QUASI-CONTACT S-R METRICS: NORMAL FORM IN $\mathbf{R}^{2n}$, WAVE FRONT AND CAUSTIC IN $\mathbf{R}^4$

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Abstract

This paper deals with sub-Riemannian metrics in the quasi-contact case. First, in any even dimension, we construct normal coordinates, a normal form and invariants, which are the analogs of normal coordinates, normal form and classical invariants in Riemannian geometry. Second, in dimension 4, and thanks to this "normal form", we study the local singularities of the exponential map.

1. Introduction

This paper is a continuation of a series of papers ([C-G-K], [A-C-G-K], [A-C-G], [A-G], [A-C-G-Z]) dealing with contact metrics. In these papers, the authors construct normal coordinates, a canonical field of normal frames and invariant tensors. These objects are the analogs of normal coordinates, normal frame and classical invariant tensors (such as the curvature tensor) in Riemannian geometry. They do this in any odd dimension. They also study in details the exponential map, the wave-front, the cut locus and the conjugate locus in dimension 3.

In this paper, we construct the same type of normal coordinates, canonical field of normal frame and invariant tensors **in the quasi-contact case** in any even dimension. We also study the exponential map and the conjugate locus in the 4-dimensional case.

A quasi-contact structure, on a $(2n+2)$-manifold M, is a sub-bundle Δ of the tangent bundle, of codimension 1, such that, if Δ is the kernel of a 1-form ω, then the kernel of $\mathrm{d}\omega_{|\Delta}$ has dimension 1. A quasi-contact sub-Riemannian metric on a $(2n+2)$-manifold M is the data of a couple (Δ, g), with Δ a quasi-contact structure and g a non-degenerate Riemannian metric on Δ.

Almost everywhere in the paper, the study is local, that is, we consider germs of quasi-contact sub-Riemannian structures at a point.

1.1. **Invariants. Let us fix q in M** and let us choose ω, defined locally, such that $\ker\omega = \Delta$. It is defined up to a non vanishing function on a neighborhood of q. The kernel $\ker \mathrm{d}\omega_{|\Delta}$ is defined without ambiguity and has dimension 1. Let us denote by Δ_0 the orthogonal for g in Δ of this kernel. We ask ω to satisfy that $(\mathrm{d}\omega_{|\Delta_0})^n = \mathrm{Vol}_{g|\Delta_0}$. Now ω is determined up to sign.

Let $A_q : \Delta_q \to \Delta_q$ be defined by $\mathrm{d}\omega_{|\Delta}(X,Y) = g(A_q(X), Y)$, where A_q is skew-symmetric wrt g. For a generic quasi-contact structure and outside a codimension 3 closed stratified subset, A_q has a set of eigenvalues of the form:

$$\{-\mathrm{i}\alpha_1, \ldots, -\mathrm{i}\alpha_n, 0, \mathrm{i}\alpha_n, \ldots, \mathrm{i}\alpha_1\},$$

where $0 < \alpha_n < \ldots < \alpha_1$ and $\prod \alpha_i = \frac{1}{n!}$ because $(\mathrm{d}\omega_{|\Delta_0})^n = \mathrm{vol}_g$. For $i = 1 \ldots n$, let us denote by δ_i the distribution defined as the 2-dimensional invariant space associated with α_i. First remark that $[\delta_i, \delta_i]$ is not contained in Δ (see lemma 4 in appendix). We denote by δ the orthogonal of δ_1 for $\mathrm{d}\omega$. It is easy to check that the intersection of $[\delta_1, \delta_1]$ and δ has dimension 1 and is transversal to Δ. We choose ν the vector field in this intersection such that $\omega(\nu) = 1$.
Remark: first, the eigenvalues of A_q are the first invariants of the problem. Second, α_1 plays a special rôle because it determines the first non trivial term in the asymptotics of the conjugate time, defined in the next section. Third, for a generic sub-Riemannian structure, the set of points q where at least two eigenvalues of A_q are equal is a codimension 3 closed stratified subset. **In all the sequel we will consider such a generic sub-Riemannian quasi-contact structure and we will assume that the pole q is outside this stratified set**.

In the sequel of this section and all along the paper, we use some terminology and some preliminaries that are the purpose of the next section.

A consequence of more general results from section 3 is:

Theorem 1. *In a neighborhood of a point q where the α_i are all distinct and non zero, there exists a local coordinate system $(x_1, y_1, \ldots, x_n, y_n, z, w)$, called a normal coordinate system, such that:*

- *Along the w-axis we have $\frac{\partial}{\partial w} = \nu$.*
- *$span(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = \delta_i$, $d\omega(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) > 0$ and $span(\frac{\partial}{\partial z}) = \ker d\omega_{|\Delta}$ along the w-axis.*
- *The lines contained in a set $S_{w_0} = \{w = w_0\}$ and containing $(0, w_0)$ are geodesics of the sub-Riemannian structure, that minimize the sub-Riemannian distance to the w-axis. The distance between (x_i, y_i, z, w) and the w-axis is $\sqrt{z^2 + \sum_i x_i^2 + y_i^2}$*
- *If we denote by P_{w_0} the orthogonal projection on S_{w_0} in this coordinate system, then its differential at a point of S_{w_0} maps the metric g on Δ at this point to a non-degenerate metric g_{w_0} on S_{w_0}. The sectional curvatures of g_{w_0}, relative to the 2-planes $\delta_i(0, w_0)$ all vanish at $(0, w_0)$.*

This normal coordinate system is unique up to rotations in the spaces $\delta_i(q)$ hence up to elements of a maximal torus T^n of $SO(\Delta_q)$. It should be noticed that this situation is much more rigid, and is simpler, than that of Riemannian geometry: in the Riemannian case, in normal coordinates, the remaining "structure group" is $SO(T_qM)$ (non Abelian). As a consequence, in our case, all the invariant tensors can be reduced to complex numbers (characters).

In such normal coordinates, we are able to construct a unique natural *normal form* for the metric, that is, a canonical choice of a field of orthonormal frames of the distribution Δ. This normal form allows to obtain a family of invariants that we discuss in section 4.

1.2. **Caustic in dimension 4.**

As in Riemannian geometry we may consider the geodesics issued from a fixed point q, that allow to define an exponential map. The first singular value of the exponential map along a geodesic is called a first conjugate point and the set of these points forms the first caustic. Contrarily to the Riemannian case, the conjugate locus associated with a point q has q in its closure. In the 4-dimensional case, with generic conditions on some invariants that we will discuss in section 4, a theorem, consequence of the main results in this paper, is:

Theorem 2 (illustration of further results). *1) For a generic point q of a generic sub-Riemannian quasi-contact structure, the local first singularities of the exponential map have type $\mathcal{A}_2$ (folds), $\mathcal{A}_3$ (cusps) and $\mathcal{D}_4^+$.*

2) The first caustic is as follows:

- *In normal coordinates, if we cut the upper conjugate locus (the $w > 0$ part) by a 3-plane $z = z_0$ with $z_0 \neq 0$ small enough, we find a surface (see figure 1).*
  - *The points where it is smooth are singular values where the singularity of the exponential map has type $\mathcal{A}_2$ (folds).*
  - *The points where the surface is not smooth are cusp points ($\mathcal{A}_3$) except at two points which are of type half $\mathcal{D}_4^+$*[1].

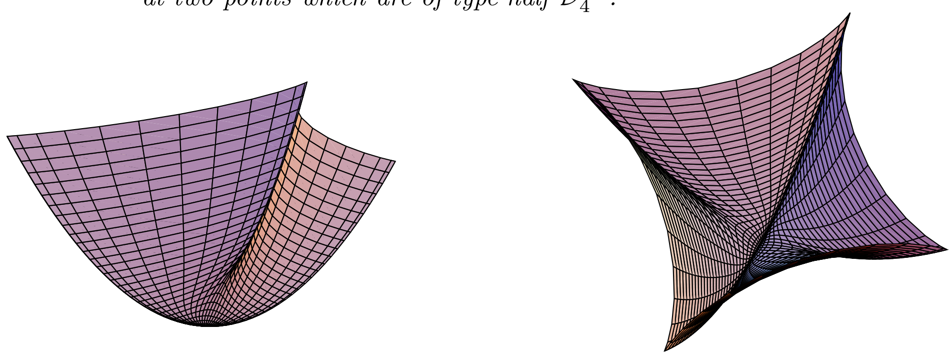

FIGURE 1. Intersection with a plane $\{z = z_0 \neq 0\}$

- *In normal coordinates, the section of the first caustic by the plane $\{z = 0\}$ is the same as the 3-dimensional contact generic caustic: see* [C-G-K] *and figure 2.*

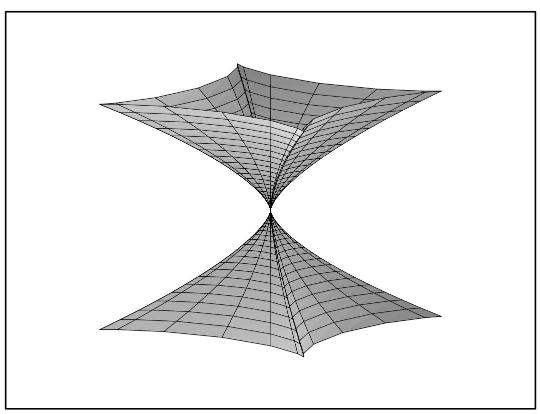

FIGURE 2. Intersection with the plane $\{z = 0\}$

[1]half $\mathcal{D}_4^+$ corresponds to the first conjugate time and the second half to the second conjugate time: at such points, they are equal.

- *In normal coordinates, if we cut the conjugate locus by a 3-plane $w = w_0$ with $w_0 \neq 0$ small enough, we find an other surface (see figure 3), such that the points where it is smooth are folds of the exponential map and the singular points of the surface are cusps.*

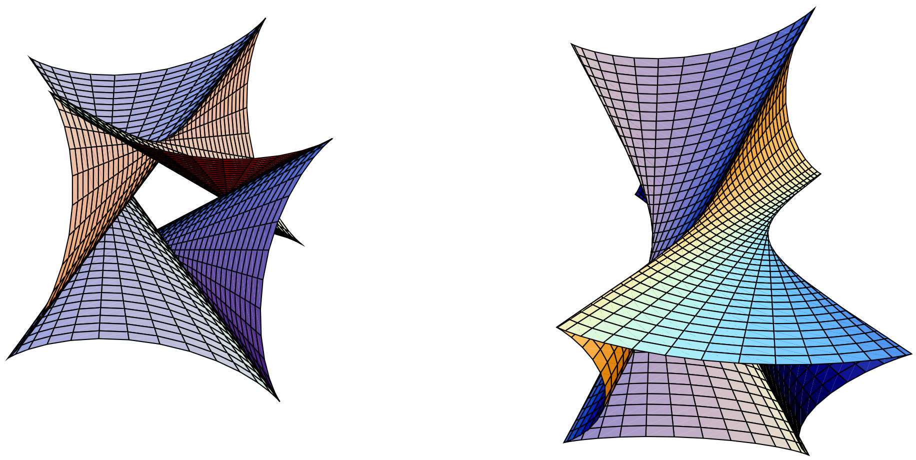

FIGURE 3. Intersection with a 3-plane $\{w = w_0 \neq 0\}$

Some computations in this paper are difficult to handle. In particular to compute the jets of the exponential map, we have used the formal calculus language **Mathematica**.

2. Preliminaries

2.1. **The Control Theory point of view.** Let q_0 be a fixed point, $\{F_i\}$ a field of orthonormal frames of the distribution Δ and $\mathcal{U} = L^2(\mathbf{R}_+, \mathbf{R}^{2n+1})$. With an element u of $\mathcal{U}$, called the control function, one can associate the curve issued from q_0 obtained by integrating $\dot{x}(t) = \sum_i u_i(t)F_i(x(t))$. The set of all such curves issued from any point of M is the set of the *admissible* curves. In particular, almost everywhere, such a curve γ satisfies $\dot{\gamma}(t) \in \Delta(\gamma(t))$. The length of such a curve, between the times t_1 and t_2, is:

$$L(\gamma) = \int_{t_1}^{t_2} \sqrt{\sum_i u_i^2(t)}\mathrm{d}t,$$

that may also be written:

$$L(\gamma) = \int_{t_1}^{t_2} \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}\mathrm{d}t.$$

A *geodesic* is an admissible curve being locally optimal for the problem of minimizing the length between two points. As in Riemannian geometry, it is a direct consequence of the Cauchy Schwarz inequality that we can search the geodesics as the trajectories minimizing the energy $E(\gamma) = \int_{t_1}^{t_2} \sum_i u_i^2(t)\mathrm{d}t$ in fixed time. Now, the Pontryagin Maximum Principle gives the candidates, called extremals, that are of two types: normal and abnormal. A strictly abnormal extremal is an abnormal one that cannot be realized as a normal one. Normal extremals are always locally C^0-optimal. In our case, there is no strictly abnormal geodesic (see lemma 4 in appendix), hence we have to compute the normal geodesics only.

Let us consider the canonical fibration $\pi : T^*M \to M$ and the canonical symplectic structure on T^*M. The Hamiltonian:

$$\mathcal{H}(\psi) = \frac{1}{2} \sup_{v \in \Delta \setminus \{0\}} (\frac{\psi(v)}{\|v\|})^2 = \frac{1}{2} \sum_i (\psi_x(F_i(x))^2),$$

is a smooth function on T^*M hence we can define its symplectic gradient $\overrightarrow{\mathcal{H}}$ for the canonical symplectic structure. Now, the fact that any geodesic can be realized as a normal extremal, together with the Maximum Principle of Pontryagin, allow to check that the geodesics are the projections, on the basis of the fibration π, of the integral curves of $\overrightarrow{\mathcal{H}}$ in T^*M.

2.2. **Some geometrical objects.** This last fact allows to define the exponential map:

$$\begin{array}{cccc} exp_{q_0} : & T^*_{q_0}M & \to & M \\ & \psi & \mapsto & \pi(\Phi_{\overrightarrow{\mathcal{H}}}(\psi)), \end{array}$$

where $\Phi_{\overrightarrow{\mathcal{H}}}$ is the flow at time 1 of $\overrightarrow{\mathcal{H}}$. This map is smooth since it is the composition of two smooth maps.

Now, we can state some definitions:

Definition 1 (Sub-Riemannian objects)**.**

- *The sphere $S(q_0, r)$ of center q_0 and radius r is the set of all the points the distance of which to q_0 is r.*
- *The wave front of center q_0 and radius r is the set of endpoints of the geodesics issued from q_0 and with length r.*
- *The conjugate locus is the set of the singular values of the exponential mapping. We also call it the caustic.*
- *The cut locus is the set of points where a geodesic loses its global optimality.*
- *Let us take $\psi_0 \in T^*_{q_0}M$, then it can exist a time $s \in \mathbf{R}^*_+$ being the smallest time such that exp_{q_0} is singular at $s\psi_0$. Then $exp_{q_0}(s\psi_0)$ is called a first conjugate point and the set of all the first conjugate points is called the first conjugate locus or first caustic.*

2.3. **Symplectic considerations.** [see [A-V-G]] Any fiber of π is lagrangian for the canonical symplectic structure of T^*M hence we say that π is a lagrangian fibration. Let us recall that a lagrangian mapping is the composition of an embedding in a lagrangian fibration, the image of which is a lagrangian submanifold, with the canonical projection on the basis of the fibration.

Now, the exponential map exp_{q_0} appears clearly to be a lagrangian mapping: actually, the flow $\Phi_{\overrightarrow{\mathcal{H}}}$ preserves the symplectic structure and $T^*_{q_0}M$ is a lagrangian submanifold hence $\Phi_{\overrightarrow{\mathcal{H}}}(T^*_{q_0}M)$ is a symplectic submanifold and, as a consequence, the exponential map being the composition of the flow, restricted to $T^*_{q_0}M$, with the fibration π, is a lagrangian map.

2.4. **Stability of the exponential map.** [see [A-V-G]] In this paper we will study the local stability, on an open set of its domain, of the exponential map. For this we may consider the exponential map either as a smooth map or as a lagrangian map. Hence we may study either its C^∞-stability or its lagrangian stability:

Definition 2.
- *A smooth map f is said to be C^∞-stable in a neighborhood of a point q if there exist a neighborhood V_q of q and a neighborhood U_f of $f_{|V_q}$ (C^∞-topology) such that for any g in U_f, there exist diffeomorphisms φ_1 and φ_2 such that $\varphi_1 \circ g \circ \varphi_2 = f$ on V_q.*
- *A lagrangian equivalence of lagrangian fibrations is a diffeomorphism between lagrangian fibrations mapping the symplectic structure to the symplectic structure and fibers to fibers.*
- *Two lagrangian mappings are Lagrange-equivalent if there exists a lagrangian equivalence between the two corresponding lagrangian fibrations mapping the first lagrangian submanifold on the second one.*
- *A lagrangian mapping f is said to be Lagrange-stable in a neighborhood of a point q if there exist a neighborhood V_q of q and a neighborhood U_f of $f_{|V_q}$ for C^∞-topology such that any g lagrangian in U_f is Lagrange-equivalent to $f_{|V_q}$.*

Let us take a (smooth or lagrangian) map f. We may consider its (smooth or Lagrange) equivalence class, that is the class of maps being equivalent (for C^∞ or Lagrange equivalence) to f. In this class, we can choose a special map that we consider as the representative of the class. We call it the normal form of the class.

In this paper we will consider three particular singularities, or classes of maps, defined in a neighborhood of a point q.

- The first one is $\mathcal{A}_2$. It is both a lagrangian equivalence class and a smooth equivalence class. It has the normal form:
$$\left\{ \begin{array}{ccc} x & \mapsto & x^2 \\ \mathbf{R} & \to & \mathbf{R} \end{array} \right.$$
at 0. If a map from $\mathbf{R}^k$ to $\mathbf{R}^k$ is equivalent to
$$(x_1, \ldots, x_{k-1}, x_k) \mapsto (x_1, \ldots, x_{k-1}, x_k^2)$$
at 0, we still say that it has singularity $\mathcal{A}_2$. If a map f from $\mathbf{R}^k$ to $\mathbf{R}^k$ is such that there is a variable x_i satisfying
$$\mathrm{Jac}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k})(0) = 0 \quad \text{but} \quad \mathrm{Jac}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial^2 f}{\partial x_i^2}, \ldots, \frac{\partial f}{\partial x_k})(0) \neq 0$$
then it has singularity $\mathcal{A}_2$ at 0. A map with singularity $\mathcal{A}_2$ at a point q may always be seen locally as a lagrangian map and is **stable** in a neighborhood of the point q **for both C^∞ and lagrangian stabilities**.
- The second one is $\mathcal{A}_3$. It is both a lagrangian equivalence class and a smooth equivalence class. It has the normal form:
$$\left\{ \begin{array}{ccc} (x, y) & \mapsto & (x^3 + xy, y) \\ \mathbf{R}^2 & \to & \mathbf{R}^2 \end{array} \right.$$
at 0. If a map from $\mathbf{R}^k$ to $\mathbf{R}^k$ is equivalent to
$$(x_1, \ldots, x_{k-1}, x_k) \mapsto (x_1, \ldots, x_{k-1}, x_k^3 + x_{k-1}x_k)$$
at 0, we still say that it has singularity $\mathcal{A}_3$. If a map f from $\mathbf{R}^k$ to $\mathbf{R}^k$ is such that we can find a coordinate system $(x_1, \ldots, x_k)$ with $E_{k-1} = \mathrm{span}(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{k-1}})$ having dimension $k-1$ and $(\frac{\partial}{\partial x_k}, \frac{\partial^2}{\partial x_k^2})$ are in E_{k-1} but $\frac{\partial^3 f}{\partial x_k^3}$ and $\frac{\partial^2 f}{\partial x_k \partial x_{k-1}}$ are not in E_{k-1}, then f has singularity $\mathcal{A}_3$. A map

with singularity $\mathcal{A}_3$ at q may always be seen locally as a lagrangian map and is **stable** in a neighborhood of the point q **for both C^∞ and lagrangian stabilities**.

- The last one is $\mathcal{D}_4^+$. It has the normal form:

$$\left\{ \begin{array}{ccc} (x,y,z) & \mapsto & (x, yz, z^2+y^2+xz) \\ \mathbf{R}^3 & \to & \mathbf{R}^3 \end{array} \right.$$

at 0. If a map from $\mathbf{R}^k$ to $\mathbf{R}^k$ is equivalent to

$$(x_1,\ldots,x_{k-1},x_k) \mapsto (x_1,\ldots,x_{k-2},x_{k-1}x_k, x_k^2+x_{k-1}^2+x_{k-2}x_k)$$

at 0, we still say that it has singularity $\mathcal{D}_4^+$ at $(x_1,\ldots,x_{k-3},0,0,0)$. A map with singularity $\mathcal{D}_4^+$ at a point q is locally a smooth and a lagrangian map because it is equivalent to the normal form which is a lagrangian map. **It is not stable as smooth map** from $\mathbf{R}^3$ to $\mathbf{R}^3$, but **it is lagrangian stable**.

Important fact: the singularity $\mathcal{A}_2$ is 2-determined, i.e., if a map has the same 2-jet as the normal form of $\mathcal{A}_2$ at a point q, then it has singularity $\mathcal{A}_2$ at point q. $\mathcal{A}_3$ is 3-determined. $\mathcal{D}_4^+$ is 2-determined, i.e., if a lagrangian map has the same 2-jet as the normal form of $\mathcal{D}_4^+$ at a point q, then it has singularity $\mathcal{D}_4^+$ at a point q.

2.5. **Nilpotent approximation.** [see [B] for more details] In the following, we will need the definition of the nilpotent approximation.

Let us go back to the general sub-Riemannian situation. Let Δ be a distribution on a manifold M^n satisfying the Chow condition (its Lie algebra evaluated at any point is the whole tangent space at this point). We define $L^s(q)=\sum_{i\leq s}\Delta^i(q)$ and we construct a coordinate system $(z_1,\ldots,z_n)$ on M such that $\mathrm{d}z_1,\ldots,\mathrm{d}z_n$ is adapted to the flag $L^1(q)\subsetneq L^2(q)\subsetneq\ldots\subsetneq L^r(p)=T_qM$ and we define the sequence (of weights):

$$w_1\leq\ldots\leq w_n$$

by $w_i=s$ if $\frac{\partial}{\partial z_i}$ belongs to $L^s(q)$ but do not belong to $L^{s-1}(q)$. Let $(X_1,\ldots,X_k)$ be a field of orthonormal frame of the distribution Δ. We say that a function has order s if all its derivatives of order less than s w.r.t. the X_i are zero but at least one derivative of order s is not zero.

Theorem 3. *There exist coordinate systems $(z_1,\ldots,z_n)$ (called privileged coordinate systems), centered at q, such that z_i has order w_i. They satisfy:*

$$d(0,(z_1,\ldots,z_n))\asymp|z_1|^{\frac{1}{w_1}}+\ldots+|z_n|^{\frac{1}{w_n}}.$$

Our normal coordinate system, described in the introduction, is a privileged coordinate system. The coordinates x_i, y_i and z have weight 1, the coordinate w has weight 2.

These weights induce gradations in $C^\infty(M)$, $\mathcal{X}^\infty(M)$, $\bigwedge^\infty(M)$ and $C^\infty(T^*M)$ in the usual way. These gradations are consistent in particular with the Lie bracket and the Poisson bracket. For instance, a monomial $z_1^{\beta_1}\ldots z_n^{\beta_n}$ has order $w_1\beta_1+\ldots+w_n\beta_n$, a function has order k at a point q if the monomial of smaller order appearing in its Taylor expansion has order k, and a vector field $\frac{\partial}{\partial z_i}$ has order $-w_i$.

Now, if we denote by $\{F_i\}$ a field of orthonormal frames of the distribution and $\widetilde{F}_i$ the homogeneous part of smallest degree (-1) of F_i, then $\{\widetilde{F}_i\}$ is said to be the nilpotent approximation of $\{F_i\}$ at q.

Theorem 4. *In the quasi-contact case, in the normal coordinates, the nilpotent approximation constructed from our "normal form" is:*

$$(\frac{\partial}{\partial x_1} + \alpha_1 \frac{y_1}{2}\frac{\partial}{\partial w}, \frac{\partial}{\partial y_1} - \alpha_1 \frac{x_1}{2}\frac{\partial}{\partial w}, \ldots, \frac{\partial}{\partial x_n} + \alpha_n \frac{y_n}{2}\frac{\partial}{\partial w}, \frac{\partial}{\partial y_n} - \alpha_n \frac{x_n}{2}\frac{\partial}{\partial w}, \frac{\partial}{\partial z}).$$

It is an invariant of the quasi-contact metric and the only invariants of the nilpotent approximation are the invariants α_i already defined. In particular, in the 4-dimensional case, there is no invariant for the nilpotent approximation.

The proof is a direct consequence of the construction of the "normal form" which is done in the next section.

Remark: let us denote by v the dual coordinate to w in T^*M. It may be proved that for the nilpotent approximation, any geodesic γ parameterized by arclength and such that $\dot{\gamma}(0) \neq \pm\frac{\partial}{\partial z}$ has conjugate time equal to $\frac{2\pi}{\alpha_1 v}$ (in any dimension).

3. Normal form in the quasi-contact case

We assume again that we are at a generic point as defined in the introduction. The form ω and the vector field ν satisfy the same conditions as in the introduction.

3.1. **normal coordinates.** The purpose of this subsection is the construction, from the structure (M, Δ, g), of a normal coordinate system around q_0. For this, we follow an idea very similar to the idea of the construction of normal forms in Riemannian geometry.

First, we define the curve Γ around which we will construct the coordinates:

$$\Gamma(0) = q_0,$$
$$\frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t} = \nu(\Gamma(t)).$$

It is transversal to the distribution. Now, we can define the subspace $A_{\Gamma(t)}$ of $T^*_{\Gamma(t)}M$ by $A_{\Gamma(t)} = \{\psi \in T^*_{\Gamma(t)}M |\ \psi(\nu(\Gamma(t))) = 0\}$, and $A_\Gamma = \cup_{t\in]-\varepsilon,\varepsilon[} A_{\Gamma(t)}$.

Proposition 1. *In a neighborhood of a point q_0, there exists a smooth coordinate system $(p_1, \ldots, p_{2n}, p_{2n+1}, w)$ such that:*

1. $\Gamma(w) = (0, w)$;
2. *The geodesics starting from $\Gamma(w_0)$ and satisfying the transversality Conditions to Γ are straight lines contained in $S_{w_0} = \{w = w_0\}$;*
3. *For s small enough $C_s = \{\sum_j p_j^2 = s^2\} = \{q |\ d(q,\Gamma) = s\}$.*

Proof: first remark that A_Γ is the union of all the impulses along Γ satisfying the Pontryagin's Transversality Condition for the problem of the minimization of the distance to Γ.

Let us choose (F_i) a field of orthonormal frames of Δ in a neighborhood of q_0 and denote by $< .,. >$ the metric on T^*M:

$$< \psi_1, \psi_2 >= \frac{1}{2}\sum_{i=1}^{2n+1} \psi_1(F_i)\psi_2(F_i),$$

$(\psi_1(w), \ldots, \psi_{2n+1}(w))$ an orthonormal frame of $A_{\Gamma(w)}$ for $<,>$ and $\mathcal{E}$ the map mapping $(p_1, \ldots, p_{2n+1}, w)$ to the point obtained by following the geodesic issued from

$A_{\Gamma(w)}$ with the impulse $\psi = \sum p_i\psi_i$ during the time 1. $\mathcal{E}$ is a local diffeomorphism hence $(p_1, \ldots, p_{2n+1}, w)$ can be used as a coordinate system.

The geodesics considered all satisfy the Transversality Condition because their impulses at time 0 are in A_Γ. Because of this, the points 1 and 2 are satisfied.

For the point 3, we know that the map we have just defined is a local diffeomorphism hence two different geodesics transversal to Γ can not reach the same point for a distance s small enough. Therefore there is no cut locus, for the local problem of minimizing the distance to Γ, close to 0. This is enough to conclude. ∎

Such a coordinate system $(p_1, \ldots, p_{2n+1}, w)$ is said to be *adapted to* Γ. It is unique up to a change of coordinates such that:

$$\begin{array}{rcl}\widetilde{w} & = & w,\\ (\widetilde{\zeta}, \widetilde{z}) & = & U(w)(\zeta, z), U(w) \in O(2n+1).\end{array}$$

We denote $q_w = (0, w)$ and we take a coordinate system *adapted* to Γ.

Now we consider the coordinate systems $(x_1, y_1, \ldots, x_n, y_n, z, w)$, *adapted* to Γ and such that $\delta_i(q_w) = \mathrm{span}\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}(q_w)$, $\mathrm{d}\omega(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) > 0$, $\frac{\partial}{\partial z}$ is a normalized generator of $\ker \mathrm{d}\omega_{|\Delta}$ along Γ. Such a coordinate system is said to be *reduced to* Γ. It is unique up to a change of coordinates such that:

$$\begin{array}{rcl}\widetilde{w} & = & w,\\ \widetilde{z} & = & z,\\ \widetilde{\zeta} & = & T(w)\zeta, T(w) \in T^n,\end{array}$$

where, for each w, $T(w)$ is a block diagonal matrix with (2×2)-blocks, and T^n is a maximal torus of $SO(2n+1)$.

Remark: if we change ω for $-\omega$, w is changes for $-w$, the orientation on each δ_i is changed, therefore if we want to keep the same orientation on M^{2n+2}, if n is odd we leave z invariant, and, if n is even, we change z for $-z$.

Now, we consider a coordinate system *reduced* to Γ. We denote $S_{w_0} = \{(\zeta, z, w) | w = w_0\}$ and $P_{w_0} : (\zeta, z, w) \mapsto (\zeta, z, w_0)$, the vertical projection on S_{w_0}. In a neighborhood of 0, the differential of P_{w_0}, at a point of S_{w_0}, is a bijection in restriction to $\Delta_{(\zeta,z,w_0)}$. It allows to push g to a Riemannian metric g_{w_0} on S_{w_0}. We have the formula:

$$g_{w_0}(X, Y) = g((\mathrm{d}P_{w_0}|_\Delta)^{-1}(X), (\mathrm{d}P_{w_0}|_\Delta)^{-1}(Y)), \forall X, Y \in T_{(\zeta,z,w_0)}S_{w_0}.$$

We denote $S^i_{w_0} = \{(\zeta, z, w) |\ z = x_j = y_j = 0 \text{ if } j \neq i \text{ and } w = w_0\}$.

Proposition 2. *Up to a change of coordinates, we can assume that this coordinate system, reduced to* Γ*, is such that the curvature of* $g_{w|S^i_w}$ *at* q_w *is 0. It is unique up to a change of coordinates such that:*

$$\begin{array}{rcl}\widetilde{w} & = & w,\\ \widetilde{z} & = & z,\\ \widetilde{\zeta} & = & T\zeta,\ T \in T^n.\end{array}$$

where T *is a* (2×2)*-blocks diagonal matrix and* T^n *is a maximal torus of* $SO(2n+1)$.

Proof: we will consider changes of coordinates *reduced* to Γ:

$$\begin{array}{rcl}\widetilde{w} & = & w,\\ \widetilde{z} & = & z,\\ \widetilde{x_j} & = & x_j\cos(\delta_j(w)) - y_j\sin(\delta_j(w)),\\ \widetilde{y_j} & = & x_j\sin(\delta_j(w)) + y_j\cos(\delta_j(w)).\end{array}$$

Observation 1: if we denote by $C(r)$ the circle in S^i_w, whose center is q_w and of radius r for g_w, then we have:

$$\mathrm{length}_{g_w}(C(r)) = 2\pi(r - \frac{c(w)r^3}{6} + \mathcal{O}(r^4)),$$

where $c(w)$ is the curvature of $g_{w|S^i_w}$ at q_w. ✓

Observation 2: if (r_i,θ_i) are polar coordinates in S^i_w, associated with the coordinate system (x_i,y_i), then the property 2 of the Γ-adapted coordinate systems allows to claim that $r_i\frac{\partial}{\partial r_i}$ is in Δ. But locally we can write $\omega = f(\mathrm{d}w - \sum_j(\mu_{x_j}\mathrm{d}x_j + \mu_{y_j}\mathrm{d}y_j) - \mu_z\mathrm{d}z)$, therefore we have $\mu_{x_i|S^i_w}x_i + \mu_{y_i|S^i_w}y_i = 0$, that implies:

$$\begin{array}{rcl}\mu_{x_i|S^i_w} & = & \frac{\overline{\alpha}^i}{2}y_i,\\ \mu_{y_i|S^i_w} & = & -\frac{\overline{\alpha}^i}{2}x_i,\end{array}$$

with $\overline{\alpha}^i_{|\Gamma} = \alpha_i$ because $\mathrm{d}\omega_{|\Gamma} = \sum_j \alpha_j \mathrm{d}x_j\wedge\mathrm{d}y_j$. Now, computations in the other coordinate system gives, at a point of S^i_w:

$$\omega = f\left(\left(1 - \frac{\overline{\alpha}^i\dot{\delta}_i(w)r_i^2}{2}\right)\mathrm{d}\widetilde{w} - \sum_j(\mu_{\widetilde{x_j}}\mathrm{d}\widetilde{x}_j + \mu_{\widetilde{y_j}}\mathrm{d}\widetilde{y_j}) - \mu_{\widetilde{z}}\mathrm{d}\widetilde{z}\right).$$

✓

Observation 3: the length of $C(r_i)$ for g_w is:

$$\int_0^{2\pi}\sqrt{g_w(r_i\frac{\partial}{\partial\theta_i}, r_i\frac{\partial}{\partial\theta_i})}\,d\theta_i.$$

The circle of radius r_i is the same for the two metrics g_w and $g_{\widetilde{w}}$ because it is the intersection of S^i_w and of $\{q|\mathrm{d}(q,\Gamma) = r_i\}$. Then we have the same formula for $g_{\widetilde{w}}$:

$$\mathrm{length}_{g_{\widetilde{w}}}(C(r_i)) = \int_0^{2\pi}\sqrt{g_{\widetilde{w}}(r_i\frac{\partial}{\partial\theta_i}, r_i\frac{\partial}{\partial\theta_i})}\,d\theta_i.$$

✓

Now, we will compute the pull back of $r_i\frac{\partial}{\partial\theta_i}$ by the two projections to compare their norms for the two metrics:

$$(P_{w|\Delta})^{-1}(r_i\frac{\partial}{\partial\theta_i}) = r_i\frac{\partial}{\partial\theta_i} - \frac{\overline{\alpha}^i r_i^2}{2}\frac{\partial}{\partial w},$$

and

$$\begin{array}{rcl}(P_{\widetilde{w}|\Delta})^{-1}(r_i\frac{\partial}{\partial\theta_i}) & = & r_i\frac{\partial}{\partial\theta_i} - (\frac{\overline{\alpha}^i r_i^2}{2}\frac{\partial}{\partial\widetilde{w}})/(1 - \frac{\overline{\alpha}^i\dot{\delta}_i(w)r_i^2}{2})\\ & = & (r_i\frac{\partial}{\partial\theta_i} - \frac{\overline{\alpha}^i r_i^2}{2}\frac{\partial}{\partial w})/(1 - \frac{\overline{\alpha}^i\dot{\delta}_i(w)r_i^2}{2}).\end{array}$$

As a consequence:

$$\begin{aligned}\|r_i \tfrac{\partial}{\partial\theta_i}\|_{g_{\widetilde{w}}} &= \|r_i \tfrac{\partial}{\partial\theta_i}\|_{g_w}/(1-\tfrac{\overline{\alpha}^i\dot{\delta}_i(w)r_i^2}{2}) \\ &= \|r_i \tfrac{\partial}{\partial\theta_i}\|_{g_w}\times(1+\tfrac{\overline{\alpha}^i\dot{\delta}_i(w)r_i^2}{2}+\mathcal{O}(r_i^3)).\end{aligned}$$

Now $\overline{\alpha}^i = \alpha_i + \mathcal{O}(r_i)$ hence we find:

$$c_i(\widetilde{w}) = c_i(w) - 3\alpha_i\dot{\delta}_i(w).$$

So, if we want to annihilate the curvatures, we have to choose:

$$\dot{\delta}_i(w) = \frac{c_i(w)}{3\alpha_i},\ \forall i = 1,\ldots,n.$$

The $\delta_i(w)$ are then uniquely determined by $(\delta_1(0),\ldots,\delta_n(0))$, solving this ordinary differential equation. ■

This coordinate system is the normal coordinate system, which is defined up to an element of T^n.

3.2. **Normal form.** Now, computing in a normal coordinate system, we want to construct a field of normal frames $\mathcal{F}$ of the distribution:

$$\mathcal{F} = \begin{pmatrix} Q \\ L \end{pmatrix}$$

where Q is a $(2n+1)\times(2n+1)$-block and L is a $1\times(2n+1)$-block. The columns of the matrix are the vector fields of the frame $\mathcal{F}$, written in the normal coordinates.

Let K be the matrix of the metric g_w in the basis $(\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_n},\frac{\partial}{\partial z})$.

Proposition 3. *K is symmetric and $K(\zeta,z,w)(\zeta,z) = (\zeta,z)$.*

Proof: the fact that K is symmetric is obvious.

Observation: The lines $t\mapsto(t\zeta,tz,w)$ are geodesics for the metric g_w on S_w. Indeed, if it would existed a smaller trajectory between q_w and (ζ,z,w) for g_w, its lifting would be a shorter trajectory between Γ and $\{q|d(q,\Gamma)=|(\zeta,z)|\}$ for g, which is impossible because our system of coordinates is adapted to Γ. ✓

hence the vector $\zeta\frac{\partial}{\partial\zeta}+z\frac{\partial}{\partial z}$ is orthogonal to the sphere of radius $|(\zeta,z)|$ for g_w. Now, this sphere is $\{q=(\overline{\zeta},\overline{z},w)|\sum\overline{\zeta}_i^2+\overline{z}^2=\sum\zeta_i^2+z^2\}$, which implies that $\zeta\frac{\partial}{\partial\zeta}+z\frac{\partial}{\partial z}$ is orthogonal for g_w to its orthogonal for the Euclidean metric associated with the coordinate system. Therefore $K(\zeta,z,w)(\zeta,z)$ is collinear to (ζ,z). On the other hand, the way we have built the coordinates adapted to Γ allows us to assume that $t\mapsto(t\frac{(\zeta,z)}{|(\zeta,z)|},w)$ is parameterized by arclength and, as a consequence, that $\zeta\frac{\partial}{\partial\zeta}+z\frac{\partial}{\partial z}$ has norm $|(\zeta,z)|$. Hence $K(\zeta,z,w)(\zeta,z)=(\zeta,z)$. ■

We can take $Q=\sqrt{K^{-1}}$. Then L is determined by ω. We denote:

$$\mathcal{F} = (F_1,G_1,\ldots,F_n,G_n,E).$$

We have $F_i|_\Gamma = \frac{\partial}{\partial x_i}$, $G_i|_\Gamma = \frac{\partial}{\partial y_i}$, $E|_\Gamma = \frac{\partial}{\partial z}$.

The nilpotent approximation gives weight 1 to ζ and z, and weight 2 to w.

We denote by Q_k and L_k the homogeneous part with order k, in the variables ζ, z and w (wrt their weights), of Q and L, K^{ii} et Q^{ii} the i^{th} blocks 2×2 on the diagonals of K and Q, L^{ii} the i^{th} couple of coordinates of L and $\overline{\zeta_i}=(0,\ldots,0,x_i,y_i,0,\ldots,0)$. Now we can state the result:

Theorem 5 (Normal form). *There is a field of orthonormal frame $\mathcal{F}$, which can be defined, as above, by a couple (Q, L) written in the normal coordinates, unique once the normal coordinates are fixed (hence unique up to an element of T^n)such that the matrices Q and L satisfy the properties:*

(1) *Q is symmetric,*
(2) $Q_0 = \mathrm{Id}_{2n+1}$,
(3) $Q(\zeta, z, w)(\zeta, z) = (\zeta, z)$,
(4) $Q_1 = 0$,
(5) $Q^{ii}(\overline{\zeta_i}, w) = \begin{pmatrix} 1 + y_i^2\beta_i(\overline{\zeta_i}, w) & -x_iy_i\beta_i(\overline{\zeta_i}, w) \\ -x_iy_i\beta_i(\overline{\zeta_i}, w) & 1 + x_i^2\beta_i(\overline{\zeta_i}, w) \end{pmatrix}$, *with* $\beta_i(0, w) = 0$,
(6) $L(\zeta, z, w)(\zeta, z) = 0$,
(7) $L_0 = 0$,
(8) $L_1 = (\frac{\alpha_1 y_1}{2}, -\frac{\alpha_1 x_1}{2}, \ldots, \frac{\alpha_n y_n}{2}, -\frac{\alpha_n x_n}{2}, 0)$,
(9) $\frac{\partial^2 L_2(2n+1)}{\partial x_1^2} + \frac{\partial^2 L_2(2n+1)}{\partial y_1^2} = \frac{\partial^2 L_2(1)}{\partial x_1 \partial z} + \frac{\partial^2 L_2(2)}{\partial y_1 \partial z}$,
(10) $\forall j \neq 1$

$$0 = \alpha_1 \left(\frac{\partial^2 L_2(2j-1)}{\partial x_1^2} + \frac{\partial^2 L_2(2j-1)}{\partial y_1^2} - \frac{\partial^2 L_2(1)}{\partial x_1 \partial x_j} - \frac{\partial^2 L_2(2)}{\partial y_1 \partial x_j} \right) + \\ + \alpha_j \left(\frac{\partial^2 L_2(1)}{\partial y_1 \partial y_j} - \frac{\partial^2 L_2(2)}{\partial x_1 \partial y_j} \right),$$

$$0 = \alpha_1 \left(\frac{\partial^2 L_2(2j)}{\partial x1^2} + \frac{\partial^2 L_2(2j)}{\partial y1^2} - \frac{\partial^2 L_2(1)}{\partial x_1 \partial y_j} - \frac{\partial^2 L_2(2)}{\partial y_1 \partial y_j} \right) + \\ + \alpha_j \left(\frac{\partial^2 L_2(2)}{\partial x_1 \partial x_j} - \frac{\partial^2 L_2(1)}{\partial y_1 \partial x_j} \right),$$

(11) $n\lambda z = \sum_i \frac{1}{\alpha_i} \left(\frac{\partial L_2(2i)}{\partial x_i} - \frac{\partial L_2(2i-1)}{\partial y_i} \right)$, *where* $\lambda = \frac{\partial f}{\partial z}(0)$.

Proof: we denote by $\mathcal{O}^k$ the terms of order k in ζ, z and w.

(1) by definition Q is symmetric.
(2) $K(0, w) = \mathrm{Id}_{2n}$ hence $Q(0, w) = \mathrm{Id}$ and $Q_0 = \mathrm{Id}$.
(3) $K(\zeta, z, w)(\zeta, z) = (\zeta, z)$ hence$Q^{-2}(\zeta, z, w)(\zeta, z) = (\zeta, z)$ that is $(Q^2(\zeta, z, w) - \mathrm{Id})(\zeta, z) = 0$. Hence $(Q(\zeta, z, w) + \mathrm{Id})(Q(\zeta, z, w) - \mathrm{Id})(\zeta, z) = 0$. But near Γ, $(Q + \mathrm{Id})$ is invertible. Hence $Q(\zeta, z, w)(\zeta, z) = (\zeta, z)$.
(4) We denote by $Q_1(i, j, k)$ the coefficient of the $\mathrm{k^{th}}$ coordinate of (ζ, z) in $Q_1(i, j)$. Then, because $Q_1(\zeta, z, w).(\zeta, z) = 0$, we have $Q_1(i, j, k) = -Q_1(i, k, j)$ and, because Q_1 is symmetric, $Q_1(i, j, k) = Q_1(j, i, k)$, for all $(i, j, k) \in (\mathbf{R}^{2n})^3$:

$$\begin{array}{rcccccc} Q_1(i,j,k) & = & -Q_1(i,k,j) & = & -Q_1(k,i,j) & = & Q_1(k,j,i) \\ & = & Q_1(j,k,i) & = & -Q_1(j,i,k) & = & -Q_1(i,j,k). \end{array}$$

Hence, for all $(i, j, k) \in (\mathbf{R}^{2n})^3$ $Q_1(i, j, k) = 0$, that means $Q_1 = 0$.
(5) The fact that $(Q^{ii}(\overline{\zeta_i}, w) - \mathrm{Id}_2)\overline{\zeta_i} = 0$ and that $(Q^{ii}(\overline{\zeta_i}, w) - \mathrm{Id}_2)$ is symmetric give:

$$Q^{ii}(\overline{\zeta_i}, w) - \mathrm{Id}_2 = \begin{pmatrix} y_i^2\beta_i(\overline{\zeta_i}, w) & -x_iy_i\beta_i(\overline{\zeta_i}, w) \\ -x_iy_i\beta_i(\overline{\zeta_i}, w) & x_i^2\beta_i(\overline{\zeta_i}, w) \end{pmatrix}.$$

Hence:

$$K^{ii}(\overline{\zeta_i},w)=\begin{pmatrix}1+y_i^2\overline{\beta_i}(\overline{\zeta_i},w) & -x_iy_i\overline{\beta_i}(\overline{\zeta_i},w)\\ -x_iy_i\overline{\beta_i}(\overline{\zeta_i},w) & 1+x_i^2\overline{\beta_i}(\overline{\zeta_i},w)\end{pmatrix},$$

where:

$$\overline{\beta_i}=\frac{-2\beta_i-\beta_i^2(2x_i^2+2y_i^2+(x_i^2+y_i^2)^2)}{(1+(x_i^2+y_i^2)\beta_i)^2}.$$

Whence:

$$g_w(r_i\frac{\partial}{\partial\theta_i},r_i\frac{\partial}{\partial\theta_i})=\begin{pmatrix}-y_i\\ x_i\end{pmatrix}^tK^{ii}\begin{pmatrix}-y_i\\ x_i\end{pmatrix},$$

which gives $g_w(r_i\frac{\partial}{\partial\theta_i},r_i\frac{\partial}{\partial\theta_i})=(r_i^2)(1+(r_i^2)\overline{\beta_i})$ and then $|(-y_i,x_i)|_{g_w}=r_i(1+\frac{(r_i^2)\overline{\beta_i}}{2})$. We deduce that $c_i(w)=-3\overline{\beta_i}(0,w)=6\beta_i(0,w)$. As a consequence $\beta_i(0,w)=0$.

(6) Because $Q(\zeta,z,w)(\zeta,z)=(\zeta,z)$ we deduce that $\sum_i x_iF_i+y_iG_i+zE$ and $\sum_i x_i\frac{\partial}{\partial x_i}+y_i\frac{\partial}{\partial y_i}+z\frac{\partial}{\partial z}$, which are both in $\Delta(\zeta,w)$, have the same projection on S_w and then that they are equal. The coordinate on $\frac{\partial}{\partial w}$ of $\sum_i x_iF_i+y_iG_i+zE$ is then 0. But this is $L(\zeta,z,w)(\zeta,z)$.

(7) This comes immediately from $\omega_{|\Gamma}=\mathrm{d}w$.

(8) Locally, ω has the form $f(\mathrm{d}w-\sum_i(\mu_{x_i}\mathrm{d}x_i+\mu_{y_i}\mathrm{d}y_i)-\mu_z\mathrm{d}z)$. Now $i_\nu\mathrm{d}\omega=0$ implies that $\mathrm{d}f_{|\Gamma}=0$. Which gives $\mathrm{d}\omega_{|\Gamma}=\sum_i(\mathrm{d}x_i\wedge\mathrm{d}\mu_{x_i}+\mathrm{d}y_i\wedge\mathrm{d}\mu_{y_i})+\mathrm{d}z\wedge\mathrm{d}\mu_z$. Now, because $Q_0=\mathrm{Id}_{2n+1}$, we have $(\mu_{x_i},\mu_{y_i})=(L_1(2i-1),L_1(2i))+\mathcal{O}^2$, $\mu_z=L_1(2n+1)$, which gives $\mathrm{d}\omega_{|\Gamma}=\sum_i(\mathrm{d}x_i\wedge\mathrm{d}L_1(2i-1)+\mathrm{d}y_i\wedge\mathrm{d}L_1(2i))+\mathrm{d}z\wedge\mathrm{d}L_1(2n+1)=\sum_i\alpha_i(\mathrm{d}x_i\wedge\mathrm{d}y_i)$.

On the other hand, $L_1(\zeta,z)=0$. With those two results, we find:

$$L_1=(\frac{\alpha_1y_1}{2},-\frac{\alpha_1x_1}{2},\ldots,\frac{\alpha_ny_n}{2},-\frac{\alpha_nx_n}{2},0).$$

(9) We set:
- $x_i^1(V)$, $y_i^1(V)$, $z^1(V)$ and $w^1(V)$ the homogeneous part of order 1 in the variable ζ,z and w of order 1, 1 and 2, of the coordinates of a vector field V,
- Z the vector field in $\ker(\mathrm{d}\omega_{|\Delta})$ such that $g(Z,Z)=1$ and $Z_{|\Gamma}=\frac{\partial}{\partial z}$.
- (X_i,Y_i) an orthonormal frame of δ_i which coincides with (F_i,G_i) along Γ.
- $\mathcal{O}^k$ the terms of order k in ζ, z and w of order 1, 1 and 2.

Yet we know that:

$$\begin{array}{rcl}F_i&=&(1+\mathcal{O}^2)\frac{\partial}{\partial x_i}+(\frac{\alpha_iy_i}{2}+L_2(2i-1)+\mathcal{O}^3)\frac{\partial}{\partial w},\\ G_i&=&(1+\mathcal{O}^2)\frac{\partial}{\partial y_i}+(\frac{-\alpha_ix_i}{2}+L_2(2i)+\mathcal{O}^3)\frac{\partial}{\partial w},\\ E&=&(1+\mathcal{O}^2)\frac{\partial}{\partial z}+(L_2(2n+1)+\mathcal{O}^3)\frac{\partial}{\partial w}.\end{array}$$

Now we calculate the coordinates of Z to the order 1:

$$\mathrm{d}\omega_{|\Delta}=\sum_i(\alpha_i\mathrm{d}x_i\wedge\mathrm{d}y_i+\mathrm{d}x_i\wedge\mathrm{d}L_2(2i-1)+\mathrm{d}y_i\wedge\mathrm{d}L_2(2i))+$$
$$+\mathrm{d}z\wedge\mathrm{d}L_2(2n+1)+\mathcal{O}^2,$$

hence:

$$\begin{array}{rcl}0=\mathrm{d}\omega(F_j,Z)&=&\alpha_jy_j^1(Z)+\frac{\partial L_2(2i-1)}{\partial z}-\frac{\partial L_2(2n+1)}{\partial x_j}+\mathcal{O}^2,\\ 0=\mathrm{d}\omega(G_j,Z)&=&-\alpha_jx_j^1(Z)+\frac{\partial L_2(2i)}{\partial z}-\frac{\partial L_2(2n+1)}{\partial y_j}+\mathcal{O}^2,\end{array}$$

which implies:

$$\begin{array}{rcl} \alpha_j y_j^1(Z) + \frac{\partial L_2(2i-1)}{\partial z} - \frac{\partial L_2(2n+1)}{\partial x_j} & = & 0, \\ -\alpha_j x_j^1(Z) + \frac{\partial L_2(2i)}{\partial z} - \frac{\partial L_2(2n+1)}{\partial y_j} & = & 0. \end{array}$$

On the other hand $g(X_1, Z) = g(Y_1, Z) = 0$ gives:

$$\begin{array}{c} z^1(X_1) = -x_1^1(Z) = \frac{1}{\alpha_1}(\frac{\partial L_2(2n+1)}{\partial y_1} - \frac{\partial L_2(2)}{\partial z}), \\ z^1(Y_1) = -y_1^1(Z) = \frac{1}{\alpha_1}(\frac{\partial L_2(1)}{\partial z} - \frac{\partial L_2(2n+1)}{\partial x_1}). \end{array}$$

Now we can calculate the coordinate on $\frac{\partial}{\partial z}$ along Γ of $[X_1, Y_1]$, which has to be zero because $[\Delta_1, \Delta_1]_{|\Gamma} = \mathrm{span}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial w})$. which gives the point 9.

(10) For $j \neq 1$, the system:

$$\begin{array}{rcccl} 0 & = & g(X_1, X_j) & = & x_1^1(X_j) + x_j^1(X_1) + \mathcal{O}^2, \\ 0 & = & g(X_1, Y_j) & = & x_1^1(Y_j) + y_j^1(X_1) + \mathcal{O}^2, \\ 0 & = & g(Y_1, X_j) & = & y_1^1(X_j) + x_j^1(Y_1) + \mathcal{O}^2, \\ 0 & = & g(Y_1, Y_j) & = & y_1^1(Y_j) + y_j^1(Y_1) + \mathcal{O}^2, \\ 0 & = & \mathrm{d}\omega(X_1, X_j) & = & \alpha_1 y_1^1(X_j) - \alpha_j y_j^1(X_1) + \frac{\partial L_2(1)}{\partial x_j} - \frac{\partial L_2(2j-1)}{\partial x_1} + \mathcal{O}^2, \\ 0 & = & \mathrm{d}\omega(X_1, Y_j) & = & \alpha_1 y_1^1(Y_j) + \alpha_j x_j^1(X_1) + \frac{\partial L_2(1)}{\partial y_j} - \frac{\partial L_2(2j)}{\partial x_1} + \mathcal{O}^2, \\ 0 & = & \mathrm{d}\omega(Y_1, X_j) & = & -\alpha_1 x_1^1(X_j) - \alpha_j y_j^1(Y_1) + \frac{\partial L_2(2)}{\partial x_j} - \frac{\partial L_2(2j-1)}{\partial y_1} + \mathcal{O}^2, \\ 0 & = & \mathrm{d}\omega(Y_1, Y_j) & = & -\alpha_1 x_1^1(Y_j) + \alpha_j x_j^1(Y_1) + \frac{\partial L_2(2)}{\partial y_j} - \frac{\partial L_2(2j)}{\partial y_1} + \mathcal{O}^2, \end{array}$$

is invertible and we find in particular:

$$\begin{array}{rcl} x_j^1(X_1) & = & \frac{\alpha_1}{\alpha_1^2-\alpha_j^2}\left(\frac{\partial L_2(2j-1)}{\partial y_1} - \frac{\partial L_2(2)}{\partial x_j}\right) - \frac{\alpha_j}{\alpha_1^2-\alpha_j^2}\left(\frac{\partial L_2(2j)}{\partial x_1} - \frac{\partial L_2(1)}{\partial y_j}\right), \\ y_j^1(X_1) & = & \frac{\alpha_1}{\alpha_1^2-\alpha_j^2}\left(\frac{\partial L_2(2j)}{\partial y_1} - \frac{\partial L_2(2)}{\partial y_j}\right) + \frac{\alpha_j}{\alpha_1^2-\alpha_j^2}\left(\frac{\partial L_2(2j-1)}{\partial x_1} - \frac{\partial L_2(1)}{\partial x_j}\right), \\ x_j^1(Y_1) & = & -\frac{\alpha_1}{\alpha_1^2-\alpha_j^2}\left(\frac{\partial L_2(2j-1)}{\partial x_1} - \frac{\partial L_2(1)}{\partial x_j}\right) - \frac{\alpha_j}{\alpha_1^2-\alpha_j^2}\left(\frac{\partial L_2(2j)}{\partial y_1} - \frac{\partial L_2(2)}{\partial y_j}\right), \\ y_j^1(Y_1) & = & -\frac{\alpha_1}{\alpha_1^2-\alpha_j^2}\left(\frac{\partial L_2(2j)}{\partial x_1} - \frac{\partial L_2(1)}{\partial y_j}\right) + \frac{\alpha_j}{\alpha_1^2-\alpha_j^2}\left(\frac{\partial L_2(2j-1)}{\partial y_1} - \frac{\partial L_2(2)}{\partial x_j}\right), \end{array}$$

which allows us to calculate $x_j([X_1, Y_1])_{|\Gamma}$ and $y_j([X_1, Y_1])_{|\Gamma}$ which also have to be zero. This gives formulas of point 10.

(11) Finally, We have:

$$\begin{array}{rcl} \alpha_i(\zeta, z, w) & = & \mathrm{d}\omega(X_i, Y_i) \\ & = & (1 + \lambda z)\alpha_i + x_i^1(X_i) + y_i^1(Y_i) + \frac{\partial L_2(2i-1)}{\partial y_i} - \frac{\partial L_2(2i)}{\partial x_i} + \mathcal{O}^2, \end{array}$$

and $g(X_i, X_i) = g(Y_i, Y_i) = 1$ implies $x_i^1(X_i) = y_i^1(Y_i) = 0$, so, if we make the product we find:

$$\begin{array}{rcl} \frac{1}{n!} & = & \prod_i \alpha_i(\zeta, z, w) \\ & = & (1 + n\lambda z)\prod_i \alpha_i + \sum_i \left(\prod_{j\neq i} \alpha_j\right)\left(\frac{\partial L_2(2i-1)}{\partial y_i} - \frac{\partial L_2(2i)}{\partial x_i}\right) + \mathcal{O}^2. \end{array}$$

Then:

$$n\lambda z = \sum_i \frac{1}{\alpha_i}\left(\frac{\partial L_2(2i)}{\partial x_i} - \frac{\partial L_2(2i-1)}{\partial y_i}\right).$$

■

In the 4-dimensional case, we give the first terms of L. In the sequel, it will become clear that the Riemannian part (Q) does not play any rôle in the face of the caustic.

$$\begin{aligned}
&L_1 = (\tfrac{y}{2}, -\tfrac{x}{2}, 0),\\
&L_2 = (-z(Re(A\mathrm{e}^{ia}(x+iy)) + B\cos(b)z + \tfrac{\lambda y}{2}), z(Im(A\mathrm{e}^{ia}(x+iy)) + B\sin(b)z + \tfrac{\lambda x}{2},\\
&\qquad\qquad Re[A\mathrm{e}^{ia}(x+iy)^2 + B\mathrm{e}^{ib}z(x+iy)]),\\
&L_3(1) = \tfrac{1}{2}(F_2(-3y^3 + x^2y) + F_3\sin(f_3)(y^3 - 3x^2y) + F_3\cos(f_3)(8xy^2)) + \mathcal{O}(z,w),\\
&L_3(2) = \tfrac{1}{2}(F_2(-x^3 + 3xy^2) + F_3\sin(f_3)(3x^3 - xy^2) - F_3\cos(f_3)(8x^2y)) + \mathcal{O}(z,w).
\end{aligned}$$

remark: This "normal form" is the analog of the normal form obtained by [A-G] in the contact case and it is very similar to it.

4. Invariants

4.1. **In the general case.** Let us remark that all the normal coordinate systems define the same metric g_w on S_w. Therefore we can define intrinsically the metric $\mathcal{Q}_w = g_w^{-\frac{1}{2}}$. Hence, if we denote:

$$B^1_{\ell,k} : \begin{cases} S^\ell(T_0S_{w=0}) \otimes S^2(T_0S_{w=0}) & \to \quad \mathbf{R} \\ (U_1 \odot \ldots \odot U_\ell) \otimes (V_1 \odot V_2) & \mapsto \quad \mathrm{D}^\ell(\frac{\partial^k}{\partial w^k}(\mathcal{Q}_w(V_1, V_2)))_{|0}(U_1 \odot \ldots \odot U_\ell) \end{cases}$$

where D^ℓ denote the ℓ^{th} derivative with respect to (ζ, z), then $B^1_{\ell,k} \otimes \omega^{\otimes k}$ is an element of $S^\ell(T_0^*S_{w=0}) \otimes S^2(T_0^*S_{w=0}) \otimes \omega^{\otimes k}$ and it is invariant by the changes of normal coordinates. Hence, if we move the basepoint of the whole construction, we obtain a tensor field. In coordinates, $B^1_{\ell,k} = \mathrm{D}^\ell(\frac{\partial^k}{\partial w^k}Q)$.

Let us denote by g^e_w the Euclidean metric defined on S_w by the normal coordinates (they all define the same), and, for any V, $V_{\mathcal{Q}}$ the vector field such that $g^e_w(V_{\mathcal{Q}}, .) = \mathcal{Q}_w(V, .)$. In terms of matrices, $V_{\mathcal{Q}} = Q.V$. Let us take the pre-image of $V_{\mathcal{Q}} = Q.V$ in Δ. Its w-coordinate is well defined, independently of the normal coordinates. In terms of matrices, it is $L.V$. Hence, if we denote:

$$B^2_{\ell,k} : \begin{cases} S^\ell(T_0S_{w=0}) \otimes T_0S_{w=0} & \to \quad \mathbf{R} \\ (U_1 \odot \ldots \odot U_\ell) \otimes (V) & \mapsto \quad \mathrm{D}^\ell(\frac{\partial^k}{\partial w^k}(L.V))_{|0}(U_1 \odot \ldots \odot U_\ell) \end{cases}$$

then $B^2_{\ell,k} \otimes \omega^{\otimes k}$ is an element of $S^\ell(T_0^*S_{w=0}) \otimes T_0^*S_{w=0} \otimes \omega^{\otimes k}$, and it is invariant by all the changes of normal coordinates. Moving the basepoint, we obtain again a tensor field. In coordinates, $B^2_{\ell,k} = \mathrm{D}^\ell\frac{\partial^k}{\partial w^k}(L)$.

4.2. **Decomposition of tensors.** All the typical fibers of the tensor bundles under consideration above have a metric structure inherited from g at 0. The action of T^n on Δ_0 induces a unitary representation of T^n on these typical fibers. T^n being abelian and compact, these unitary representations are unitarily equivalent to a finite direct sum of characters.

Therefore, all our invariant tensors can be reduced to real and complex numbers, the modules of which are independent elementary invariants of the sub-Riemannian structure. This decomposition can be done as follows.

All our tensors are covariant symmetric tensors over Δ. The space of covariant symmetric tensors of degree k over Δ can be canonically identified to the space

$S^k(\Delta^*)$ of homogeneous polynomial of degree k over Δ^*. This space can be identified to the set $\widetilde{S}^k(\Delta^*)$ of real parts of homogeneous polynomials of degree k of the complex variables $z_j = x_j + iy_j$ and $\bar{z_j}$ $(j = 1, \ldots, n)$, and of the real variable z:

$$\widetilde{S}^k(\Delta^*) = \{Re[P_k(z_1, \bar{z_1}, \ldots, z_n, \bar{z_n}, z)]\}.$$

The action of T^n on these spaces is the natural one induced by

$$\mathcal{X}(\theta)z_j = \mathrm{e}^{i\theta_j} z_j.$$

A decomposition of this action of T^n on $\widetilde{S}^k(\Delta^*)$ in characters is the following:

- A polynomial $P_k(x_1, y_1, \ldots, x_n, y_n, z)$ can be written in a unique way:
$$P_k(x_1, y_1, \ldots, x_n, y_n, z) = \sum_{\substack{I,\,J,\,\ell \\ \ell + \sum_i I_i + J_i = k}} Re(\Lambda_{I,J,\ell}(\prod_i z_i^{I_i} \bar{z}_i^{J_i}) z^\ell)$$
with $\overline{\Lambda_{I,J,\ell}} = \Lambda_{J,I,\ell}$.
- The character corresponding to $z^\ell \prod_i z_i^{I_i} \bar{z}_i^{J_i}$ is $\mathrm{e}^{(I_1 - J_1)\theta_1 + \ldots + (I_n - J_n)\theta_n}$.

4.3. **In dimension 4.** Let us have a look to the special case of dimension 4 that we will study in more details in the following. We define complex and real numbers $A\mathrm{e}^{ia}$, $B\mathrm{e}^{ib}$, $C\mathrm{e}^{ic}$, $D\mathrm{e}^{id}$ and λ: the real $\lambda = \frac{\partial \omega(\frac{\partial}{\partial w})}{\partial z}(0)$; the complexes $A\mathrm{e}^{ia}$ and $B\mathrm{e}^{ib}$ are such that the third coordinate of L_2 satisfies :

$$L_2[3] = Re[A\mathrm{e}^{ia}(x + iy)^2 + B\mathrm{e}^{ib} z(x + iy)];$$

the complexes $C\mathrm{e}^{ic}$ and $D\mathrm{e}^{id}$ are such that the two first coordinates of L_3 satisfy :

$$L_3[1] = Re[C\mathrm{e}^{ic}(x^2 + y^2)(x + iy) + D\mathrm{e}^{id}(x + iy)^3] + O(z, w),$$

$$L_3[2] = Re[iC\mathrm{e}^{ic}(x^2 + y^2)(x + iy) - iD\mathrm{e}^{id}(x + iy)^3] + O(z, w).$$

Actually, under a change of normal coordinates $\widetilde{x} + i\widetilde{y} = (x + iy)\mathrm{e}^{i\theta}$, $L_2[3]$ is unchanged hence it is an invariant, as well as $\cos(\theta)L_3[1] + \sin(\theta)L_3[2]$ and $-\sin(\theta)L_3[1] + \cos(\theta)L_3[2]$. As a consequence, A, B, C, D and $a - d$ are invariants of the sub-Riemannian structure. So we can define the invariant $\mathcal{I}nv = \mathrm{Re}[96D^e i(d - a) - 45A^2\lambda]$.

As for the first invariants $(\alpha_1, \ldots, \alpha_n)$, for a generic quasi-contact metric, outside a codimension 1 closed stratified subset, we can assume that A, B and $\mathcal{I}nv$ are not null. **We will assume in the following that 0 is such a point**.

From now, we are dealing with the 4-dimensional case.

5. Computation of exponential mapping jets

In this section, we describe how to compute the exponential mapping exp_0 in the 4-dimensional case: $n = 1$.

From now, we will set $\bar{p}, \bar{q}, \bar{r}$ and $\bar{v}$ the dual coordinates, in the fibers of $T^*\mathbf{R}^4$, of x, y, z and w.

As shown in lemmas 1 and 2, and thanks to Pontryagin Maximal Principle, it's enough to integrate the trajectories of $\overrightarrow{\mathcal{H}}$ in $T^*\mathbf{R}^4$:

$$\begin{cases} (\dot{x}, \dot{y}, \dot{z}, \dot{w}, \dot{\bar{p}}, \dot{\bar{q}}, \dot{\bar{r}}, \dot{\bar{v}})(s) = \overrightarrow{\mathcal{H}}(x, y, z, w, \bar{p}, \bar{q}, \bar{r}, \bar{v}), \\ (x, y, z, w, \bar{p}, \bar{q}, \bar{r}, \bar{v})(0) = (0, 0, 0, 0, \bar{p}_0, \bar{q}_0, \bar{r}_0, \bar{v}_0), \end{cases}$$

where $(\bar{p}_0, \bar{q}_0, \bar{r}_0, \bar{v}_0)$ is in $\mathcal{H}^{-1}(\frac{1}{2}) \cap T_0^*\mathbf{R}^4$. Actually, if we denote by $\mathcal{E}$ the map mapping $(\bar{p}_0, \bar{q}_0, \bar{r}_0, \bar{v}_0, s)$ to the point obtained by following the geodesic with initial condition $(\bar{p}_0, \bar{q}_0, \bar{r}_0, \bar{v}_0)$ during time s, then:

$$\mathcal{E}(\bar{p}_0, \bar{q}_0, \bar{r}_0, \bar{v}_0, s) = exp_0(s\bar{p}_0, s\bar{q}_0, s\bar{r}_0, s\bar{v}_0).$$

Hence we can construct all the geodesics only by considering those starting from $\mathcal{H}^{-1}(\frac{1}{2}) \cap T_0^*\mathbf{R}^4$. They are parameterized by arclength.

5.1. **First reparameterization.** Our goal is to study the local conjugate locus. Any geodesic being locally optimal, the geodesics that will have interest for us will be those with $\bar{v}_0$ close to $+\infty$ or $-\infty$, which will create two parts of conjugate locus, one for $\bar{v}_0$ near $+\infty$, the other one for $-\infty$. Hence, a natural change of parameterization of the initial covector (or impulse) is to set $\rho = \frac{1}{\bar{v}}$ and $\rho_0 = \frac{1}{\bar{v}_0}$. Now we are working with ρ_0 in a neighborhood of 0. We keep in mind that there are two parts: $0 < \rho_0 < \varepsilon$ and $-\varepsilon < \rho_0 < 0$. We also denote $p = \rho\bar{p}$, $q = \rho\bar{q}$ and $r = \rho\bar{r}$.

5.2. **Reparameterization of time.** Further calculation will show that the conjugate arclength will be close to $2\pi\rho$, hence there is a natural time-reparameterization which is $t = \int_0^s \bar{v}\ \mathrm{d}u$, where we integrate along the geodesics. Because $\bar{v}$ is not zero along the geodesics we are looking at, this reparameterization is well defined and invertible: $s = \int_0^t \rho \mathrm{d}u$. This change of time-parameterization will send the conjugate time close to 2π.
Remark: in the sequel, we will study only the geodesics with $\rho_0 > 0$ and $t \geq 0$ (which are the same as those with $\rho_0 < 0$ and $t \geq 0$). The study of the geodesics with $t \leq 0$ is equivalent to the one we will do.

5.3. **Two different parameterizations of the set of initial conditions.** Now, we set $p_0 = \rho_0\bar{p}_0$, $q_0 = \rho_0\bar{q}_0$ and $r_0 = \rho_0\bar{r}_0$. They satisfy $p_0^2 + q_0^2 + r_0^2 = \rho_0^2$. Hence (p_0, q_0, r_0) lives in $S^2(\rho_0)$ and we can parameterize it locally by a couple of angles (φ, θ) which will depend on the part of the sphere we are looking at.

Anyway, we can define a new exponential mapping:

$$exp_2(\rho_0, t, \varphi, \theta) = exp_{q_0}(\bar{p}_0, \bar{q}_0, \bar{r}_0, \bar{v}_0)$$

In all the sequel, we will use two different parameterizations. The first one (denoted by $\mathcal{P}_1$) is:

$$\begin{array}{ccl} [-\frac{\pi}{2}, \frac{\pi}{2}] & \rightarrow & S^2(\rho_0) \\ (\theta, \varphi) & \mapsto & (p_0 = \rho_0\cos(\theta)\cos(\varphi), q_0 = \rho_0\cos(\theta)\sin(\varphi), r_0 = \rho\sin(\theta) \end{array}$$

which is not singular outside $p_0 = q_0 = 0$ ($|\theta| = \frac{\pi}{2}$). The second one (denoted by $\mathcal{P}_2$) is:

$$(\theta, \varphi) \mapsto (p_0 = \rho_0\cos(\theta)\sin(\varphi), q_0 = \rho_0\sin(\theta), r_0 = \rho_0\cos(\theta)\cos(\varphi)),$$

and we will use it near $p_0 = q_0 = 0$ where it is not singular.

5.4. **New differential system.** The interest of the changes of variables we have done in the previous subsections appears clearly in the form of the differential system we have to integrate after these changes. Let us recall that we denote by (F, G, E) our normal form, $\bar{\psi} = (\bar{p}, \bar{q}, \bar{r}, \bar{v})$ and $\psi = (p, q, r, 1)$. Then:

$$\mathcal{H} = \frac{1}{2}\left[(\bar{\psi}.F)^2 + (\bar{\psi}.G)^2 + (\bar{\psi}.E)^2\right].$$

Now, with our new variables, easy computations show that we are integrating the new differential system:

$$\left\{\begin{array}{lcl}\frac{dx}{dt} & = & \psi.(F_xF+G_xG+E_xE)\\ \frac{dy}{dt} & = & \psi.(F_yF+G_yG+E_yE)\\ \frac{dz}{dt} & = & \psi.(F_zF+G_zG+E_zE)\\ \frac{dw}{dt} & = & \psi.(F_wF+G_wG+E_wE)\\ \frac{dp}{dt} & = & (\psi.F)(\psi.(p\frac{\partial F}{\partial w}-\frac{\partial F}{\partial x}))+(\psi.G)(\psi.(p\frac{\partial G}{\partial w}-\frac{\partial G}{\partial x}))+(\psi.E)(\psi.(p\frac{\partial E}{\partial w}-\frac{\partial E}{\partial x}))\\ \frac{dq}{dt} & = & (\psi.F)(\psi.(q\frac{\partial F}{\partial w}-\frac{\partial F}{\partial y}))+(\psi.G)(\psi.(q\frac{\partial G}{\partial w}-\frac{\partial G}{\partial y}))+(\psi.E)(\psi.(q\frac{\partial E}{\partial w}-\frac{\partial E}{\partial y}))\\ \frac{dr}{dt} & = & (\psi.F)(\psi.(r\frac{\partial F}{\partial w}-\frac{\partial F}{\partial z}))+(\psi.G)(\psi.(r\frac{\partial G}{\partial w}-\frac{\partial G}{\partial z}))+(\psi.E)(\psi.(r\frac{\partial E}{\partial w}-\frac{\partial E}{\partial z}))\\ \frac{d\rho}{dt} & = & \rho\left((\psi.F)(\psi.\frac{\partial F}{\partial w})+(\psi.G)(\psi.\frac{\partial G}{\partial w})+(\psi.E)(\psi.\frac{\partial E}{\partial w})\right)\end{array}\right.$$

with $(x,y,z,w,p,q,r,\rho)(0)=(0,0,0,0,p_0,q_0,r_0,\rho_0)$, $p_0^2+q_0^2+r_0^2=\rho_0^2$, and where K_x, K_y, K_z and K_w denote the coordinates of K in $\mathbf{R}^4$.
Remark: we want to compute the jets of order k with respect to ρ_0 of the map exp_2. From the considerations we had in introduction, we know that x, y and z have weight 1 and w has weight 2.

Now, our differential system has the form:

$$\dot{Y}=A.Y+B_2(Y)+\ldots+B_m(Y)+B_{(m+1)}(Y),$$

where A is linear, B_j is homogeneous of order j in the variables $Y=(x,y,z,w,p,q,r,\rho)$ with the weights (1,1,1,2,1,1,1,1), and $B_{(j)}$ is a $\mathcal{O}^j(Y)$.

5.5. **Integration of the new differential system.** The solution satisfies:

$$Y(t)=\exp(t.A).Y_0+\int_0^t \exp((t-u)A).(B_2+\ldots+B_m+B_{(m+1)})(Y(u))\mathrm{d}u.$$

It allows us to compute by itering integration. Actually, with this formula, we can see that, if we denote by $\mathrm{Hg}(m,\mathcal{G})$ the homogeneous part of order m of a map $\mathcal{G}$, with respect to ρ_0, then:

$$\mathrm{Hg}(1,Y(t))=\exp(t.A).Y_0,$$
$$\mathrm{Hg}(m,Y(t))=\sum_{i=2}^m\int_0^t \exp((t-u)A).\mathrm{Hg}(m,B_i(\sum_{j=1}^{m-i+1}\mathrm{Hg}(j,Y(u))))\mathrm{d}u,$$

if $m\geq 2$.

Now, we have a recursive method to construct the jets of the exponential mapping exp_2. You can find the **Mathematica** calculation at appendix 10.5. Before we study it, we will first look again at the nilpotent approximation.

6. Nilpotent Approximation

The nilpotent approximation is the distribution given by its normal form ($F=\{\frac{\partial}{\partial x}+\frac{y}{2}\frac{\partial}{\partial w}\}, G=\{\frac{\partial}{\partial y}-\frac{x}{2}\frac{\partial}{\partial w}\}, E=\{\frac{\partial}{\partial z}\}$). The results on the normal form show that we can see the other sub-Riemannian normal forms as local perturbations of this one. An easy computation gives exp_2 in the nilpotent case:

$$\begin{array}{rcl}x_{\mathcal{N}}(t,\varphi,\theta,\rho_0) & = & \rho_0(\sin(\varphi)-\sin(\varphi-t))\cos(\theta),\\ y_{\mathcal{N}}(t,\varphi,\theta,\rho_0) & = & \rho_0(\cos(\varphi-t)-\cos(\varphi))\cos(\theta),\\ z_{\mathcal{N}}(t,\varphi,\theta,\rho_0) & = & \rho_0 t\sin(\theta),\\ w_{\mathcal{N}}(t,\varphi,\theta,\rho_0) & = & \frac{1}{2}\rho_0^2\cos(\theta)^2(t-\sin(t)),\end{array}$$

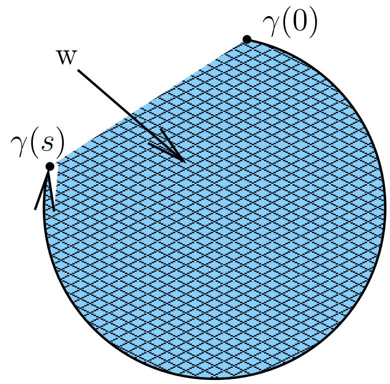

Figure 4. Projected geodesic

which gives when parameterized by the arclength s:

$$\begin{array}{rcl} x_{\mathcal{N}}(s,\varphi,\theta,\rho_0) & = & \rho_0(\sin(\varphi)-\sin(\varphi-\frac{s}{\rho_0}))\cos(\theta), \\ y_{\mathcal{N}}(s,\varphi,\theta,\rho_0) & = & \rho_0(\cos(\varphi-\frac{s}{\rho_0})-\cos(\varphi))\cos(\theta), \\ z_{\mathcal{N}}(s,\varphi,\theta,\rho_0) & = & s\sin(\theta), \\ w_{\mathcal{N}}(s,\varphi,\theta,\rho_0) & = & \frac{1}{2}\rho_0^2\cos(\theta)^2(\frac{s}{\rho_0}-\sin(\frac{s}{\rho_0})). \end{array}$$

Remark: we are using the parameterization $\mathcal{P}_1$ of S^2, with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, which is singular at $|\theta| = \frac{\pi}{2}$. It only has consequences on the calculus of the Jacobian.

Proposition 4. *The cut-time in the nilpotent case is $t = 2\pi$ ($s = 2\pi\rho_0$) for the geodesics with $|\theta| \neq \frac{\pi}{2}$.*

Proof: first remark that, if we project a geodesic on the (x, y)-plane, it is an arc of circle of length $s\cos(\theta)$ for the Euclidean metric, and of radius $\rho_0 \cos(\theta)$. The second thing is that w is the area described by the projection on the (x, y)-plane (see figure 4).

Now, let us take a geodesic γ with $\cos(\theta) \neq 0$ and with $\frac{s}{\rho_0} < 2\pi$. Let us suppose that there is a shorter geodesic $\bar{\gamma}$ between 0 and $\gamma(s)$. First, $s\sin(\theta) = \bar{s}\sin(\bar{\theta})$ because they have the same z-coordinate at $\gamma(s)$. Because $\bar{s} < s$, we found $\sin(\bar{\theta}) > \sin(\theta)$. But, because of the isoperimetric inequality, and because γ is an arc of circle not closed, it is the smallest curve describing the area w. But $\bar{\gamma}$ describes the same area because $\bar{w} = w$. Therefore $\cos(\bar{\theta}) > \cos(\theta)$. This is in contradiction with $\sin(\bar{\theta}) > \sin(\theta)$. Hence the geodesic γ is the shortest pass between 0 and $\gamma(s)$.

Now, if we take $t > 2\pi$, the projection on (x, y) is no more a solution of the isoperimetric problem so it cannot be optimal. ■

Remark: for $|\theta| = \frac{\pi}{2}$, the trajectory does not depend on φ or ρ_0. It is optimal because any geodesic with $|\theta| < \frac{\pi}{2}$ do not intersect the line $\{x = y = w = 0\}$.

Proposition 5. *In the nilpotent case, the first conjugate time is $t = 2\pi$ for the geodesics with $|\theta| \neq \frac{\pi}{2}$.*

Proof: the Jacobian of exp_2 is $2\rho_0^4 t\cos(\theta)^3(t\cos(\frac{t}{2}) - 2\sin(\frac{t}{2}))\sin(\frac{t}{2})$. ■

Remark: for $|\theta| = \frac{\pi}{2}$, the trajectory does not depend on φ or ρ_0, hence the first conjugate-time is 0.

As a consequence, in the nilpotent case, the first conjugate-locus is the plane $\{x = y = 0\}$ and the cut-locus is $\{x = y = 0\} - \{x = y = w = 0\}$. The figure 5 shows a part of the intersection of a wavefront with the 3-plane $\{y = 0\}$.

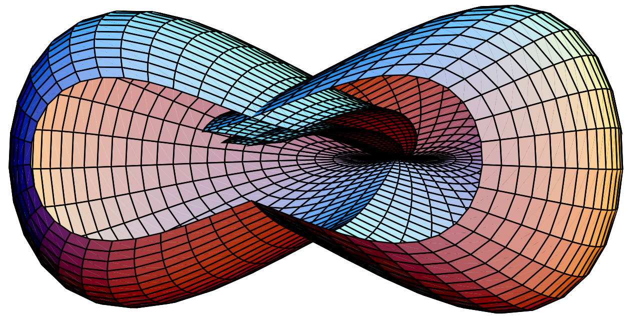

FIGURE 5. Wave-front

It is easy to show that the z axis is an abnormal geodesic, which is not strict. Generically, there will not be this abnormal geodesic (see the paper from Agrachev and Gauthier [A-G2]).

7. Estimation of conjugate time

In this section, we prove the following:

Theorem 6. *for any $\epsilon > 0$ there exists a $\eta > 0$ such that for all $\rho_0 > 0$ satisfying $\rho_0 < \eta$, then the Jacobian of the exponential map exp_2 is positive for $t \in]0, 2\pi - \epsilon]$ and it has no more than 2 roots (counted with multiplicity) in $]2\pi - \epsilon, 2\pi + \epsilon[$.*

To prove that the Jacobian of the exponential map is positive for t positive close to 0, we need the lemma:

Lemma 1. *With the new differential system and the normal form, one can compute that:*

$$\begin{array}{l}
x = p_0 t + \frac{q_0}{2}t^2 + \frac{t^3}{3}(p_2 - p_0 r_0 A\cos(a) - r_0^2 B\cos(b) - \frac{q_0 r_0}{2}(\lambda - 2A\sin(a))) + \mathcal{O}(t^4),\\
y = q_0 t - \frac{p_0}{2}t^2 + \frac{t^3}{3}(q_2 + q_0 r_0 A\cos(a) + r_0^2 B\cos(b) + \frac{p_0 r_0}{2}(\lambda + 2A\sin(a))) + \mathcal{O}(t^4),\\
z = r_0 t + \mathcal{O}(t^3)\\
p = p_0 + t(\frac{q_0}{2}) + t^2 p_2 + \mathcal{O}(t^3)\\
q = q_0 - t(\frac{p_0}{2}) + t^2 q_2 + \mathcal{O}(t^3)\\
r = r_0 + \mathcal{O}(t^2).
\end{array}$$

with:

$$\begin{array}{rcl}
p_2 &=& -\frac{1}{2}(Ar_0(p_0\cos(a) - q_0\sin(a)) + \frac{\lambda}{2}q_0 r_0 + B\cos(b)r_0^2 + \frac{1}{2}p_0),\\
q_2 &=& \frac{1}{2}(Ar_0(p_0\sin(a) + q_0\cos(a)) + \frac{\lambda}{2}p_0 r_0 + B\sin(b)r_0^2 - \frac{1}{2}q_0).
\end{array}$$

In particular, we can write $x = x^1 t + x^2 t^2 + \mathcal{O}(t^3)$ with x^1 and x^2 polynomial of degree 1, with respect to the variable ρ_0. We can do the same for y and z. The same method allows to compute w and to write it $w = w^3 t^3 + w^4 t^4 + \mathcal{O}(t^5)$ with w^3 and w^4 polynomial of degree 2 and 3, with respect to the variable ρ_0.

Proof: one just have to integrate by iteration the trajectories of $\overrightarrow{\mathcal{H}}$. Left to the reader. ■

Proof of the theorem: we will work on two different domains:

7.1. **If** $\cos(\theta) > M\rho_0$**, with** M **large enough (in the parameterization** $\mathcal{P}_1$ **of** $S^2(\rho_0)$**).** Because the parameterization is degenerate for $\cos(\theta) = 0$, it is clear that we can factorize $\cos(\theta)$ in the expression of the Jacobian. Computing the jets of the Jacobian of exp_2, one can found:

$$4\rho_0^4 \cos(\theta)(\cos(\theta)^2 \mathcal{J}_4 + \rho_0 \cos(\theta)\, jac_5(t, \varphi, \theta) + \rho_0^2 jac_6(\rho_0, t, \varphi, \theta)).$$

where $\mathcal{J}_4 = t(\sin(\frac{t}{2}) - \frac{t}{2}\cos(\frac{t}{2}))\sin(\frac{t}{2})$, jac_5 and jac_6 are smooth functions.

Furthermore, the previous lemma 1 insure that we can factorize t^6 in jac_5 and in jac_6. Hence we can write the Jacobian:

$$4\rho_0^4\, t^5 \cos(\theta)^3 (\widetilde{\mathcal{J}_4}(t) + \frac{\rho_0}{\cos(\theta)} \widetilde{jac_5}(t, \varphi, \theta) + \frac{\rho_0^2}{\cos(\theta)^2} \widetilde{jac_6}(\rho_0, t, \varphi, \theta)),$$

where $\widetilde{\mathcal{J}_4}(t) = \frac{(\sin(\frac{t}{2}) - \frac{t}{2}\cos(\frac{t}{2}))\sin(\frac{t}{2})}{t^4}$ is smooth as well as $\widetilde{jac_5}$ and $\widetilde{jac_6}$. $\frac{\rho_0}{\cos(\theta)}$ is bounded by $\frac{1}{M}$.

The function $\widetilde{\mathcal{J}_4}$ is strictly positive for $t \in [0, 2\pi[$, $\widetilde{\mathcal{J}_4}(2\pi) = 0$ and $\widetilde{\mathcal{J}_4}'(2\pi) < 0$. Now, because $\widetilde{jac_5}$ and $\widetilde{jac_6}$ are bounded, as well as their derivatives with respect to t, and because we can assume $\frac{1}{M}$ small enough, then the Jacobian have the same property as $\widetilde{\mathcal{J}_4}$: for any ϵ_1, it exists η_1 such that, if $\frac{\rho_0}{\cos(\theta)} < \eta_1$, the Jacobian has only one root which is in $[2\pi - \epsilon_1, 2\pi + \epsilon_1]$, where its derivative with respect to t is strictly negative. In fact it is possible to prove that, on this domain, the first time can be written $2\pi + \rho_0\tau$ with τ smooth.

We have proved the theorem in restriction at this first domain. Let us prove it on the second domain:

7.2. **If** $\theta^2 + \varphi^2 < 2M^2 * \rho_0^2$ **(in the parameterization** $\mathcal{P}_2$ **of** $S^2(\rho_0)$**).** In that case, we can make a change of variables: $\widetilde{\varphi} = \frac{\varphi}{B\rho_0}$ and $\widetilde{\theta} = \frac{\theta}{B\rho_0}$. Now, computations in appendix show that we can write:

$$jac = \rho_0^6(\mathcal{J}_a(t, \widetilde{\theta}, \widetilde{\varphi}) + \rho_0 jac_7(\rho_0, t, \widetilde{\theta}, \widetilde{\varphi})).$$

with $\mathcal{J}_a$ (denoted by FFF_a in appendix) and jac_7 smooth. We show, in appendix 10.6, that for $t \in [0, 2\pi]$, $\mathcal{J}_a \geq \mathcal{J}_b$ (denoted by FFF_b in appendix) where

$$\mathcal{J}_b = 9\,\sin(\frac{t}{2})^2 \frac{(4t^3\cos(\frac{t}{2}) + (12 - 12t^2 + t^4)\sin(\frac{t}{2}) - 4\sin(\frac{3t}{2}))}{t(\frac{t}{2}\cos(\frac{t}{2}) - \sin(\frac{t}{2}))}.$$

Let suppose as known that $\mathcal{J}_b$ is strictly positive for $t \in]0, 2\pi[$ (we will show it at the end of this section with lemma 2). Now, it is easy to show that we can write $\mathcal{J}_b = t^7\mathcal{J}_c$, with $\mathcal{J}_c$ smooth and strictly positive for $t \in [0, 2\pi[$: 7 is just the first integer k such that $\frac{\partial^k \mathcal{J}_b}{\partial t^k}$ is not 0. Furthermore the lemma 1 allows to assume that jac_7 can be written $t^7\widetilde{jac_7}$, with $\widetilde{jac_7}$ smooth.

Hence, for any ϵ_2, if ρ_0 is small enough, then $\rho_0^6(\mathcal{J}_b + \rho_0 jac_7)$ is strictly positive for $t \in]0, 2\pi - \epsilon_2]$. But it minimizes the Jacobian for $t \in [0, 2\pi]$, hence the Jacobian is strictly positive for $t \in]0, 2\pi - \epsilon_2]$.

Now, computations show that (see appendix 10.6), if we set $\bar\theta = \widetilde\theta - 6\sin(b)$ and $\bar\varphi = \widetilde\varphi + 6\cos(b)$:

$$\begin{array}{rcl}\mathcal{J}_a(t=2\pi) & = & 0,\\ \frac{\mathrm{d}\mathcal{J}_a}{\mathrm{d}t}(t=2\pi) & = & -4\pi^2(\bar\varphi^2+\bar\theta^2),\\ \frac{\partial^2\mathcal{J}_a}{\partial t^2}(t=2\pi) & = & 4\pi(18-\bar\varphi^2-\bar\theta^2-6(\bar\varphi+\pi\bar\theta)\cos(b)-6(\pi\bar\varphi-\bar\theta)\sin(b)).\end{array}$$

Now if $\bar\theta^2+\bar\varphi^2 \leq 0.0001$ we have $\frac{\partial^2\mathcal{J}_a}{\partial t^2}(t=2\pi) > 4\pi * 17$. And, because $\{(\bar\theta,\bar\varphi)|\bar\theta^2+\bar\varphi^2\leq 0.0001\}$ is compact, it is also true for $t\in[2\pi-\epsilon_3,2\pi+\epsilon_3]$ with ϵ_3 small enough. Hence, on the interval $[2\pi-\epsilon_3,2\pi+\epsilon_3]$, the Jacobian will be a convex function of t, and hence it can not have more than 2 roots.

If $\bar\theta^2+\bar\varphi^2\in[0.0001,M^2]$, then $\frac{\mathrm{d}\mathcal{J}_a}{\mathrm{d}t}(t=2\pi) < -4\pi^2 0.0001$. Because $\{(\bar\theta,\bar\varphi)|\bar\theta^2+\bar\varphi^2\in[0.0001,M^2]\}$ is compact, it is also true for $t\in[2\pi-\epsilon_4,2\pi+\epsilon_4]$ with ϵ_4 small enough. As a consequence, on the interval $[2\pi-\epsilon_4,2\pi+\epsilon_4]$, the Jacobian will be strictly decreasing, and hence it can not have more than 1 root.

Now, if we take, $\epsilon_2=\min(\epsilon_3,\epsilon_4)$, we get the result: for ρ_0 small enough, the Jacobian has no root in the interval $]0,2\pi-\epsilon_2]$ and no more than 2 in the interval $]2\pi-\epsilon_2,2\pi+\epsilon_2[$. Moreover, because the Jacobian is positive on the interval $]0,2\pi-\epsilon_2]$, its derivative is non-positive at the first conjugate time in the second interval, if it has in the interval $]0,2\pi+\epsilon_2]$.

Lemma 2. *The function:*

$$\mathcal{J}_b = 9\,\sin(\frac{t}{2})^2\frac{(4t^3\cos(\frac{t}{2})+(12-12t^2+t^4)\sin(\frac{t}{2})-4\sin(\frac{3t}{2}))}{t(\frac{t}{2}\cos(\frac{t}{2})-\sin(\frac{t}{2}))}$$

is positive for $t\in]0,2\pi[$.

Proof: $\sin(\frac{t}{2})^2/(t(\frac{t}{2}\cos(\frac{t}{2})-\sin(\frac{t}{2})))$ is negative for $t\in]0,2\pi[$ hence it's enough to prove that $k(t)=(4t^3\cos(\frac{t}{2})+(12-12t^2+t^4)\sin(\frac{t}{2})-4\sin(\frac{3t}{2}))$ is negative for $t\in]0,2\pi[$. But $k(0)=k'(0)=0$, hence it's enough to prove that k" is negative for $t\in]0,2\pi[$. $k"(t)=(3t^3\cos(\frac{t}{2})+(-27+3t^2-t^4/4)\sin(\frac{t}{2})+9\sin(\frac{3t}{2}))$.

Let us remark that $(-27+3t^2-t^4/4)\leq 18$ for any t.

First, if $t\in[\pi,3\pi/2]$, we have $\sin(\frac{t}{2})\geq\frac{1}{\sqrt2}$ and $\sin(\frac{3t}{2})\leq\frac{1}{\sqrt2}$ therefore $k"(t)\leq\frac{-18}{\sqrt2}+\frac{9}{\sqrt2}<0$.

Secondly, if $t\in[3\pi/2,2\pi]$, $\cos(\frac{t}{2})\leq-\frac{1}{\sqrt2}$ hence $k"(t)\leq-\frac{3t^3}{\sqrt2}+9\leq-\frac{3(3\pi/2)^3}{\sqrt2}+9<0$.

Finally, for $t\in]0,\pi]$, it is a little more intricate. k" is equal to its Taylor series:

$$\begin{array}{rcl}k"(t) & = & \sum_{i\geq3}\frac{(-1)^it^{2i+1}}{2i+1}(3^{2i+3}-27-(2i+1)2i(16i^2+24i-4))\\ & = & P(t)+\sum_{i\geq7}\frac{(-1)^it^{2i+1}}{2i+1}(3^{2i+3}-27-(2i+1)2i(16i^2+24i-4))\end{array}$$

where $P(t)=t^7(p_0-p_2t^2+p_4t^4-p_6t^6)$ with $P(\pi)<0$, $p_0<0$ and $p_2\pi^2<p_0<p_4\pi^4<p_6\pi^6<0$.

It is easy to show that $\sum_{i\geq7}\frac{(-1)^it^{2i+1}}{2i+1}(3^{2i+3}-27-(2i+1)2i(16i^2+24i-4))$ can be written $\sum_{i\geq7}(-1)^ia_i$ where $(a_i)_i$ is a real sequence decreasing to 0, therefore it is negative. For P, let us make the change of variable $u=t/\pi$. then $Q(u)=P(t)=u^7(q_0-q_2u^2+q_4u^4-q_6u^6)$ with $Q(1)<0$, $q_0<0$ and $q_2<q_0<q_4<q_6<0$. $Q(1)<0$ implies $-q_2<q_6-q_0-q_4$ hence we can write $-q_2=q_6-q_0-q_4-\epsilon$ with $\epsilon>0$. Hence we find $Q(u)=u^7(1-u^2)(q_0-q_4u^2+q_6u^2(1+u^2)-\epsilon u^2)$ which is clearly negative. Hence $P(t)<0$ and $k"(t)<0$. ∎

Conclusion: Joining the two cases, we can see that, for any positive ε as small as we want, if we suppose $\rho_0 > 0$ small enough, then for any point of $S^2(\rho_0)$, the Jacobian is positive on the interval of time $]0, 2\pi - \varepsilon]$ and has no more than two roots on the interval $[2\pi - \varepsilon, 2\pi + \varepsilon]$. The theorem is proved on both domains.

8. The first conjugate locus

In this section, we prove the following:

Theorem 7. *For $\rho_0 > 0$ small enough and for any initial condition (φ, θ) in $S^2(\rho_0)$, there is a first conjugate time $T_c(\rho_0, \varphi, \theta)$, the exponential map is locally Lagrange stable at $(\rho_0, \varphi, \theta, T_c(\rho_0, \varphi, \theta))$ and the only singularities existing are $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{D}_4^+$.*

For this, we work on the jets of the exponential map, computing with **Mathematica**. We separate four sub-domains of the set of initial conditions $S^2\times]0, \eta[$, which union is $S^2\times]0, \eta[$. We need to work on different domains, first because we need the parameterization to be smooth (this implies at least two domains); second because on some part we know a priori the existence of conjugate time and on some other we do not; and third because of the geometry of the problem: close to the abnormal direction of the nilpotent approximation, the computation is different because the first stable jet of the exponential map appears at a higher order than for other initial conditions.

The domains will be defined as follows:

- First domain (with parameterization $\mathcal{P}_1$): $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$.
- Second domain (with parameterization $\mathcal{P}_1$): $\theta \in [-\frac{\pi}{2} + M_2\rho_0, -\frac{\pi}{6}] \cup [\frac{\pi}{6}, \frac{\pi}{2} - M_2\rho_0]$, M_2 being defined later. In fact we will work only on the second interval, the calculus being the same on both interval and giving the same results.
- Third domain (with parameterization $\mathcal{P}_2$): $\{(\theta - \frac{3B\rho_0\sin(b)}{\pi})^2 + (\varphi + \frac{3B\rho_0\cos(b)}{\pi})^2 \leq 2M_2^2\rho_0^2\} \cap \{(\theta - \frac{3B\rho_0\sin(b)}{\pi})^2 + (\varphi + \frac{3B\rho_0\cos(b)}{\pi})^2 \geq M_4\rho_0^3\}$, M_4 being defined later. In fact, we should also consider the symmetric domain but, as for the previous domain, the computations are the same and leads to the same results.
- Last domain (with parameterization $\mathcal{P}_2$): $\{(\theta - \frac{3B\rho_0\sin(b)}{\pi})^2 + (\varphi + \frac{3B\rho_0\cos(b)}{\pi})^2 \leq M_4\rho_0^3\}$. As, in the two previous cases, we could consider also the symmetric domain but it does not bring anything more.

Let us remark that the union of the four domain is really the complete sphere if we fix M_2 large enough.

The singularity $\mathcal{D}_4^+$, which doesn't appear in the generic cases of contact sub-Riemannian structures of dimension 3 (see [A-C-G-K]), appears in the last domain, close to the abnormal direction of the nilpotent case.

8.1. First domain: $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ in $\mathcal{P}_1$.

We have proved in section 6 that, in this domain, the first conjugate time exists and is close to 2π. But then, the Nilpotent approximation allows to check that $w = \frac{1}{2}\rho_0^2\cos(\theta)^2(t - \sin(t)) + \rho_0^3 W_3(t, \varphi, \theta, \rho_0)$, hence, close to $t = 2\pi$, we have $w > 0$ and $\frac{\partial w}{\partial \rho_0} > 0$. Consequently we can change the variable ρ_0 for $h = \sqrt{\frac{w}{\pi}}$. Now, on the domain, thanks to nilpotent approximation, we can check that, close to $t = 2\pi$, we have $\frac{\partial z(h,\theta,\varphi,t)}{\partial \theta} > 0$ hence we can change θ for z. In fact we will use the variable $\theta_0 = \arctan(\frac{z}{2\pi h})$.

Now, if we denote by exp_3 the new exponential map (after the double change of variables), we have:

$$exp_3(h,\theta_0,\varphi,t) = (x(h,\theta_0,\varphi,t), y(h,\theta_0,\varphi,t), 2\pi h\tan(\theta_0), \pi h^2),$$

hence we just have to annihilate the Jacobian of $exp_{h,\theta_0} : (\varphi,t) \to (x(h,\theta_0,\varphi,t), y(h,\theta_0,\varphi,t))$ to find the conjugate time.

We set $T=\frac{t-2\pi}{h}$ and we compute with the jets of the Jacobian of exp_{h,θ_0}. We find (in appendix 10.7) $T = T_1 + h\,T_2 + \mathcal{O}(h\theta_0, h^2)$, where:

$$\begin{array}{rcl} T_1 &=& 2\pi(\pi\lambda - 3A\cos(a+2\varphi))\tan(\theta_0),\\ T_2 &=& \frac{\pi}{6}(-135A^2+72F_2+12F_1\pi-15\lambda^2-2\pi^2\lambda^2+72F_3\sin(f_3)). \end{array}$$

We want to study the first conjugate locus at (h,θ_0,φ_0) hence we set $\tau = T - (T_1(h,\theta_0,\varphi_0) + h * T_2(h,\theta_0,\varphi_0))$ and $\tilde\varphi = \varphi - \varphi_0$, and we still denote by exp_{h,θ_0} the restricted exponential map. The computations of 10.7 show that we can write:

$$exp_{h,\theta_0}(\tilde\varphi,\tau) = \overline{exp_{h,\theta_0}}(\tilde\varphi,\tau) + \mathcal{O}(h^3\theta_0, h^4),$$

where, if we make a good affine change of coordinates on (x,y) (depending on the point we are looking at) and a good polynomial change of the variables $(\tau,\tilde\varphi)$ fixing $(0,0)$, we have:

$$\overline{exp_{h,\theta_0}}(\tilde\varphi,\tau) = (h^2\tau, h^2\tilde\varphi\tau + \frac{h^2}{4}(\tilde\varphi^2 k(\varphi_0) + \frac{1}{3}\tilde\varphi^3\dot k(\varphi_0))) + h^2\theta_0\mathcal{O}^4(\tilde\varphi,\tau) + h^3\mathcal{O}^4(\tilde\varphi,\tau)$$

and

$$k(\varphi_0) = -36A\pi\sin(a+2\varphi_0)\tan(\theta_0) + \pi h(72F_3\cos(f_3-2\varphi_0) + 24F_3\cos(f_3+2\varphi_0)$$
$$-48F_2\sin(2\varphi_0) + 45A\lambda\cos(a+2\varphi_0)).$$

8.1.1. *Let us first assume that* $\theta_0 \neq 0$. Then, denoting $\tilde\tau = \frac{\tau}{\tan(\theta_0)}$, we can write:

$$exp_{h,\theta_0}(\tilde\varphi,\tilde\tau) = h^2\tan(\theta_0)(app(\tilde\varphi,\tilde\tau) + \frac{h}{\tan(\theta_0)}R(h,\theta_0,\tilde\varphi,\tilde\tau)),$$

where

$$app : (\tilde\varphi,\tilde\tau) \mapsto (\tilde\tau, \tilde\varphi\tilde\tau + \frac{1}{4}(\tilde\varphi^2 k_1(\varphi_0) + \frac{1}{3}\tilde\varphi^3\dot k_1(\varphi_0))) + \mathcal{O}^4(\tilde\varphi,\tau),$$

$k_1(\varphi_0) = -36A\pi\sin(a+2\varphi_0)$ and all is smooth. The map *app* has singularity $\mathcal{A}_2$ when $k_1(\varphi_0)\neq 0$ and $\mathcal{A}_3$ when $k_1(\varphi_0)=0$.

Now, let us consider the map:

$$(\tilde\varphi,\tilde\tau) \mapsto \frac{exp_{h\tan(\theta_0),\theta_0}(\tilde\varphi,\tilde\tau)}{h^2\tan(\theta_0)} = app(\tilde\varphi,\tilde\tau) + \frac{h}{\tan(\theta_0)}R(h\tan(\theta_0),\theta_0,\tilde\varphi,\tilde\tau).$$

It exists η_1 such that if $|\frac{h}{\tan(\theta_0)}| \leq \eta_1$ then this map is locally equivalent to *app*. Therefore exp_{h,θ_0} is locally equivalent to *app*.

8.1.2. *Now we assume $h \geq \eta_1 \tan(\theta_0)$.* If we denote $\tilde{\tau} = \frac{\tau}{h}$, we can write:

$$exp_{h,\theta_0}(\tilde{\varphi}, \tilde{\tau}) = h^3(app_2(\tilde{\varphi}, \tilde{\tau}) + hR_2(h, \theta_0, \tilde{\varphi}, \tilde{\tau})),$$

where:

$$app_2 : (\tilde{\varphi}, \tilde{\tau}) \mapsto (\tilde{\tau}, \tilde{\varphi}\tilde{\tau} + \frac{1}{4}(\tilde{\varphi}^2 k_2(\varphi_0) + \frac{1}{3}\tilde{\varphi}^3 \dot{k}_2(\varphi_0))) + \mathcal{O}^4(\tilde{\varphi}, \tilde{\tau}),$$

$$k_2(\varphi_0) = -36A\pi \sin(a + 2\varphi_0)\tan(\theta_1) + \pi(72F_3\cos(f_3 - 2\varphi_0) + 24F_3\cos(f_3 + 2\varphi_0) \\ -48F_2 \sin(2\varphi_0) + 45A\lambda\cos(a + 2\varphi_0))$$

and $h\tan(\theta_1) = \tan(\theta_0)$ which implies $|\tan(\theta_1)| \leq \frac{1}{\eta_1}$.

Lemma 3. *For all θ_1, k_2 is not identically equal to 0.*

Proof: we have:

$$k_2(\varphi_0) = \pi\cos(2\varphi_0)(-36A\sin(a)\tan(\theta_1) + (96F_3\cos(f_3) + 45A\cos(a)\lambda)) + \\ +\pi\sin(2\varphi_0)(-36A\cos(a)\tan(\theta_1) + (48F_3\sin(f_3) - 48F_2 - 45A\sin(a)\lambda)).$$

The function k_2 can be identically equal to 0 only if the coefficients of $\cos(2\varphi_0)$ and $\sin(2\varphi_0)$ can be 0 together. And this is possible only if the matrix:

$$\begin{pmatrix} A\sin(a) & 96F_3\cos(f_3) + 45A\cos(a)\lambda \\ A\cos(a) & 48F_3\sin(f_3) - 48F_2 - 45A\sin(a)\lambda \end{pmatrix}$$

has rank 1. But this condition is obtained only if $\mathcal{I}nv = 0$ which is not the case for us because of the assumption we made in section 4. ■

Let us assume that we are in the generic situation. Because k_2 satisfies $\ddot{k}_2 + 4k_2 = 0$ and is not identically equal to 0, $k_2(\varphi_0)$ and $\dot{k}_2(\varphi_0)$ can not be both equal to 0. Consequently the map app_2 has singularity $\mathcal{A}_2$ when $k_2(\varphi_0)$ is not zero and $\mathcal{A}_3$ when $k_2(\varphi_0)$ is zero.

Now let us consider the map:

$$(\tilde{\varphi}, \tilde{\tau}) \mapsto \frac{exp_{h,\theta_0}(\tilde{\varphi}, \tilde{\tau})}{h^3} = app_2(\tilde{\varphi}, \tilde{\tau}) + hR_2(h, \theta_0, \tilde{\varphi}, \tilde{\tau}).$$

It exists η_2 such that if $0 < h \leq \eta_2$ then the previous map is locally equivalent to app_2. Hence, if $0 < h \leq \eta_2$ then exp_{h,θ_0} is locally equivalent to app_2.

We have proved that for any (θ, φ) such that $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, it exists $\eta(\theta, \varphi)$ such that if $0 < \rho_0 < \eta(\theta, \varphi)$ then the exponential map is locally equivalent to its jet at $(\rho_0, \theta, \varphi, T_c(\rho_0, \theta, \varphi))$, and have singularity $\mathcal{A}_2$ or $\mathcal{A}_3$. But the set of (θ, φ) such that $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ is compact hence we can find η such that if $0 < \rho_0 < \eta$ then the exponential map has only singularity $\mathcal{A}_2$ or $\mathcal{A}_3$ in the domain. The theorem is proved in the first domain.

8.2. **Second domain:** $\theta \in [\frac{\pi}{6}, \frac{\pi}{2} - M_2\rho_0]$, again in $\mathcal{P}_1$.

As before, thanks to previous section, we know the existence of a first conjugate time which is close to 2π hence the nilpotent approximation allows to check that $\frac{\partial z}{\partial \rho_0} > 0$ close to the conjugate time if ρ_0 is small enough. We change the variable ρ_0 for z. We denote by exp_4 the new exponential map after this change of variable and we have:

$$exp_4(z, \widetilde{\theta}, \varphi, t) = (x(z, \widetilde{\theta}, \varphi, t), y(z, \widetilde{\theta}, \varphi, t), z, w(z, \widetilde{\theta}, \varphi, t)),$$

hence we just have to annihilate the Jacobian of

$$(\widetilde{\theta}, \varphi, t) \mapsto (x(z, \widetilde{\theta}, \varphi, t), y(z, \widetilde{\theta}, \varphi, t), w(z, \widetilde{\theta}, \varphi, t))$$

to compute the conjugate time. We denote by t_1 its value of order 1 (computed with the jet) with respect to z and we set $\tau = t - t_1$. We also change $\widetilde{\theta}$ for $h = \tan(\widetilde{\theta})$ and φ for $\widetilde{\varphi} = \varphi - \varphi_0$. We study exp_4 in a neighborhood of $(z, h, \varphi_0, \tau = 0)$. Forgetting the z-coordinate, we can write:

$$exp_{4,z}(h, \widetilde{\varphi}, \tau) = z^2(app_{3,1}(h, \widetilde{\varphi}, \tau) + (zx_3, zy_3, zhw_3 + z^2 w_4)).$$

with x_3, y_3, w_3 and w_4 smooth. After some affine changes of coordinates on (x, y) (depending on the point we are looking at) and polynomial changes of the variables $\widetilde{\varphi}$ and τ fixing $(0, 0)$, the computation of 10.8 gives:

$$app_{3,1}(h, \widetilde{\varphi}, \tau) = (x(z, h, \widetilde{\varphi}, \tau), y(z, h, \widetilde{\varphi}, \tau), w(z, h, \widetilde{\varphi}, \tau))$$

with

$$\begin{array}{rcl} x(z, h, \widetilde{\varphi}, \tau) & = & \frac{h\tau}{2\pi}, \\ y(z, h, \widetilde{\varphi}, \tau) & = & \frac{-h\widetilde{\varphi}(-2t + 6A\widetilde{\varphi}^2\cos(a+2\varphi_0) + 9A\widetilde{\varphi}\sin(a+2\varphi_0))}{4\pi}, \\ w(z, h, \widetilde{\varphi}, \tau) & = & \frac{h^2}{4\pi}. \end{array}$$

Now, $w = \frac{z^2h^2}{4\pi}(1 + \frac{z}{h}W_1 + \frac{z^2}{h^2}W_2)$ with W_1 and W_2 smooth. Therefore, because W_1 and $W_2$are bounded on the domain, if $|\frac{z}{h}|$ is sufficiently small then $w > 0$ and we have $\widetilde{h} = \sqrt{\frac{4\pi w}{z^2}} = h(1 + \widetilde{W})$ with $\widetilde{W}$ a $\mathcal{O}(z, \frac{z}{h}$ and hence we can write $h = \widetilde{h}(1 + \widetilde{H})$ with $\widetilde{H} = \mathcal{O}(z, \frac{z}{h})$. M_2 is fixed such a way $\frac{z}{h}$ is small enough on the domain. Now we can write:

$$exp_{4,z}(\widetilde{h}, \widetilde{\varphi}, \tau) = z^2(app_{3,1}(\widetilde{h}, \widetilde{\varphi}, \tau) + (zx_3, zy_3, 0)),$$

with

$$app_{3,1} = (\frac{\widetilde{h}\tau}{2\pi}, \frac{-\widetilde{\widetilde{h}\varphi}(-2t + 6A\widetilde{\varphi}^2\cos(a + 2\varphi_0) + 9A\widetilde{\varphi}\sin(a + 2\varphi_0))}{4\pi}, \frac{\widetilde{h}^2}{4\pi}).$$

Therefore the singularity of the jet is $\mathcal{A}_2$ or $\mathcal{A}_3$ (if $\sin(a + 2\varphi_0) = 0$, and then, if z is small enough, the singularity of the exponential map is $\mathcal{A}_2$ or $\mathcal{A}_3$ at the point we consider. But the set of (θ, φ) of the domain is a compact set hence we can conclude that on the domain there is a η such that if $0 < z < \eta$ then the exponential map has first singularity $\mathcal{A}_2$ or $\mathcal{A}_3$. Because $\rho_0 \asymp z$, we find the same condition on ρ_0 on the domain. The theorem is proved on the domain.

From now, we use the second parameterization of S^2.
Remark: in the sequel, we change a little our point of view. In the two first domains, we knew the existence of the first conjugate time and the jets allowed to estimate it. From now, we don't know a priori the existence of the conjugate time. We have informations about it in the case it exists but we don't have yet the proof of its existence. Hence we first study the jet, we find that it has first conjugate time for any initial condition, we prove that its singularity at this first conjugate time is stable, and finally, by stability, we obtain that the exponential map itself, as a perturbation of the jet, has also singularity, of the same type.

8.3. **Third domain:** we look at the conjugate time of the jet close to 2π. First, as for the previous domain, we can change ρ_0 for z and t for $\tau = t - 2\pi$.

Because we can ask ρ_0 to be as small as we want, then $|1 - \frac{z}{\rho_0}|$ can be as close to 0 as we want on the domain. Therefore we can define this third domain with ρ_0 or z equivalently.

Then we change θ for $\widetilde{\theta} = -\frac{3B\sin(b)}{\pi} - L\sin(l) + \frac{\theta}{z}$, φ for $\widetilde{\varphi} = \frac{3B\cos(b)}{\pi} - L\cos(l) + \frac{\varphi}{z}$ and τ for $\widetilde{\tau} = \frac{\tau}{z} - \pi\lambda + 3A\cos(a+2l)$ (hear, L and l determine the point we are looking at in the domain and $z(\pi\lambda - 3A\cos(a+2l))$ is the conjugate value for the variable τ computed with the jet).

Remark: now the domain can be contained in the set of (L,l,z) such that $L \in [\frac{\sqrt{M_4 z}}{2}, 2M_2]$ if ρ_0 is small enough.

After a linear transformation depending on z one can find (we forget the z-coordinate):

$$exp_5(z,\widetilde{\theta},\widetilde{\varphi},\widetilde{\tau}) = app_4(\widetilde{\theta},\widetilde{\varphi},\widetilde{\tau}) + (\frac{z}{L}x_4, \frac{z}{L}y_4, \frac{z}{L^2}w_5),$$

where x_4, y_4 and w_5 are smooth functions. We make an affine change of coordinates on (x,y,w) depending on (L,l,z) and some polynomial changes of the variables $(\widetilde{\theta},\widetilde{\varphi},\widetilde{\tau})$ fixing $(0,0,0)$ (see appendix 10.9) and we find:

- when $a+2l = \pi[2\pi]$, then:

  $$app_4(\widetilde{\theta},\widetilde{\varphi},\widetilde{\tau}) = ((2+2\widetilde{\varphi}-\widetilde{\varphi}^2)\widetilde{\tau}, \widetilde{\theta}(3A\widetilde{\theta}^2+\widetilde{\tau}), \widetilde{\varphi}) + \mathcal{O}^4(\widetilde{\theta},\widetilde{\varphi},\widetilde{\tau}).$$

  app_4 has singularity $\mathcal{A}_3$ at $\widetilde{\theta} = \widetilde{\varphi} = \widetilde{\tau} = 0$.
- when $a+2l = 0[2\pi]$, then:

  $$app_4(\widetilde{\theta},\widetilde{\varphi},\widetilde{\tau}) = (2+2\widetilde{\varphi}-\widetilde{\varphi}^2)\widetilde{\tau}, \widetilde{\theta}(-3A\widetilde{\theta}^2+\widetilde{\tau}), \widetilde{\varphi}) + \mathcal{O}^4(\widetilde{\theta},\widetilde{\varphi},\widetilde{\tau}).$$

  app_4 has singularity $\mathcal{A}_3$ at $\widetilde{\theta} = \widetilde{\varphi} = \widetilde{\tau} = 0$.
- when $\sin(a+2l) \neq 0$ then:

  $$\mathrm{jac}(\frac{\partial app_4}{\partial \widetilde{\varphi}}, \frac{\partial app_4}{\partial \widetilde{\tau}}, \frac{\partial app_4}{\partial \widetilde{\theta}}) = 0$$

  and

  $$\mathrm{jac}(\frac{\partial app_4}{\partial \widetilde{\varphi}}, \frac{\partial app_4}{\partial \widetilde{\tau}}, \frac{\partial^2 app_4}{\partial \widetilde{\theta}^2}) \neq 0.$$

  This prove that app_4 has singularity $\mathcal{A}_2$ at $\widetilde{\theta} = \widetilde{\varphi} = \widetilde{\tau} = 0$.

Now, at each point of the domain, we have that if $\frac{z}{L}$ is small enough then the exponential map is equivalent to its jet hence it has the same singularity. The set of (L,l,z) such that $L \in [\frac{\sqrt{M_4 z}}{2}, 2M_2]$ being compact, there is a common η such that if $\frac{z}{h} < \eta$, then the first singularities of the exponential map are $\mathcal{A}_2$ and $\mathcal{A}_3$.

8.4. **Last domain.** In this case, we make the changes of variables as for the third domain, to introduce L and l. Then we change z for U^2, L for $\widetilde{L} = \frac{L}{U}$, $\widetilde{\theta}$ for $\vartheta = \frac{\widetilde{\theta}}{U}$ and $\widetilde{\varphi}$ for $\varphi = \frac{\widetilde{\varphi}}{U}$.

Remark: now the domain can be contained in the set of $(\widetilde{L},l,U)$ such that $\widetilde{L} \in [0, 2\sqrt{M_4}]$.

After a linear transformation on the exponential map (depending on U), we get(we forget the z coordinate):

$$exp_6 = app_5(\vartheta,\varphi,t) + U(x_8,y_8,w_{11}),$$

where x_8, y_8 and w_{11} are smooth functions.

After some linear changes of coordinates on (x,y,w) depending on $(U,\widetilde{L},l)$ and some polynomial changes of variables fixing $(\vartheta = 0, \varphi = 0)$ and sending the conjugate time for the jet to 0 (see 10.10), we find:

- If $a+2l=0[2\pi]$:
$$app_5(\vartheta,\varphi,t)=(A\varphi,t\vartheta-6A\pi^3\vartheta^3,t)+\mathcal{O}^4(\vartheta,\varphi,t),$$
hence app_5 has singularity $\mathcal{A}_3$ at $(0,0,0)$.
- If $a+2l=\pi[2\pi]$ and $27AB^2-2\widetilde{L}^2\pi^3<0$:
$$app_5(\vartheta,\varphi,t)=(-A\varphi,t\vartheta-6A\pi^3\vartheta^3,t)+\mathcal{O}^4(\vartheta,\varphi,t),$$
hence app_5 has singularity $\mathcal{A}_3$ at $(0,0,0)$.
- If $a+2l=\pi[2\pi]$ and $27AB^2-2\widetilde{L}^2\pi^3=0$:
$$app_5(\vartheta,\varphi,t)=(t,A\varphi\vartheta,3A(3\varphi^2+\vartheta^2)+2\varphi t)+\mathcal{O}^4(\vartheta,\varphi,t),$$
hence app_5 has singularity $\mathcal{D}_4^+$ at $(0,0,0)$.
- If $a+2l=\pi[2\pi]$ and $27AB^2-2\widetilde{L}^2\pi^3>0$ and $\widetilde{L}>0$:
$$\mathrm{jac}(\frac{\partial app_5}{\partial\vartheta},\frac{\partial app_5}{\partial t},\frac{\partial app_5}{\partial\varphi})=0$$
and
$$\mathrm{jac}(\frac{\partial app_5}{\partial\vartheta},\frac{\partial app_5}{\partial t},\frac{\partial^2 app_5}{\partial\varphi^2})\neq 0.$$
This implies that app_5 has singularity $\mathcal{A}_2$ at $(0,0,0)$.
- If $\sin(a+2l)\neq 0$ and $\widetilde{L}>0$: $app_{5,5}$ is such that, if we denote $app_5=(\bar{x},\bar{y},\bar{w})$, $\upsilon_1=\frac{\partial\bar{y}}{\partial\vartheta}$, $\upsilon_2=\frac{\partial\bar{y}}{\partial\varphi}$ and if we use the new variables $\bar{\theta}$ and $\bar{\varphi}$ such that $\vartheta=\upsilon_1\bar{\theta}+\upsilon_2\bar{\varphi}$ and $\varphi=-\upsilon_1\bar{\varphi}+\upsilon_2\bar{\theta}$, then $\frac{\partial app_5}{\partial\bar{\varphi}}=0$ but $\mathrm{Det}(\frac{\partial app_5}{\partial\bar{\theta}},\frac{\partial app_5}{\partial t},\frac{\partial^2 app_5}{\partial\bar{\varphi}^2})\neq 0$, which implies that app_5 has singularity $\mathcal{A}_2$ at $(0,0,0)$.

In all the different cases, exp_6 can be seen as a lagrangian perturbation of a stable lagrangian map. The set of $(\widetilde{L},l)$ being compact, we can find η such that if $U<\eta$ then for any $(\widetilde{L},l,U)$, we have that the exponential map has only singularities of type $\mathcal{A}_2$, $\mathcal{A}_3$ or $\mathcal{D}_4^+$ at the first conjugate locus. Therefore there is a η' such that if $\rho_0<\eta'$ we have the same conclusion.

This ends the proof of the theorem.

9. Images of the conjugate locus

In this section, we present some pictures of the intersection of the first caustic with two special 3-subspaces of $\mathbf{R}^4$.

In figure 6, one can find pictures of the intersection of the caustic with $\{z = z_0\}$ with $z_0 \neq 0$. One can see two singularities of type $\mathcal{D}_4^+$ where two of the lines of cusp points stop.

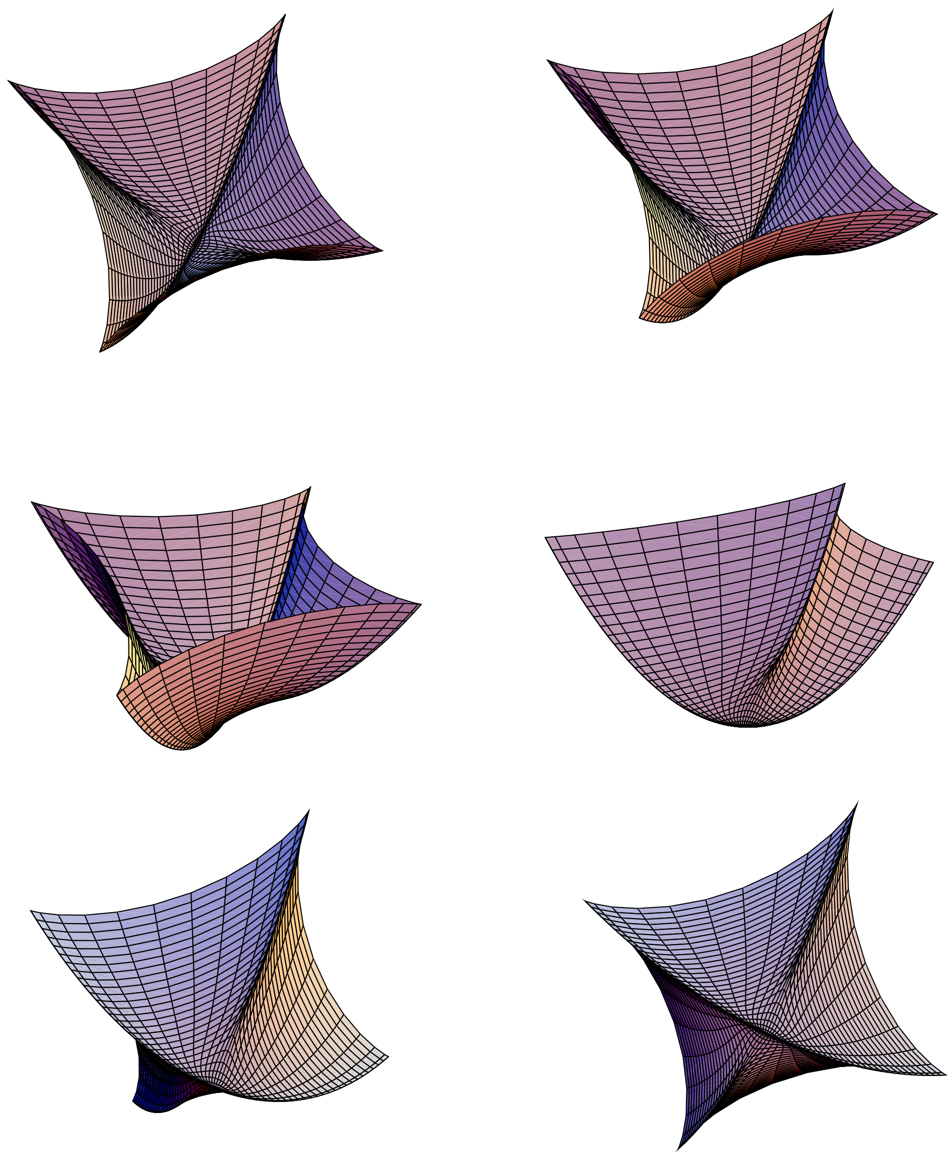

Figure 6. Pictures corresponding to the last domain