A finite-volume version of Aizenman-Higuchi theorem for the 2d Ising model

Loren COQUILLE

University of Geneva
joint work with Y.Velenik

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Outlook of the talk

1. Introduction
   - Gibbs measures
   - Known results for the 2d Ising model
2. Statement of the theorem
3. Main tools
4. Sketch of the proof
   - Philosophy
   - The proof in 3 lemmata

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Gibbs measures in finite volume

Definition

$\Lambda \Subset \mathbb{Z}^2, \quad \sigma \in \{-1,+1\}^\Lambda$

Proba of the config. σ

$$\mu^{\omega}_{\Lambda,\beta}(\sigma) = \frac{1}{Z^{\omega}_{\Lambda,\beta}} e^{-\beta H^{\omega}_{\Lambda}(\sigma)}$$

E.g. Ising Hamiltonian

$$H^{\omega}_{\Lambda}(\sigma) = -\sum_{\substack{i,j\in\Lambda \\ i\sim j}} \sigma_i\sigma_j - \sum_{\substack{i\in\Lambda \\ j\in\partial\Lambda}} \sigma_i\omega_j$$

Markov property

$$\mu^{\omega}_{\Lambda,\beta}(\sigma | \sigma_{\text{restr. to } \Lambda\setminus\Delta} = \tilde{\omega}) = \mu^{\omega\cup\tilde{\omega}}_{\Delta,\beta}(\sigma)$$

Gibbs measures in infinite volume

Definition : DLR

μ is ∞-volume Gibbs measure

$$\Leftrightarrow$$

$$\mu(\sigma | \sigma_{\text{restr. to } \Lambda^c} = \omega) = \mu^{\omega}_{\Lambda,\beta}(\sigma)$$

- for μ-a.e. ω
- for all $\Lambda \Subset \mathbb{Z}^2$

Existence

Take $(\Lambda_n)_n \uparrow \mathbb{Z}^2$ and $(\omega_n)_n$ b.c.

$$\mu^{\omega_n}_{\Lambda_n,\beta} \to \mu_\beta$$
$$\Leftrightarrow$$
$$\mu^{\omega_n}_{\Lambda_n,\beta}(f) \to \mu_\beta(f)$$

for all f local function.

Compacity argument $\Rightarrow$ $\exists$ acc. pts

Uniqueness

NO in general !

E.g. (Ising) : $\langle \sigma_0 \rangle^+_\beta \neq \langle \sigma_0 \rangle^-_\beta$ for $\beta > \beta_c$

$\mathcal{G}_\beta$ is a Choquet simplex

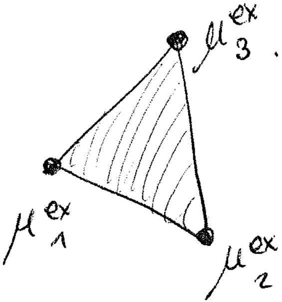
For a given model (Hamiltonian), let

$\mathcal{G}_\beta = \{$ infinite volume Gibbs measures at inverse temperature $\beta\}$

then, $\mathcal{G}_\beta$ is a Choquet simplex, i.e. :

- compact
- convex
- metrisable
- unique decomposition onto a set of extremal measures $\mathcal{G}_\beta^{ex}$:

  $\forall \mu \in \mathcal{G}_\beta, \; \mu \stackrel{!}{=} \sum_i a_i \mu_i^{ex}$ with $\sum_i a_i = 1$



Known results for the 2d Ising model

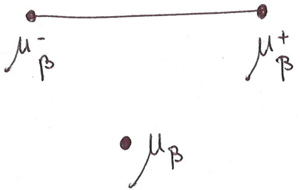
- $\mu_\beta^\pm = \lim_{n\to\infty} \mu_{\Lambda_n,\beta}^\pm \in \mathcal{G}_\beta^{ex}$
- 1975 : Messager, Miracle-Sole

$$\mu \in \mathcal{G}_\beta \text{ translation invariant}$$
$$\Downarrow$$
$$\mu = a\mu_\beta^+ + (1-a)\mu_\beta^-$$

- 1980 : Aizenman $\amalg$ Higuchi
  $\forall \mu \in \mathcal{G}_\beta,\ \mu$ is tranlation invariant

$$\beta \le \beta_c \Rightarrow \mathcal{G}_\beta = \left[\mu_\beta^-, \mu_\beta^-\right]$$

$$\beta > \beta_c \Rightarrow \mathcal{G}_\beta = \{\text{ the ! measure }\}$$



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Finite volume approach : how do we see AH theorem in a physical system ?

Theorem (C. - Velenik)

Let $\beta > \beta_c$.
For every $\xi < 1/2$ and $0 < \delta < 1/2 - \xi$, as n tends to infinity, there exists a constant $a^{n,\omega}(\beta) \in [0, 1]$ such that,

$$\mu^{\omega}_{\Lambda_n,\beta}(f) = a^{n,\omega}\mu^{+}_{\beta}(f) + (1 - a^{n,\omega})\mu^{-}_{\beta}(f) + O(\|f\|_{\infty}\, n^{-\delta})$$

where the O notation is uniform in the boundary condition ω and in function f having support in $\Lambda_{n^{\xi}}$

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Main tools

- **FKG inequality :**
  Increasing functions are positively correlated.
- **Kramers-Wannier duality :** ($\tanh \beta^* = e^{-2\beta}$)

$$\frac{Z^{\pm,(x,y)}_{\Lambda,\beta}}{Z^{+}_{\Lambda,\beta}} = \langle \sigma_x \sigma_y \rangle^{free}_{\Lambda^*,\beta^*}$$

Strict positivity of the surface tension at inverse temperature β
$\Leftrightarrow$
Strict positivity of the inverse correlation length at dual inverse temperature β^*.

- **BK inequality :** Writing $\frac{Z^{\omega}_{\Lambda,\beta}}{Z^{+}_{\Lambda,\beta}} = \sum_{\underline{\gamma} \sim \omega} q_{\Lambda,\beta}(\underline{\gamma})$ We have

$$\sum_{\substack{\gamma_1 : x_1 \to y_1 \\ \gamma_2 : x_2 \to y_2}} q(\gamma_1, \gamma_2) \leq \sum_{\gamma_1 : x_1 \to y_1} q(\gamma_1) \sum_{\gamma_2 : x_2 \to y_2} q(\gamma_2)$$

Main tools (bis)

- **Finite volume estimates for the 2-points function (Ornstein-Zernike prefactor):**
  For all $x, y \in \partial\Lambda_n$ such that $\overline{xy} \cap \Lambda_{n/2} \neq \emptyset$, we have

$$\frac{C_1}{n^{1/2}} e^{-\tau_\beta(x-y)} \leq \frac{Z^{\pm(x,y)}_{\Lambda_n,\beta}}{Z^{+}_{\Lambda_n,\beta}} \leq \frac{C_2}{|x-y|^{1/2}} e^{-\tau_\beta(x-y)}$$

- **Exponential relaxation in pure phases :**
  For every $\beta > \beta_c$, uniformly in the local function f,

$$|\mu^{+}_{\Lambda,\beta}(f) - \mu^{+}_{\beta}(f)| \leq ||f||_\infty |Supp(f)| e^{-C \cdot d(Supp(f),\Lambda^c)}$$

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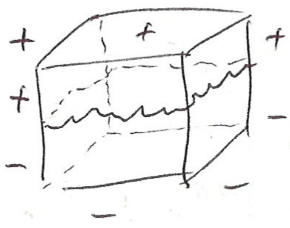
Key remarks

3D : How to build non-translation invariant measure ?



Take $(\Lambda_n)_n \uparrow \mathbb{Z}^3$ with Dobrushin b.c.

$\Rightarrow$

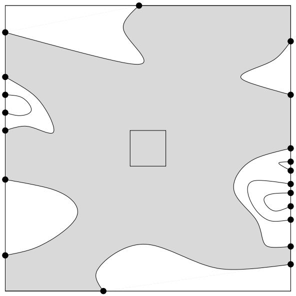
Bounded fluctuations for large β

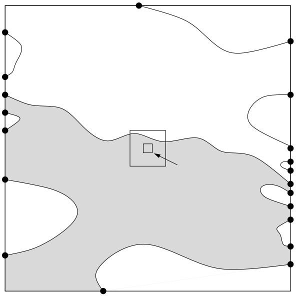
2D : All $\mu \in \mathcal{G}_\beta$ are translation invariant (AH)

Unbounded fluctuations of the Dobrushin interface $\forall \beta$

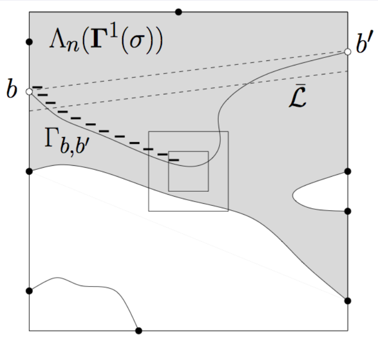
What about arbitrary boundary condition ω ?

Our goal is to show that, for arbitrary b.c. ω, and for sufficiently large systems, every local function *f* finds its support either *deep* in the "$+$ phase" or *deep* in the "– phase"
... and to quantify this *deep*.





Lemma 1



Take $a < 1$. If the interface $\Gamma_{bb'}$ reaches the box Λ_{n^a}, then $\overline{bb'}$ must intersect Λ_{2n^a} with probability $1 - o_n(1)$.

More precisely,

$$\mu^{\omega}_{\Lambda_n,\beta}\left(\begin{array}{c}\exists\ \Gamma_{bb'} \text{ such that } \Gamma_{bb'} \cap \Lambda_{n^a} \neq \emptyset \\ \text{and } \overline{bb'} \cap \Lambda_{2n^a} = \emptyset\end{array}\right) \leq e^{-C(\beta)\ n^{2a-1}}$$

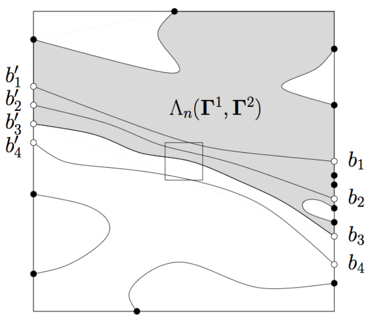
Idea of the proof

- Condition on interfaces above $\overline{bb'}$.
- Notice that the event "$\Gamma_{bb'}$ reaches the box Λ_{n^a}" is decreasing (wlog there exists a path of - spins reaching the little box)
- Use FKG inequality and Markov property to work in the box Λ_n with $\pm(b, b')$ boundary condition. Work with surface tension :

$$\mu_{\Lambda_n,\beta}^{\pm(b,b')}(\Gamma_{bb'} \cap \mathcal{L} \neq \emptyset) = \underbrace{\frac{Z_{\Lambda_n,\beta}^{\pm,(b,b')}(\Gamma_{bb'} \cap \mathcal{L} \neq \emptyset)}{Z_{\Lambda_n,\beta}^{+}}}_{(1)} \underbrace{\frac{Z_{\Lambda_n,\beta}^{+}}{Z_{\Lambda_n,\beta}^{\pm(b,b')}}}_{(2)}$$

- Use BK inequality to bound (1) from above, and a soft finite volume estimate to bound $(2)^{-1}$ from below.

Lemma 2



The number N_{cross} of interfaces intersecting Λ_{n^a} is either 0 or 1 with probability $1 - o_n(1)$.

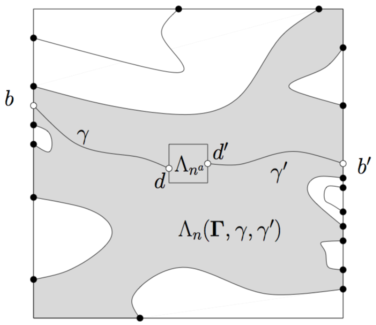
More precisely,

$$\mu^{\omega}_{\Lambda_n,\beta}(N_{cross} \geq 2) \leq e^{-C(\beta)\, n}$$

Idea of the proof

- Use Lemma 1 to notice that endpoints of crossing interfaces are "quasi" diametraly opposed.
- Condition on interfaces above the first crossing.
- In the random box (colored in grey on the picture), compare, for the two remaining interfaces, the cost of crossing (soft infinite volume upper bound) and of following the boundary (soft estimate, equivalent to a partial change of boundary condition outside the box Λ_n).

Lemma 3



Take $b < a/2$. If there is exactly one crossing interface Γ, then it will miss a box Λ_{n^b} with probability $1 - o_n(1)$.

More precisely,

$$\mu^{\omega}_{\Lambda_n,\beta}(N_{cross} = 1 \text{ and } \Gamma \cap \Lambda_{2n^b} \neq \emptyset) \leq C\, n^{b-a/2}$$

Idea of the proof

Essentially the same as for Lemma 1, but applied to the box Λ_{n^a} :

- By Lemma 1, d and d' are "quasi" diametraly opposed with high probability.
- Use the refined finite volume estimate (Ornstein-Zernike prefactor) to get the right lower bound.
- Key point : $\Gamma_{dd'}$ has a Brownian bridge behavior in the scaling limit, so we have a good lower bound on the fluctuations of this interface.

Conclusion : interpretation of the constants $a^{n,\omega}$

For f local function "living" in Λ_{n^b},

$$\mu^{\omega}_{\Lambda_n,\beta}(f) = \mu^{\omega}_{\Lambda_n,\beta}(\mathcal{I}^{+})\mu^{+}_{\beta}(f) + \mu^{\omega}_{\Lambda_n,\beta}(\mathcal{I}^{-})\mu^{-}_{\beta}(f) + O_{\beta}(\|f\|_{\infty} n^{b-a/2})$$

where $\mathcal{I}^{\pm} = \mathcal{I}^{\pm}_0 \cup \mathcal{I}^{\pm}_1$,

$$\mathcal{I}^{\pm}_0 = \{0 \text{ crossing interface}, \Lambda_{n^b} \subset \pm phase\}$$
$$\mathcal{I}^{\pm}_1 = \{1 \text{ crossing interface}, \Lambda_{n^b} \subset \pm phase\}$$