outline of the talk

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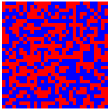
Ising model

Spin = state of a site $x \in \mathbb{Z}^d$

$$\sigma_x \in \{-1, +1\}$$



Λ

Hamiltonian = energy of a spin configuration

$$\mathcal{H}(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y \rightarrow \begin{cases} \text{●● or ●● contribution = -1} \\ \text{●● or ●● contribution = +1} \end{cases}$$

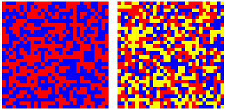
Gibbs measure on $\{-1, +1\}^{\Lambda}$:

$$\mathbb{P}_{\Lambda, T}(\sigma) = \frac{1}{Z} \exp\left(-\frac{1}{T}\mathcal{H}(\sigma)\right), \quad T = \text{temperature}$$

Ising model / Potts model

Spin = state of a site $x \in \mathbb{Z}^d$

$$\sigma_x \in \{{\color{red}-1}, {\color{blue}+1}\}$$
$$\sigma_x \in \{{\color{red}1}, {\color{blue}2} \ldots, {\color{yellow}q}\}$$



Λ

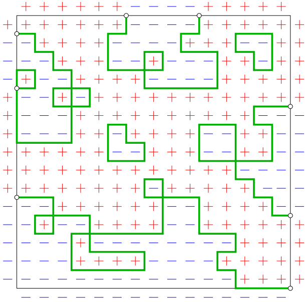
Hamiltonian = energy of a spin configuration

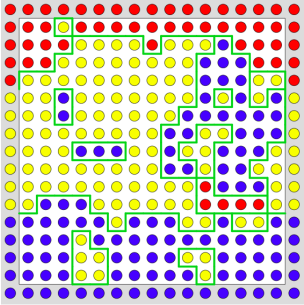
$$\mathcal{H}(\sigma) = -\sum_{x\sim y} \sigma_x \sigma_y \to \begin{cases} {\color{red}\bullet\bullet} \text{ or } {\color{blue}\bullet\bullet} & \text{contribution} = \text{-1} \\ {\color{red}\bullet}{\color{blue}\bullet} \text{ or } {\color{blue}\bullet}{\color{red}\bullet} & \text{contribution} = +1 \end{cases}$$

$$\mathcal{H}(\sigma) = -\sum_{x\sim y} \delta_{\sigma_x=\sigma_y} \to \begin{cases} {\color{red}\bullet\bullet} \text{ or } {\color{blue}\bullet\bullet} \text{ or } {\color{yellow}\bullet\bullet} \text{ or } \ldots & \text{contribution} = \text{-1} \\ {\color{red}\bullet}{\color{blue}\bullet} \text{ or } {\color{blue}\bullet}{\color{yellow}\bullet} \text{ or } {\color{yellow}\bullet}{\color{red}\bullet} \text{ or } \ldots & \text{contribution} = 0 \end{cases}$$

$$\mathbb{P}_{\Lambda,T}(\sigma) = \frac{1}{Z} \exp\left(-\frac{1}{T}\mathcal{H}(\sigma)\right), \quad T = \text{temperature}$$

Boundary conditions

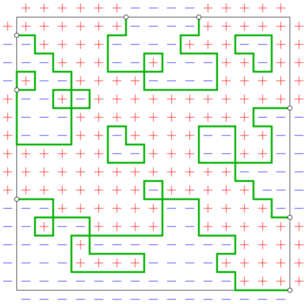


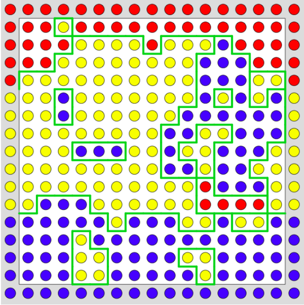


$$\textbf{Ising :}\ \mathcal{H}^{\omega}(\sigma) = -\sum_{x\sim y}\sigma_x\sigma_y - \sum_{\substack{x\in\Lambda, y\in\partial\Lambda\\ x\sim y}}\sigma_x\omega_y$$

$$\textbf{Potts :}\ \mathcal{H}^{\omega}(\sigma) = -\sum_{x\sim y}\delta_{\sigma_x=\sigma_y} - \sum_{\substack{x\in\Lambda, y\in\partial\Lambda\\ x\sim y}}\delta_{\sigma_x=\omega_y}$$

Boundary conditions





$$\textbf{Ising}: \ \mathcal{H}^{\omega}(\sigma) = -\sum_{x\sim y} \sigma_x \sigma_y - \sum_{\substack{x\in\Lambda, y\in\partial\Lambda \\ x\sim y}} \sigma_x \omega_y$$

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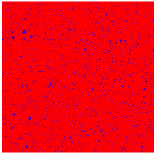
Outline of the talk

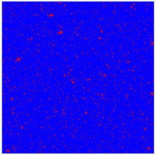
Monochromatic boundary conditions

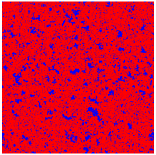
Ising : $\mathbb{P}^{+}_{\Lambda,T}$ and $\mathbb{P}^{-}_{\Lambda,T}$

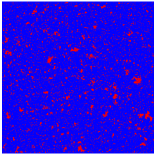
Potts : $\mathbb{P}^{i}_{\Lambda,T}$, with $i = 1, \dots, q$.

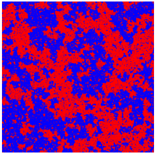


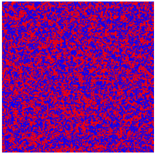
$T \simeq 0$



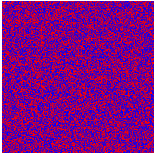








$T \gg 1$



Phase transition

Ising

There exists $T_c(d) \in (0, \infty)$ s.t.

- If $T > T_c$ then

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{E}^+_{\Lambda, T}(\sigma_0) = 0$$

- If $T < T_c$ then

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{E}^+_{\Lambda, T}(\sigma_0) > 0$$

Potts

There exists $T_c(d, q) \in (0, \infty)$ s.t.

- If $T > T_c$ then

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}^i_{\Lambda, T}(\sigma_0 = i) = 1/q$$

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$$\mathbb{E}_T^+(\sigma_0) = 0$$

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$$-\mathbb{E}_T^- = \mathbb{E}_T^+(\sigma_0) > 0$$

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Remarks

- Existence of the monochromatic phases : FKG inequality for Ising, coupling with the random-clusters model for Potts.
  They are translation invariant.

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Remarks

- Existence of the monochromatic phases : FKG inequality for Ising, coupling with the random-clusters model for Potts.
  They are translation invariant.
- Non-triviality of T_c : Peierls in $d = q = 2$, monotonicity in d and in q.

Outline of the talk

Infinite volume Gibbs measures

Weak limits approach

$$\mathcal{G}_T = \left\{ \begin{array}{c} \text{accumulation points of sequences } (\mathbb{P}_{\Lambda_n}^{\omega_n})_n \\ \text{with } \Lambda_n \uparrow \mathbb{Z}^d \text{ as } n \to \infty \end{array} \right\}$$

Weak topology : $\mathbb{P}_{\Lambda_n}^{\omega_n} \to \mathbb{P} \quad \Leftrightarrow \quad \mathbb{E}_{\Lambda_n}^{\omega_n}(f) \to \mathbb{E}(f) \quad \forall f$ local

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Dobrushin-Lanford-Ruelle approach

$$\tilde{\mathcal{G}}_T = \left\{ \begin{array}{ll} \mathbb{P} : & \text{for all } \Lambda \Subset \mathbb{Z}^d, \text{ and } \mathbb{P}\text{-a.e. } \omega, \\ & \mathbb{P}(\sigma \,|\, \sigma = \omega \text{ on } \Lambda^c) = \mathbb{P}^{\omega}_{\Lambda}(\sigma) \end{array} \right\}$$

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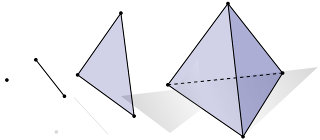
⚠ $\tilde{\mathcal{G}}_T$ is a simplex, whose extremal elements belong to $\mathcal{G}_T$!

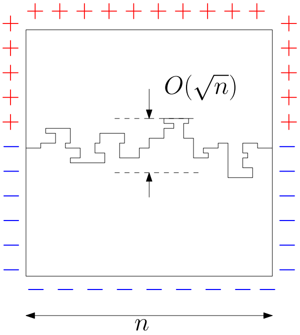


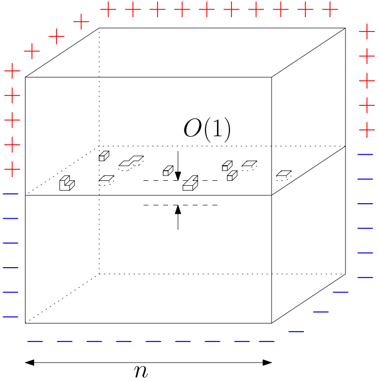
Outline of the talk

Known results for the Ising model

1972 For $0 < T \ll T_c$ and $d = 2$ resp. 3,

[Galavotti] $\mathbb{P}^{\pm} = (\mathbb{P}^{+} + \mathbb{P}^{-})/2$, [Dobrushin] $\mathbb{P}^{\pm}$ is not translation invariant





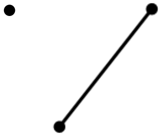
Known results for the Ising model

1980-81 In $d = 2$ for all $T > 0$ [Aizenman] [Higuchi]

$$\mathcal{G}_T = \left\{\alpha \mathbb{P}_T^+ + (1-\alpha)\mathbb{P}_T^-, \text{ with } \alpha \in [0,1]\right\}$$

$$T \geq T_c \quad \Rightarrow \quad \mathcal{G}_T = \{\mathbb{P}_T\}$$

$$T < T_c \quad \Rightarrow \quad \mathcal{G}_T = [\mathbb{P}_T^-, \mathbb{P}_T^+]$$



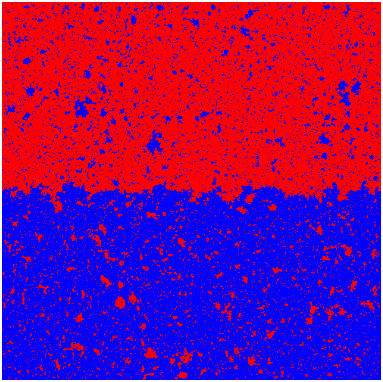
(Proof by contradiction, in the infinite volume setting, gives little information about large but finite volumes.)

Known results for the Ising model

1979 In $d = 2$ and for $T \ll T_c$ [Higuchi]

2005 In $d = 2$ and $T < T_c$ [Greenberg, Ioffe]

Under diffusive scaling,
the Dobrushin interface
weakly converges
to a Brownian bridge.



Known results for the Potts model

1986 For $d \geq 2$, $q > q_0(d)$, and $T < T_c$ [Martirosian]

If $\mathbb{P} \in \mathcal{G}_T$ is translation invariant, then $\mathbb{P} = \sum_{i=1}^{q} \alpha_i \mathbb{P}^i_T$.

Known results for the Potts model

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$$\text{If } \mathbb{P} \in \mathcal{G}_T \text{ is translation invariant, then } \mathbb{P} = \sum_{i=1}^{q} \alpha_i \mathbb{P}_T^i .$$

2008 For $d = 2$, and $T < T_c$ (*) [Campanino, Ioffe, Velenik]

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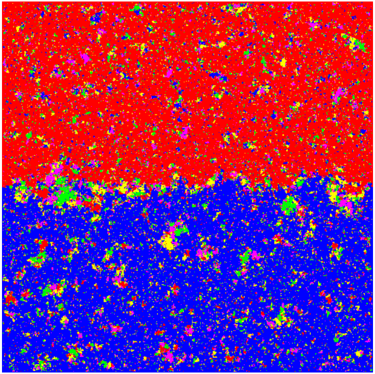

Outline of the talk

Characterization of $\mathcal{G}_T$ for the Potts model on $\mathbb{Z}^2$

Theorem (C., Duminil-Copin, Ioffe, Velenik)

For $q \geq 2$ and $T < T_c(q)$,

$$\mathcal{G}_T = \left\{ \sum_{i=1}^{q} \alpha_i \mathbb{P}_T^i, \ \textit{with} \ \sum_{i=1}^{q} \alpha_i = 1 \right\}$$

Characterization of $\mathcal{G}_T$ for the Potts model on $\mathbb{Z}^2$

Theorem (C., Duminil-Copin, Ioffe, Velenik)

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What about criticality $T = T_c = 1/\log(1+\sqrt{q})$?

- [Duminil-Copin, Sidoravicius, Tassion] uniqueness for $q = 2, 3, 4$.
- [Laanait, Messager, Miracle-Solé, Ruiz, Shlosman] for $q > 25$ there are $q + 1$ extremal phases
- **Conjecture** : $q + 1$ extremal phases for all $q > 4$.

Finite volume stronger result

Theorem (C., Duminil-Copin, Ioffe, Velenik)

Let $q \geq 2$ and $T < T_c(q)$.
For all $\varepsilon > 0$, there exists a constant $C_\varepsilon < \infty$ such that:
for every boundary condition ω there exist q positive constants $\alpha_1^n, \dots, \alpha_q^n$ depending only on (n, ω, T, q), such that

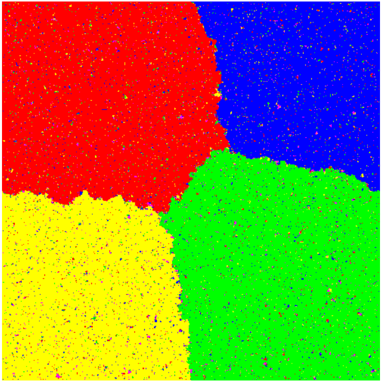
$$\left| \mathbb{P}^{\omega}_{\Lambda_n, T}(f) - \sum_{i=1}^{q} \alpha_i^n \mathbb{P}^i_T(f) \right| \leq C_\varepsilon \|f\|_\infty \cdot n^{-\frac{1}{2}+\varepsilon}$$

uniformly on functions f which have support in Λ_{n^ε}.

Outline of the talk

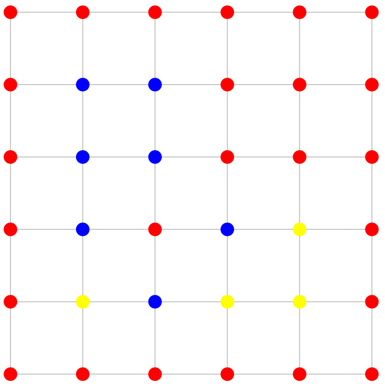
Philosophy

- Uniformly on the given boundary condition, a finite number of interfaces reach the half box,
- They are concentrated around "minimal surfaces",
- But they fluctuate enough so that locally, no phase coexistence is possible.



Edwards-Sokal coupling with the random cluster model

Let $\sigma \sim \mathbb{P}^{\bullet}_{\Lambda,T,q}$ and $p_T = 1 - \exp(-1/T)$.



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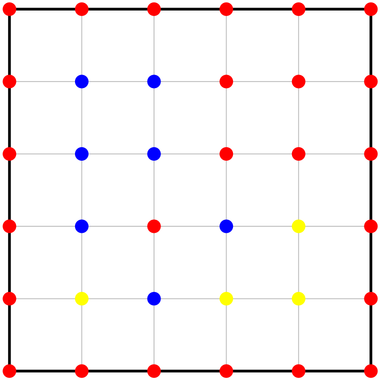
Sample $\eta \in \{0,1\}^{\mathcal{E}_\Lambda}$ as follows:

- set $\eta(e) = 1$ on $\partial\Lambda$.



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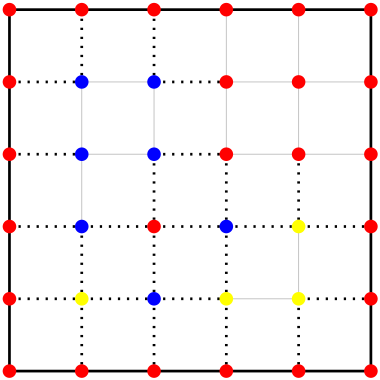
For each $e = [i,j]$,

- if $\sigma_i \neq \sigma_j$, set $\eta(e) = 0$



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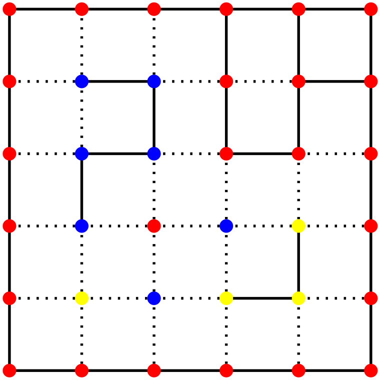
- if $\sigma_i = \sigma_j$, set
  $\eta(e) = 1$ with proba p_T
  $\eta(e) = 0$ with proba $1 - p_T$



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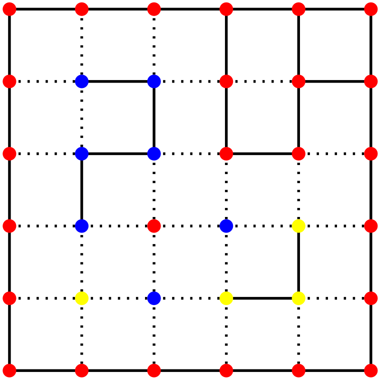

Then
$\eta \sim \phi^1_{\Lambda,p_T,q} \propto p_T^{o(\eta)}(1-p_T)^{c(\eta)} q^{\kappa(\eta)}$.

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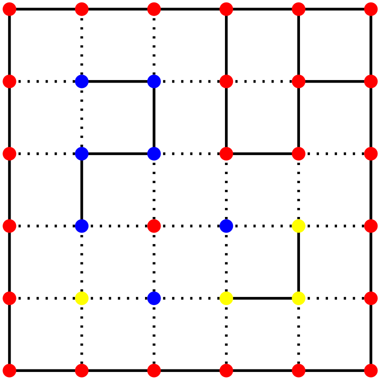

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$\eta \sim \phi^1_{\Lambda,p_T,q} \propto p_T^{o(\eta)}(1-p_T)^{c(\eta)} q^{\kappa(\eta)}$.

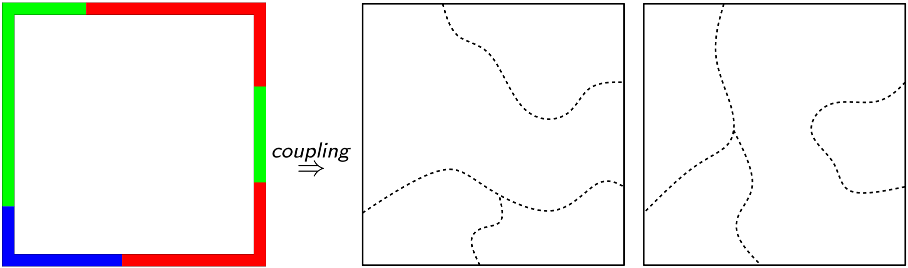
$$\frac{\mathbb{P}^i_T(\sigma_0 = i) - 1/q}{1 - 1/q} = \phi^1_{p_T,q}(0 \leftrightarrow \infty).$$

Edwards-Sokal coupling with the random cluster model

Let $T < T_c(q)$, then $p_T > p_c(q)$, and

$$\boxed{\mathbb{P}^{\omega}_{\Lambda,T} \text{ can be coupled to } \phi^{1}_{\Lambda,p_T,q}(\,\cdot\,|Cond(\omega))}$$

$$Cond(\omega) = \left\{ \begin{array}{c} \exists \text{ interfaces disconnecting the parts of } \partial\Lambda \\ \text{which have different color in } \omega \end{array} \right\}$$



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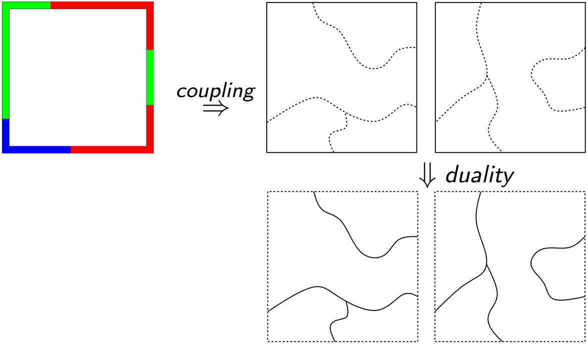
$$Cond(\omega) = \left\{ \begin{array}{c} \exists \text{ open dual paths disconnecting the parts of } \partial\Lambda \\ \text{which have different color in } \omega \end{array} \right\}$$



$\phi^1_{\Lambda,p_T,q}(\,\cdot\,|Cond(\omega))$

is dual to

$\phi^0_{\Lambda^\star,p^\star_T,q}(\,\cdot\,|Cond(\omega))$

with $p^\star_T < p_c(q)$.

Reformulation of the theorem in terms of the FK model

Theorem

For $q \geq 2$ and $p < p_c(q)$, uniformly on the Potts configuration ω,

$$\phi^0_{\Lambda_n,p,q}(\mathcal{C} \cap \Lambda_{n^\varepsilon} = \varnothing \mid Cond(\omega)) = O(n^{-\frac{1}{2}+\varepsilon})$$

where

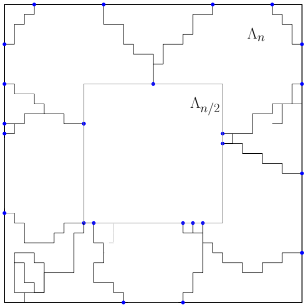
- $Cond(\omega) = \left\{ \begin{array}{c} \exists \text{ open paths disconnecting the parts of } \partial\Lambda \\ \text{which have different color in } \omega \end{array} \right\}$
- *$\mathcal{C}$ is the union of the clusters starting at the color changes.*

Step 1: Macroscopic flower domains

Uniformly in ω, a finite number of interfaces reach $\Lambda_{n/2}$ whp.

Claim : there exists a constant M such that

$$\phi^0_{\Lambda_n}\left(\exists m \in [\frac{n}{2}, n] : |\mathcal{C} \cap \partial\Lambda_m| \leq M \,\middle|\, Cond(\omega)\right) \geq 1 - e^{-cn}$$



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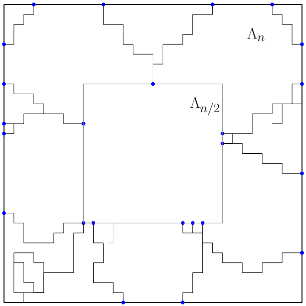

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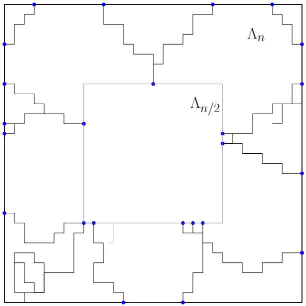

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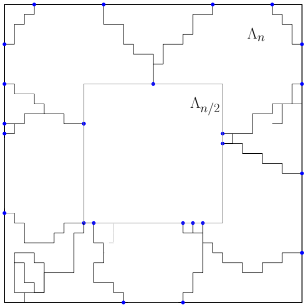

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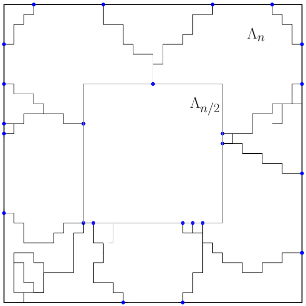

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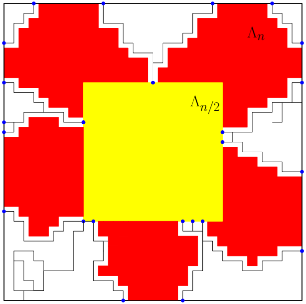

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Step 2: Concentration around Steiner forests

Theorem (Campanino, Ioffe, Velenik, 2008)

$$Let \quad \tau_p(\hat{x}) = -\lim_{n\to\infty} \frac{1}{n} \log \phi_{p,q}(0 \leftrightarrow \lfloor n\hat{x} \rfloor) \quad and \quad \tau_p(x) = |x|\tau_p(\hat{x}).$$

For all $p < p_c(q)$ (*), τ_p is a norm which satisfies the sharp triangle inequality

$$\tau_p(x+y) - \tau_p(x) - \tau_p(y) \geq \kappa_p(|x+y| - |x| - |y|)$$

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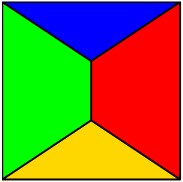
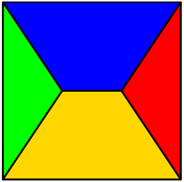
Geometrical consequence: "Minimal surfaces" connecting a finite number of points are forests (union of trees), with internal vertices of degree 3.





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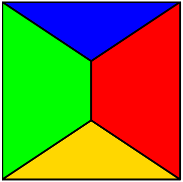
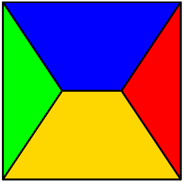
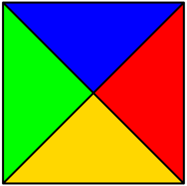




⚠ The L_∞ norm allows degree 4 !



Step 2: Concentration around Steiner forests

Large deviation analysis

In $\Lambda_{n/2}$, the remaining interfaces stay in a

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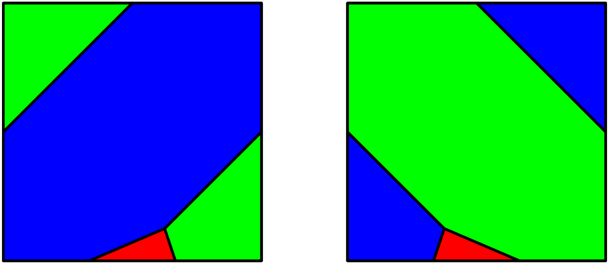
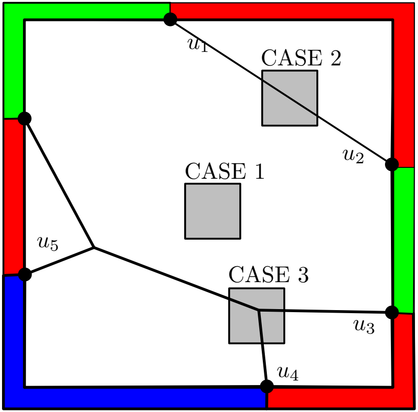
Example with 2 Steiner forests :



Step 3: Fluctuations

Three cases remain to be analysed...



- **Case 1:** exponential relaxation into pure phases.

$$\phi(\mathcal{C} \cap \Lambda_{\delta n} \neq 0) \leq e^{-Cn}$$

- **Case 2:** Brownian scaling of the interfaces between 2 phases.

$$\phi(\mathcal{C} \cap \Lambda_{n^{\varepsilon}} \neq 0) \leq O(n^{-1/2+\varepsilon})$$

- **Case 3:** remains to analyse the fluctuations of triple points.

Step 3: Fluctuations

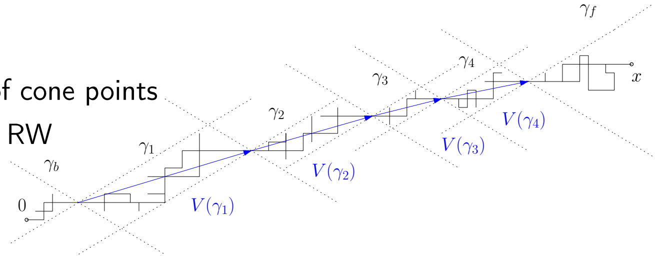
Theorem (Campanino, Ioffe, Velenik, 2008)

For $p < p_c$ (*),

$$\phi_{p,q}(0 \leftrightarrow x) = \frac{\Psi(\hat{x})}{\sqrt{|x|}} e^{-\tau(\hat{x})|x|}(1 + o(1)) \quad \textit{as } |x| \to \infty.$$

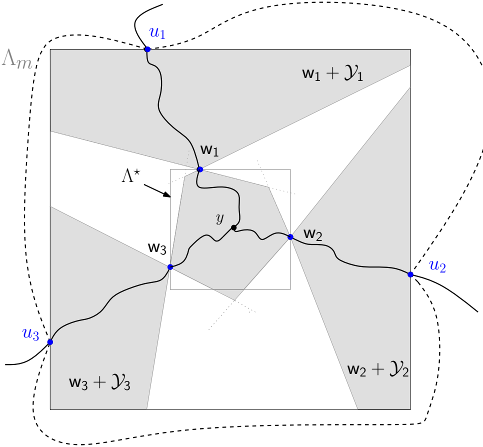
Idea of the proof:

- positive density of cone points
- effective directed RW
- Brownian scaling



Step 3: Fluctuations

Conditioning on $u_1 \leftrightarrow u_2 \leftrightarrow u_3$, the event :



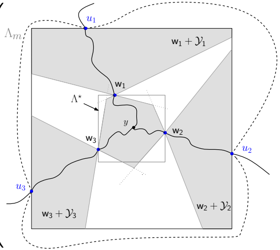
happens for some $y \in \Lambda_{\delta n}$ and some

$\Lambda^\star$ of sidelength $O(n^\varepsilon)$

with probability

$$\geq 1 - \exp(-C n^\varepsilon).$$

Let x be the Steiner triple point between u_1, u_2 and u_3.
For a fixed y, a Taylor expansion of the exponential contribution gives :



$$\phi\left(\text{[diagram]} \,\middle|\, u_1 \leftrightarrow u_2 \leftrightarrow u_3 \right) \sim \exp\left(-C\frac{|x-y|^2}{n} \right)$$

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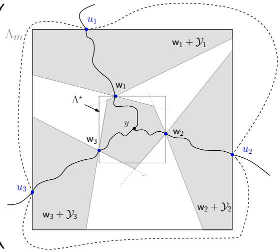

$$\phi\left(\cdot \;\middle|\; u_1 \leftrightarrow u_2 \leftrightarrow u_3 \right) \simeq O\left(\frac{1}{n}\right) \exp\left(-C\frac{|x-y|^2}{n}\right)$$

Prefactor : all $y \in \Lambda_{n^{1/2}}$ contribute the same.

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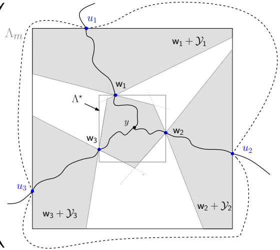

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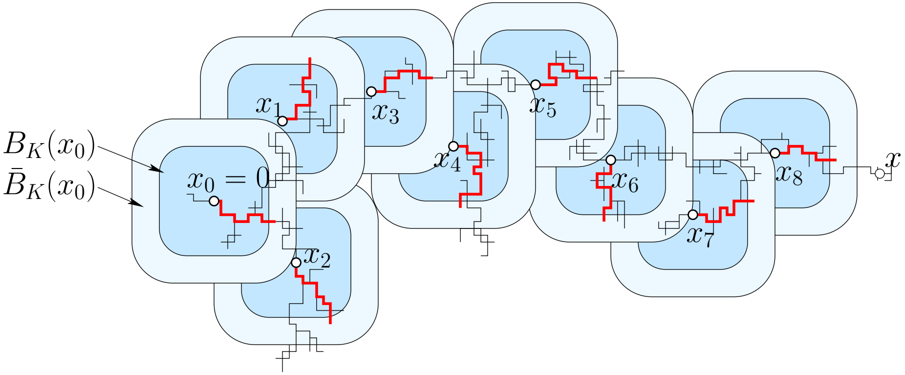
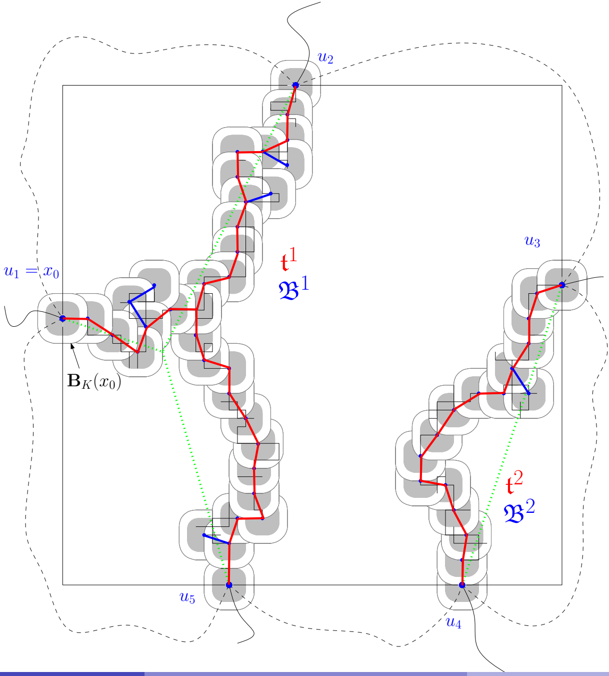
Prefactor : all $y \in \Lambda_{n^{1/2}}$ contribute the same.

A Brownian estimate allows us to conclude :

$$\phi(\mathcal{C} \cap \Lambda_{n^\varepsilon} \neq \varnothing | u_1 \leftrightarrow u_2 \leftrightarrow u_3) \leq O(n^{-1/2+\varepsilon})$$

Thank you !


$B_K(x_0)$
$\bar{B}_K(x_0)$
$x_0 = 0$
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8
x


u_2
u_3
$u_1 = x_0$
$\mathfrak{t}^1$
$\mathfrak{B}^1$
$\mathbf{B}_K(x_0)$
$\mathfrak{t}^2$
$\mathfrak{B}^2$
u_5
u_4

Coarse graining at scale K

$$\phi(E|Cond(\omega)) = \sum_{\mathcal{F}\sim E} \frac{\phi(\mathcal{F}_K = \mathcal{F})}{\phi(Cond(\omega))}$$

$$\begin{aligned}
\phi(\mathcal{F}_K = \mathcal{F}) &\leq \phi\left(\bigcap_{x_i\in\mathcal{F}} x_i \leftrightarrow \partial B_K(x_i)\right)\\
&\leq \phi\left(x_1 \leftrightarrow \partial B_K(x_1) \middle| \bigcap_{i=2}^{|\mathcal{F}|} x_i \leftrightarrow \partial B_K(x_i)\right) \phi\left(\bigcap_{i=2}^{|\mathcal{F}|} x_i \leftrightarrow \partial B_K(x_i)\right)\\
&\overset{FKG}{\leq} \prod_i \phi^1_{B_K}(x_i \leftrightarrow \partial B_K(x_i))\\
&\leq exp(-K|\mathcal{F}|(1 - o_K(1)))
\end{aligned}$$