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Ricci flow with surgery

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These notes provides some details on the lectures 2,3,4 on the Ricci flow with surgery. They are not complete and probably contains some inaccuracies. Interested readers can find most exhaustives explanations on the Perelman's papers in [KL].

1 Lecture 2: classification of κ-solutions

The aim of these lecture is to give the classification and the description of 3-dimensional κ-solutions. Let $\kappa > 0$ and $(M^n, g(t))$ a solution of the Ricci flow. M^n is supposed oriented.

definition 1.1. *$(M, g(t))$ is a κ-solution if*

- *$g(t)$ is an ancient solution of the Ricci flow*

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)}, \qquad -\infty < t \leq 0.$$

- *for each t, $g(t)$ is a complete, non flat metric of bounded curvature and non negative curvature operator.*

- *for each t, $g(t)$ is κ-noncollapsed on all scales, i.e. if $|Rm(g(t))| \leq \frac{1}{r^2}$ on $B = B(p, t, r)$, then*

$$\frac{vol_{g(t)}(B)}{r^n} \geq \kappa$$

Exemples: S^3 and $\mathbb{S}^2 \times \mathbb{R}$ with their standard flow are κ-solutions for some $\kappa > 0$. But $\mathbb{S}^2 \times \mathbb{S}^1$ with the standard flow is not a κ-solution. It is κ-collapsed at very negative times.

Some properties of κ-solutions:

- All curvatures of $g(t)$ at x are controlled by the scalar curvature $R(x,t)$.
- For each point x in M, $R(x,t)$ is nondecreasing.

  It's a consequence of the trace Harnack inequality [H93] (compare with Carlo Sinestrari notes [S05] (6.6)

  $$\frac{\partial R}{\partial t} + 2 < X, \nabla R > +2Ric(X,X) \geq 0,$$

  where X is an arbitrary vector field. Thus

  $$\sup_{M\times]\infty,0]} R(.,.) = \sup_M R(.,0) < \infty$$

  and all curvatures are uniformly bounded on $M\times]-\infty,0]$.
- $R(x,t) > 0$ for any (x,t).

  It follows from the integrated version of the Harnack Inequality,

  $$R(x_2,t_2) \geq exp\left(-\frac{d^2_{t_1}(x_1,x_2}{2(t_2-t_1}\right) R(x_1,t_1),$$

  for any $t_1 < t_2$. Indeed, if $R(x_2,t_2) = 0$ for some point (x_2,t_2), then $R(x_1,t_1) = 0$ for any point (x_1,t_1) with $t_1 < t_2$. Thus $g(t)$ would be flat for any t.

Tools: compactness theorem, asymptotic solitons, splitting

compactness theorem Given any κ-solution $(M^3, g(t))$ and $(x_0,t_0) \in M\times]-\infty,0]$, one defines the *normalized* κ-solution at (x_0,t_0) by

$$g_0(t) = R(x_0,t_0)g(t_0 + \frac{t}{R(x_0,t_0)}).$$

We have done a shift in time and a parabolic rescaling such that $R_{g_0}(x_0,0) = 1$. The motivation is :

theorem 1.2 ([P03]I.11.7, [KL]40). *For any $\kappa > 0$, the set of pointed normalized κ-solutions*

$$\{(M, g(.), x), R(x, 0) = 1\}$$

is compact.

The same result holds with the normalization $R(x, 0) \in [c_1, c_2]$, $0 < c_1 \leq c_2 < \infty$.

Asymptotic solitons Perelman defines an asymptotic soliton $(M_{-\infty}, g_{-\infty}, x_{-\infty})$ of an n-dimensional κ-solution $(M, g(t)$ as follows. Pick a sequence $t_k \to -\infty$.

theorem 1.3 ([P03]I.11.2). *there exists $x_k \in M$ such that $(M, \frac{1}{-t_k} g(t_k - t_k t), x_k)$ (sub) converge to a non flat gradient shrinking soliton$(M_{-\infty}, g_{-\infty}, x_{-\infty})$, called an asymptotic soliton of the κ-solution.*

Recall that a Ricci flow $(M, g(t))$ on (a, b), $a < 0 < b$, is a gradient shrinking soliton if there exists a decreasing function $\alpha(t)$, diffeomorphisms of M ψ_t generated by $\nabla_{g(t)} f_t$ such that

$$g(t) = \alpha(t) \psi_t^* g(0), \qquad \forall t \in (a, b).$$

The proof strongly uses the reduced length and reduced volume introduced in [P03]ch.7.

corollary 1.4 (of the compactness theorem). *Any 3-dimensional asymptotic soliton is a κ-solution.*

Proof: The sequence $\tau_k R(x_k, t_k)$ has a limit $R(x_{-\infty}, 0) \in (0, +\infty)$. Thus the asymptotic soliton is a parabolic rescaling of the limit of $(M, R(x_k, t_k) g(t_k + \frac{t}{R(x_k, t_k)}), x_k)$, a κ-solution. Thus a 3-asymptotic solitons are particular κ-solutions. Due to their self-similarity, they are much easier to classify.

Strong maximum principle the following will give splitting arguments

theorem 1.5 ([H86]). *Let $(M^3, g(t))$ a Ricci flow on $[0, T)$ such that sectional curvatures of $g(a)$ are ≥ 0. Then precisely one of the following holds*

a) *For every $t \in (0, T)$, $g(t)$ is flat.*

b) *For every $t \in (0, T)$, $g(t)$ has a local isometric splitting $\mathbb{R} \times N^2$, where N^2 is a surface with positive curvature.*

c) *For every $t \in]a, b[$, $g(t)$ has > 0 curvature.*

In case b), the universal covering is isometric $\mathbb{R} \times N^2$.

classification of 3-asymptotic solitons

proposition 1.6. *The only asymptotic solitons are $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times_{\mathbb{Z}_2} \mathbb{R}$ where the $\mathbb{Z}_2$-action is given by the relation $(x, s) \sim (-x, -s)$, and finite quotients of S^3, with their standard flows.*

Proof: Consider an asymptotic soliton $(M_{-\infty}, g_{-\infty}, x_{-\infty}) = (M, g(t), x)$ of a κ-solution. By the strong maximum principle 1.5 and the non flatness, either $g(t)$ has strictly positive curvature either it splits locally.

Consider the non compact case. The strictly positive curvature is ruled out by

theorem 1.7 ([P03]II.1.2). *There is no complete oriented 3-dimensional non compact κ-noncollapsed gradient shrinking soliton with bounded (strictly) positive curvature.*

Thus $(M, g(t))$ has a local splitting and $(\tilde{M}, \tilde{g}(t)) = (N^2 \times \mathbb{R}, h(t) + dx^2)$. As the splitting is preserved by the flow $(N^2, h(t))$ is a Ricci flow with strictly positive curvature. It is an exercice to check that it is a κ-solution.

Now there is

theorem 1.8 ([P02]I.11.2). *there is only one oriented 2-dimensional κ-solution - the round sphere.*

proof: (heuristic). Suppose that N^2 is compact. It can be shown that the asymptotic soliton $N^2_{-\infty}$ is also compact (same arguments as in [CK04], prop 9.23), thus diffeomorphic to S^2. By [H88], a metric with positive curvature on S^2 gets more rounder under the Ricci flow. More precisely, the curvatures pinching - the ratio of the minimum scalar curvature and the maximum- improves, i.e. converge to 1. On the other hand $(N^2_{\infty}, h_{-\infty}(t))$ evolves by diffeomorphims and dilations hence the curvatures pinching is constant. Thus for any $t \leq 0$, $h_{-\infty}(t)$ has constant curvature. Now the curvatures pinching of $(N^2, h(t))$ improves under the flow as $t \to 0$ and is arbitrary close to 1 when $t \to -\infty$, as the asymptotic "initial condition" $(N^2_{-\infty}, h_{-\infty(0)})$ is the round sphere. The non compact case is ruled out by [KL][.37]. In fact, they give a proof of 1.8 without solitons. □.

Thus $(\tilde{M}, \tilde{g}(t)) = \mathbb{S}^2 \times \mathbb{R}$ with a round cylindrical flow. The only non compact oriented quotient is $\mathbb{S}^2 \times_{\mathbb{Z}_2} \mathbb{R} = \mathbb{RP}^3 - \overline{\mathbb{B}^3}$.

Now consider the compact case. If $(M, g(t))$ has strictly positive curvature, by [H82] M is diffeomorphic to a round S^3/Γ and $g(t)$ gets more rounder under the flow. By self-similarity of the metric, it is the round one, as above. We cannot have a local splitting because the only oriented isometric compact quotients of $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{S}^2 \times_{\mathbb{Z}} \mathbb{S}^1 = \mathbb{RP}^3 \# \mathbb{RP}^3$, are not κ-solutions. □

classification of κ-solutions We have the following

theorem 1.9. *Any κ-solution $(M, g(t))$ is diffeomorphic to one of the following.*

a $\mathbb{S}^2 \times \mathbb{R}$ *or* $\mathbb{S}^2 \times_{\mathbb{Z}_2} \mathbb{R} = \mathbb{RP}^3 - \overline{\mathbb{B}^3}$, *and $g(t)$ is the round cylindrical flow.*

b $\mathbb{R}^3$ *and $g(t)$ has strictly positive curvature.*

c *A finite isometric quotient of the round S^3 and $g(t)$ has positive curvature. Moreover, $g(t)$ is round if and only if the asymptotic soliton is compact. If the asymptotic soliton is non compact, M is diffeomorphic to $\mathbb{S}^3$ or $\mathbb{RP}^3$.*

Proof of theorem 1.9: Apply again the strong maximum principle to the κ-solution $(M, g(t))$. If $g(t)$ locally splits, we have the same classification as for asymptotic soliton. Suppose $g(t)$ has strictly positive curvature. If it is compact, M is diffeomorphic to a finite quotient of the round S^3. If its asymptotic soliton $M_{-\infty}$ is compact, it is the round flow on a finite quotient of S^3 by the above classification. Thus the asymptotic initial condition is round and $(M, g(t)$ is itself a round flow. In a noncompact case M is diffeomorphic to $\mathbb{R}^3$ by a theorem of Gromoll and Meyer [GM89]. The cases of strictly positive curvature needs more geometrical control. The proof will be finished below.

More on κ-solutions

We describe the geometry of κ-solutions, which is useful for non round flows. We 'll see that large parts of these κ-solution looks like round cylinders.

definition 1.10. *Let $B(x,t,r)$ denotes the open metric ball of radius r, with respect to $g(t)$.*
*Fix some $\varepsilon > 0$. A ball $B(x,t,\frac{r}{\varepsilon})$ is an ε-**neck**, if after rescaling by $\frac{1}{r^2}$, it is ε-close in the $C^{[\varepsilon^{-1}]}$ topology to the corresponding subset of the standard neck $\mathbb{S}^2 \times (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$, where $\mathbb{S}^2$ has constant scalar curvature one. One says that x is the **center** of the ε-neck.*

For example, any point in $\mathbb{S}^2 \times \mathbb{R}$ is center of an ε-neck but $(x,0) \in \mathbb{S}^2 \times_{\mathbb{Z}_2} \mathbb{R}$ is not center of an ε-neck.

definition 1.11. *Let $(M, g(.))$ be a κ-solution. For every $\varepsilon > 0$ and time t, let $M_\varepsilon(t)$ $(= M_\varepsilon)$ be the set of points which **are not** center of an ε-neck at time t.*

The geometry of the κ-solutions is described by the

proposition 1.12 ([KL]42.1, strong version of [P02]I.11.8). *For all $\kappa > 0$, for $0 < \varepsilon < \varepsilon_0$, there exists $\alpha = \alpha(\varepsilon, \kappa)$ with the property that for any κ-solution$(M, g(.))$, and any time t precisely one of the following holds,*

A. *$M_\varepsilon = \emptyset$ and $(M, g(.)) = \mathbb{S}^2 \times \mathbb{R}$ is the round cylindrical flow. So every point at every time is center of an ε-neck for all $\varepsilon > 0$.*

B. *$M_\varepsilon \neq \emptyset$, M is non compact and for all $x, y \in M_\varepsilon$, we have $R(x)d^2(x,y) < \alpha$.*

C. *$M_\varepsilon \neq \emptyset$, M is compact and there is a pair of points $x, y \in M_\varepsilon$ such that $R(x)d^2(x,y) > \alpha$,*

$$M_\varepsilon \subset B(x, \alpha R(x)^{-1/2}) \cup B(y, \alpha R(y)^{-1/2}),$$

and every $z \in M \backslash M_\varepsilon$ satisfies $R(z)d^2(z, \overline{xy}) < \alpha$.

D. *$M_\varepsilon \neq \emptyset$, M is compact and there is a point $x \in M_\varepsilon$ such that $R(x)d^2(x,z) < \alpha$ for any $z \in M$.*

Preliminary lemmas

A useful fact is

lemma 1.13. *Let $(M, g(.))$ be a κ-solution which contains a line for some t. Then $M = \mathbb{S}^2 \times \mathbb{R}$ and $g(t))$ is the round cylindrical flow.*

proof: Apply the Toponogov splitting theorem ([BBI]10.5.1). If there is a line at some time t, there is a splitting $(M, g(t)) = (N^2(t) \times \mathbb{R})$ and the result follows from the classification of 3-dim. κ-solutions.$\square$

We give some consequences of the compactness theorem 1.2. Roughly the ratio of the scalar curvature at two points x, y of any κ-solution is controlled by the normalized distance $R(x,t)d^2_{g(t)}(x,y)$. Note that this expression is invariant by space dilation.

lemma 1.14. *There exists $\alpha : [0, +\infty[\to [1, +\infty[$ depending only on κ such that for any κ-solution $(M, g(.))$, for each x, y in M,*

$$\alpha^{-1}\left(R(x,t)d^2_{g(t)}(x,y)\right) \leq \frac{R(y,t)}{R(x,t)} \leq \alpha\left(R(x,t)d^2_{g(t)}(x,y)\right)$$

Proof: One can define α on each $[n, n+1[$, $n \in \mathbb{N}$. Suppose that's not true for some integer n. There is a sequence $(M_k, g_k(.))$ of κ-solutions, times t_k and points x_k, y_k in M_k such that $n \leq R(x_k, t_k)d^2_{g(t_k)}(x,y)) < n+1$ and $\frac{R(y_k,t_k)}{R(x_k,t_k)} \to 0$ or $(\frac{R(y_k,t_k)}{R(x_k,t_k)} \to +\infty)$. Normalize $g_k(.)$ in $\tilde{g_k}(t) = R(x_k, t_k)g_k(t_k + \frac{t}{R(x_k,t_k)})$. One obtains a sequence of pointed κ-solutions $(M_k, \tilde{g_k}(.), x_k)$ such that $\tilde{R}(x_k, 0) = 1$ and $\tilde{d}^2(x_k, y_k) < n+1$. By the compacity theorem, one can extract a convergent subsequence to a κ-solution $(M_\infty, g_\infty(.), x_\infty)$. Let $y_\infty \in M_\infty$ be the limit of y_k. Then $R_\infty(y_\infty, 0) = \lim \tilde{R}(y_k, 0) \in \{0, \infty\}$ and we have a contradiction. $\square$

One can give another formulation (see [KL]36.1.5)

lemma 1.15. *There exists $\beta : [0, +\infty[\to [0, +\infty[$, continuous, depending only on κ such that $\lim_{s\to+\infty} \beta(s) = +\infty$, and for every κ-solution $(M, g(.))$ and $x, y \in M$, we have $R(y)d^2(x,y) \geq \beta\left(R(x)d^2(x,y)\right)$.*

Proof: exercice.

remark 1.16. *in [P02] and [KL], these results are established* before *the compactness theorem. Here we use the compactness theorem as a black box. We have not the time for a proof.*

The pattern to use the compactness theorem is the following. You want to show that some points in κ-solutions have a nice geometry. Suppose they have not. Consider a sequence of bad points. Take a limit. Show that the limit contains a line. Thus the limit is the round cylindrical flow and the geometry is controlled. So it is just before the limit. Let ε_0 be a fix small constant, say $\varepsilon_0 = \frac{1}{10000}$.

lemma 1.17 ([KL]42.2). *For all $\kappa > 0$, for $0 < \varepsilon < \varepsilon_0$, there exists $\alpha = \alpha(\varepsilon, \kappa)$ with the followings property. Suppose $(M, g(.))$ is any κ-solution, $x, y, z \in M$ and at time t we have $x, y \in M_\varepsilon$ and $R(x)d^2(x, y) \geq \alpha$. Then at time t either $R(x)d^2(x, z) < \alpha$ or $R(y)d^2(y, z) < \alpha$ or $(R(z)d^2(z, \overline{xy}) < \alpha$ and $z \notin M_\varepsilon)$.*

Proof: Suppose not for some κ, ε. There exists a sequence of κ-solutions $(M_k, g_k(.))$, $t_k \in]-\infty, 0]$, $x_k, y_k, z_k \in M$, $x_k, y_k \in M_\varepsilon$ such that, with quantities computed at time t_k, $R(x_k)d^2(x_k, y_k) \to +\infty$ and

$$R(x_k)d^2(x_k, z_k) \to +\infty, R(y_k)d^2(y_k, z_k) \to +\infty \; and (R(z_k)d^2(z_k, \overline{x_k y_k}) \to +\infty \; or \; z_k \in M_\varepsilon).$$

Consider first the case where $R(z_k)d^2(z_k, \overline{x_k y_k}) \to \infty$ (up to a subsequence). We define $z'_k \in \overline{x_k y_k}$ as a point closest from z_k. We want to prove that $\overline{x_k y_k}$ converge to a line in the limit space of the (renormalized) sequence $(M_k, g_k(.), z'_k)$.

Claim: $R(z'_k)d^2(z'_k, x_k) \to +\infty$.

If not, suppose that $R(z'_k)d^2(x_k, z'_k) \leq c$ for a subsequence. Normalize (i.e. shift time + parabolic rescaling) $g_k(.)$ such that $R(z'_k, 0) = 1$. Here we use the same notation for the normalized metric. Thus we have $d^2(x_k, z'_k) \leq c$. On the other hand, as the ratio $\frac{R(x_k)}{R(z'_k)}$ is controlled, $d^2(x_k, y_k) \to +\infty$, $d^2(z'_k, y_k) \to +\infty$ and $d^2(z'_k, z_k) \to +\infty$. Extract a subsequence such that $(M_k, g_k(.), z'_k)$ converge to a κ-solution $(M_\infty, g_\infty(.), z'_\infty)$. Thus the segments $\overline{x_k y_k}$ and $\overline{z'_k, z_k}$ converge to rays $\overline{x_\infty \xi}$ and $\overline{z'_\infty \eta}$, where $z'_\infty \in \overline{x_\infty \xi}$. Note that $angle_{z'_\infty}(\xi, \eta) = \lim angle'_{z'_k}(y_k, z_k) \geq \frac{\pi}{2}$ where $angle'$ is the comparison angle.
Now we say that there exists $r_0 \geq 0$ such that every $u \in \overline{z'_\infty \xi}$ with $d(z'_\infty, u) \geq r_0$ is the center of an ε-neck. If not, consider a sequence $u_k \in \overline{z'_\infty \xi}$ such that $d(z'_\infty, u_k) \to \infty$. Thus $R(z'_\infty)d^2(z'_\infty, u_k) \to \infty$. By lemma 1.15, $R(u_k)d^2(z'_\infty, u_k) \to +\infty$ also. Consider a sequence of normalized κ-solution $(M_\infty, g_{\infty,k}(.), u_k)$ such that $R(u_k, 0) = 1$. Thus there is a convergent subsequence and the ray $\overline{z'_\infty \xi}$ converge to a line in the limit. Thus the limit is the round cylindrical flow by 1.13 and u_k is the center of an ε-neck for large k.
Let $u_0 \in \overline{z'_\infty \xi}$ such that every point u in $\overline{u_0 \xi}$ is the center of an ε-neck. One can

take u_0 far enough such that z'_∞ is not in the ε-neck centered at u_0. Indeed, as ε_0 is small, the length of this ε-neck is approximatively

$$\frac{2}{\varepsilon\sqrt{R(u_0)}} = \frac{2}{\varepsilon\sqrt{R(u_0)}d(z'_\infty, u_0)}d(z'_\infty, u_0) \leq \frac{d(z'_\infty, u_0)}{10}$$

if $\sqrt{R(u_0)}d(z'_\infty, u_0)$ is sufficiently large. Clearly, z'_∞ is in none of the ε-neck centered on$\overline{u_0\xi}$. The point u_0 is included in an embedded 2-sphere S_0, image of a sphere $\mathbb{S}^2 \times \{*\}$ by the ε-approximation with the standard neck. Now every curve from $u \in \overline{u_0\xi}$ to z'_∞ must exit from all ε-neck centered on $\overline{u_0\xi}$ on the left side - the side of u_0 which is closer to z'_∞- and thus must intersect S_0. That means that S_0 separates M_∞. Moreover M_∞ has at least two ends $\overline{z'_\infty\xi}$ and $\overline{z'_\infty\eta}$. Thus $(M_\infty, g_\infty(0))$ has a line. Indeed, one can consider a sequence of geodesic segments with extremities in each end and extract a convergent subsequence with the help of the intersection with S_0. Thus $(M_\infty, g_\infty(.), z'_\infty)$ is the round cylindrical flow. Thus x_k is the center of an ε-neck for large k, contradicting the hypothesis. That proves the claim. The same argument shows that $R(z'_k)d^2(z'_k, y_k) \to +\infty$.

The normalized sequence $(M_k, g_k(.), z'_k)$ converges to $(M_\infty, g_\infty(.), z'_\infty)$ and $\overline{x_k y_k}$ converge to a line in $(M_\infty, g_\infty(0))$. Thus the limit is the round cylindrical flow. The segment $\overline{z'_k z_k}$ is orthogonal to $\overline{x_k y_k}$. Thus its limit is othogonal to the line, hence is a segment $\overline{z'_\infty z_\infty}$ of bounded length. Thus $R(z_k)d^2(z_k, z'_k)$ remains bounded, proving the first case.

Now it is clear that there is α such that $z_k \notin M_\varepsilon$. If not, the same construction as above produces a limit z_∞ is in a round cylindrical flow, thus $z_k \notin M_\varepsilon$ for large k.

Proof of proposition 1.12

Let $\kappa, \varepsilon > 0$ and $(M, g(.))$ a κ-solution.
Case 1 $M_\varepsilon = \emptyset$, i.e. every point is center of an ε-neck. Fix some $x_0 \in M$, an ε-neck $U_0 \sim \mathbb{S}^2 \times [\frac{-1}{\sqrt{R(x_0)\varepsilon}}, \frac{1}{\sqrt{R(x_0)\varepsilon}}]$ and let S be the image of $\mathbb{S}^2 \times \{0\}$. One shows that if S separates M, $(M, g(.))$ is the round cylindrical flow. The other case leads to a contradiction.
Suppose that S separates. Choose point x_1 in the left side of ∂U_0. There is an ε-neck U_1 centered at x_1, thus one can choose x_2 in the left side of ∂U_1 (the side not in U_0). Repeating the argument, one define a sequence (x_k, U_k) on the left of U_0 and a sequence (y_k, V_k) on the right. Every segment $\overline{x_k y_k}$

cross all $U_0, \dots, U_{k-1}, V_1 \dots V_{k-1}$. Now, the length of each neck U_l is roughly $\frac{2}{\sqrt{R(x_l)\varepsilon}}$. Either $R(x_l) \le c$ for all integer l and then $d(x_k, x_0) \ge \frac{k}{2\sqrt{c\varepsilon}} \to \infty$. Either $R(x_l) \to \infty$ for a subsequence and then $R(x_l)d^2(x_l, x_0) \to \infty$ thus $R(x_0)d^2(x_0, x_l) \to \infty$. Using the same argument on the right, one conclude that $\ell(\overline{x_k y_k}) \to \infty$. As all segments $\overline{x_k y_k}$ intersects U_0, there exists a convergent subsequence and the limit is the line. Thus $(M, g(.))$ is the round cylindrical flow as in A.

Suppose that S does not separate. Let $\tilde{M}$ the universal cover of M and $\tilde{S}$ a lift of S. We claim that $\tilde{S}$ separates. Using a segment between sides of ∂U_0, one can take a loop γ in M intersecting S transversally in one point. Thus γ is homotopically non trivial. There is a lift $\tilde{\gamma}$ of γ intersecting $\tilde{S}$ transversally in one point, with extremities $x_1 \neq x_2$. If $\tilde{S}$ does not separate, there is a curve disjoint from $\tilde{S}$ between x_1, x_2. Thus there is a loop in $\tilde{M}$ intersecting $\tilde{S}$ transversally in one point. This is not possible, thus $\tilde{M}$ separates. By the previous argument, $(\tilde{M}, \tilde{g(.)})$ with the universal flow (which is a κ-solution) is the round cylindrical flow. Thus $(M, g(t))$ is a quotient of $\mathbb{S}^2 \times \mathbb{R}$ by a group of isometries, which contains translations, as S must separate. Thus $(M, g(.))$ is covered by $\mathbb{S}^2 \times \mathbb{S}^1$ with the round cylindrical flow, but this is not a κ-solution.

Case 2 $M_\varepsilon \neq \emptyset$ and there exists $x, y \in M_\varepsilon$ sucht that $R(x)d^2(x,y) \ge \alpha$. By the previous lemma, for $z \in M$ either we have $z \in B(x, \alpha R(x)^{-1/2}) \cup B(y, \alpha R(y)^{-1/2})$ either $R(z)d^2(z, \overline{xy}) < \alpha$ and $z \notin M_\varepsilon$. Thus we have C.

Case 3 $M_\varepsilon \neq \emptyset$ and for any $x, y \in M_\varepsilon$, we have $R(x)d^2(x,y) < \alpha$. If M is non compact, we have B.

If we suppose that M compact, we want to have D. We argue by contradiction. Fix a point $x \in M_\varepsilon$. Let z the point of M such that $R(z)d^2(x,z)$ is maximal and suppose that $R(z)d^2(z,x) \ge \alpha$. Thus $z \notin M_\varepsilon$, that is z is center of an ε-neck. Consider the middle sphere S of the neck. Either S separates, either S does not.

If S separates, M_ε is on one side. Indeed, if there were points on each side, any geodesic joigning opposite points should intersect S, thus would have length $\ge \frac{2\alpha}{\sqrt{R(z)}}$, which is not possible in our case. Now if M_ε is on one side, z is not maximal.

If S does not separate, then all lift $\tilde{S} \subset \tilde{M}$ separates, as in case 1. There is a non trivial loop $\gamma \in M$, which intersects S transversally in one point. As a lift $\tilde{\gamma}$ hits an infinite number of $\tilde{S}$ on both sides, it is easy to construct a line in $(\tilde{M}, \tilde{g}(t))$. Thus $(M, g(.))$ is covered by the round cylindrical flow and we get a contradiction as in case 1. □

end of the proof of 1.9 Note that κ-solutions on $\mathbb{S}^2 \times_{\mathbb{Z}_2} \mathbb{R}$ and $\mathbb{B}^3$ are described by case B of proposition 1.12. Recall that we suppose the κ-solution compact and its asymptotic soliton non compact. As case D imply compactness of the asymptotic soliton, the κ-solution satisfies C. We know that $M_{-\infty} = \mathbb{R} \times \mathbb{S}^2$ or $\mathbb{R} \times_{\mathbb{Z}_2} \mathbb{S}^2$. Suppose that $M_{-\infty} = \mathbb{R} \times \mathbb{S}^2$. Choose a sequence (y_k, t_k) where $t_k \to -\infty$ and $y_k \in M_\varepsilon(t_k)$. Shift in time and parabolic rescale g_k in $\tilde{g}_k$ such that $\tilde{R}(y_k, 0) = 1$. Let $(M_\infty, g_\infty(.), y_\infty)$ be a limit of a subsequence of $(M, \tilde{g}_k, y_k)$ as $k \to \infty$. We claim that the limit M_∞ is non compact. If not, the normalized distance $R.d^2$ is uniformly bounded along the subsequence $(M, \tilde{g}_k, y_k)$. The limit of $(M, g_k(.), x_k)$ is a κ-solution thus $-t_k R(y_k, t_k$ remains controlled in $(0, \infty)$. Then $M_{-\infty}$ is homothetic to M_∞ and compact- a contradiction. Now M_∞ is not the round cylinder, otherwise y_k would be center of an ε-neck for large k. Thus $M_\infty = B^3$ or $\mathbb{R}P^3 - \overline{B^3}$ and $g_\infty(0)$ is of type B. Thus for large $r > 0$, $B(y_\infty, r, 0) = B^3$ or $\mathbb{R}P^3 - \overline{B^3}$ and the boundary $\partial B(y_\infty, r, 0)$ is contained in an ε-neck. As $(M, g(.))$ is of type C and arguing in the same way for the two "components" of $M_\varepsilon(t)$, one deduce that for large k, $M_\varepsilon(t_k)$ lies in the union of two disjoints balls, each diffeomorphic to B^3 or $\mathbb{R}P^3 - \overline{B^3}$, that are joined by a long tube. Thus M is diffeomorphic to the gluing of two such balls to a large piece of the asymptotic soliton. As M has finite fundamental group, only one of the balls can be $\mathbb{R}P^3 - \overline{B^3}$, so $M = S^3$ or $M = \mathbb{R}P^3$. In the case where $M_{-\infty} = \mathbb{R} \times_{\mathbb{Z}_2} \mathbb{R}$, the same arguments shows that $M = \mathbb{R}P^3$. □

2 Lecture 2: Canonical neighborhoods theorem

Canonical neighborhoods

definition 2.1. *the parabolic neighborhood $P(x, t, r, \Delta t)$ is the set of (x', t') with $x' \in B(x, t, r)$ and $t' \in [t, t + \Delta t]$ or $t' \in [t + \Delta t, t]$, according to the sign of Δt.*

definition 2.2. *A parabolic neighborhood $P(x, t, \frac{r}{\varepsilon}, -r^2)$ is called a **strong** ε-**neck** if, after shifting in time and parabolic scaling by $\frac{1}{r^2}$, it is ε-close to the parabolic neighborhood $P(x, 0, \frac{1}{\varepsilon}, -1)$ of the round cylindrical flow $\mathbb{S}^2 \times \mathbb{R}$.*

definition 2.3. *A metric with bounded curvature on $\mathbb{B}^3$ or $\mathbb{RP}^3 - \overline{\mathbb{B}^3}$ such that each point outside some compact subset is center of an ε-neck is called an ε-**cap**.*

According to Perelman, an ""important conclusion " of the classification is the following. There exists some $\kappa_0 > 0$ such that each κ-solution is a κ_0-solution or a quotient of the round sphere. Moreover, each κ-solution has local canonical geometry.

theorem 2.4 ([P03]II.1.5 [KL]53.). *There exists some $\kappa_0 > 0$ such that any κ-solution is a κ_0-solution or a quotient of the round sphere. This implies at each point of every κ-solution*

$$|\nabla R| \leq \eta R^{3/2},\ |\frac{\partial R}{\partial t}| \leq \eta R^2 \tag{1}$$

for some universal η. Moreover there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, one can find $C_1 = C_1(\varepsilon) > 0$, $C_2 = C_2(\varepsilon) > 0$ such that for each point (x,t) in any κ-solution, there is a radius $r \in [\frac{1}{C_1 R(x,t)^{1/2}}, \frac{C_1}{R(x,t)^{1/2}}]$ and a neighborhood B, $B(x,t,r) \subset B \subset B(x,t,2r)$ which falls into one of the four categories:

a. *B is the maximal time slice of a strong ε-neck*

b. *B is an ε-cap*

c. *B is a closed manifold diffeomorphic to S^3 or $\mathbb{R}P^3$.*

d. *B is a closed manifold of constant positive sectional curvature.*

furthermore, the scalar curvature in B at time t is in $[\frac{R(x,t)}{C_2}, C_2 R(x,t)]$, its volume in cases 1), 2), 3) is greater than $C_2 R(x,t)$, and in case 3), the sectional curvature in B at time t is greater than $\frac{R(x,t)}{C_2}$.

proof: Non compact asymptotic solitons, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times_{\mathbb{Z}_2} \times \mathbb{R}$, are κ_0'-solutions for a universal $\kappa_0' > 0$. Using the monotonicity of reduced volume, one find some $\kappa_0 >$ for κ-solutions with non compact asymptotic soliton, i.e all κ-solutions except spherical ones (see [CZ05]).

remark 2.5. *In cases c), d), B is equal to M.*

definition 2.6. *Let $\Phi : (-\infty, \infty) \to (0, \infty)$ an increasing function such that $\frac{\Phi(s)}{s} \to 0$ as $s \to \infty$. One says that a solution of the Ricci flow has Φ-almost non negative curvature if $Rm(x,t) \geq -\Phi(R(x,t))$ for any (x,t).*

According to the Hamilton-Ivey pinching theorem ([[H95],[1]],see [KL][appendix C]), any 3-dimensional solution of the Ricci flow has Φ-almost non negative curvature for some function Φ. More precisely, there is a universal function Φ such that any solution with $R(x,0) \geq -1$, $Rm(x,0) \geq -\Phi(R(x,0))$, has Φ-almost non negative curvature. By scaling, any solution satisfies the initial pinching conditions. Observe that the scalar curvature bounds all curvatures

$$R + 2\Phi(R) \geq Rm \geq -\Phi(R).$$

The main result of [P02] is

theorem 2.7 ([P02]I.12.1, (canonical neighborhoods theorem)). *Given $\varepsilon > 0$,$\kappa > 0$ and a function Φ as above, there exists $r_0 = r_0(\varepsilon, \kappa, \Phi) > 0$ with the following property. If $g(t)$, $0 \leq t \leq T$, is a solution to the Ricci flow on a closed 3-dimensional manifold M, which has Φ-almost non negative curvature and is κ-noncollapsed at scales $< r_0$, then for any point (x_0, t_0) with $t_0 \geq 1$ and $Q_0 = R(x_0, t_0) \geq \frac{1}{r_0^2}$, the parabolic neighborhood $P(x_0, t_0, \frac{1}{\sqrt{\varepsilon Q_0}}, -\frac{1}{\varepsilon Q_0})$ is ε-close after parabolic scaling by factor Q_0 to the corresponding subset of a κ-solution.*

Roughly speaking, a point (x_0, t_0) with high scalar curvature has a neighborhood with almost canonical geometry. The size of the neighborhood in space-time is controlled by the scalar curvature at (x_0, t_0).

Proof of the theorem By contradiction. Suppose we have sequences $r_k \to 0$, (M_k, g_k) solutions to the Ricci flow on $[0, T_k]$, $x_k \in M_k$, $1 \leq t_k \leq T_k$ such that $Q_k := R(x_k, t_k) \geq \frac{1}{r_k^2}$ but the scalings of $P(x_k, t_k, \frac{1}{\sqrt{\varepsilon Q_k}}, -\frac{1}{\varepsilon Q_k})$ are not ε-close to the corresponding subset of a κ-solution. The idea is to show that scalings of (M_k, g_k, x_k) by factor Q_k converge to a κ-solution. In some sense, the proof will be an induction on the scale of curvature. There are four steps. Choose bad points with almost maximal curvature among bad points. Show that the rescaled metric have bounded curvature on balls. Show that the limit has bounded non negative curvature and the convergence extends to a backward time interval. Extend the interval to $(-\infty, 0]$.

step 1: choose better bad points We look for a bad point with almost maximal curvature among previous bad points in space-time. Choose $H_k \to \infty$ such that $\frac{H_k}{Q_k} \leq \frac{1}{10}$. Fix the integer k. Note that by compactness, the scalar curvature is bounded on $M_k \times [0, t_k]$. We call $(x,t) \in M_k \times [\frac{1}{2}, t_k]$ a bad point if $Q := R(x,t) \geq \frac{1}{r_k^2}$ and $P(x, t, \frac{1}{\sqrt{\varepsilon Q}}, -\frac{1}{\varepsilon Q})$ is not ε-close to the corresponding subset of a κ-solution. All others points in $M_k \times [0, t_k]$ are good points.

Claim: there exists a bad point (x'_k, t'_k) such if $(x,t) \in M_k \times [t'_k - \frac{H_k}{Q'_k}, t'_k]$, where $Q'_k := R(x'_k, t'_k)$, and $R(x,t) \geq 2Q'_k$, (x,t) is a good point.

Possibly, the assertion on (x,t) is empty. To prove the claim, we begin with the bad point (x_k,t_k). If the claim hold with (x_k,t_k), we take $(x'_k,t'_k)=(x_k,t_k)$. If not, there is a bad point in $M_k\times[t_k-\frac{H_k}{Q_k},t_k]$ with scalar curvature $\geq 2Q_k$. Replace (x_k,t_k) by this point and repeat the procedure until it stop.□

Note (x_k,t_k) the bad point we have choose. Consider $(M_k,\bar{g}_k(.),(x_k,0))$ the rescaled and shifted sequence, i.e. $\bar{g}_k(t)=Q_kg(t_k+\frac{t}{Q_k})$ defined on $[-H_k,0]\to(-\infty,0]$ as $k\to\infty$. Note $\bar{R}_k$ the scalar curvature of the rescaled metric.

Step 2 *Claim: for any $\rho>0$, the scalar curvature $\bar{R}_k$ is uniformly bounded on balls $B(x_k,\rho)\subset(M_k,\bar{g}_k(0))$.*

By definition, $\bar{R}(x_k,0)=1$ and any $(y,t)\in M_k\times[-H_k,0]$ with $\bar{R}_k(y,t)\geq 2$ has a canonical neighborhood. In particular, the estimates (1) applies with the universal constant 2η at any such (y,t). For any $(x,t)\in M_k\times[-H_k,0]$, set $Q=\bar{R}_k(x,t)+2$. Then

$$\bar{R}_k(x',t')\leq 4Q,\qquad \forall(x',t')\in P(x,t,\frac{1}{2\eta Q^{1/2}},-\frac{1}{8\eta Q}).\tag{2}$$

Indeed, consider a point (x',t') in the parabolic neighborhood such that $\bar{R}_k(x',t')\geq 2$. Consider a path static in space between (x',t') and (x',t) and $g(t)$-geodesic between (x',t) and (x,t). Integrate the estimates along the path until the first point with scalar curvature 2 or (x,t). This gives the upper bound for $\bar{R}_k(x',t')$.

Now consider

$$\rho_0=sup\{\rho>0,\bar{R}_k(.,0)\text{ is uniformly bounded on }B(x_k,\rho)\subset(M_k,\bar{g}_k(0))\}.$$

By the argument above, $\rho_0>\frac{1}{4\eta}>0$. We want $\rho_0=\infty$. Suppose this is not true. Using the κ-non collapsing assumption and uniform curvature bounds on balls $B(x_k,\rho)$ due to the Φ-pinching, we obtain $C^{1,\alpha}$ pointed Gromov-Hausdorff convergence of $(B(x_k,\rho_0),\bar{g}_k(0),x_k)$ to a non complete manifold (Z,g_∞,x_∞) of non negative curvature. In fact, the bounds above implies convergence of pieces of Ricci flow on $B(x_k,\rho)\times[-\tau(\rho),0]$ and thus smoothness of $(B(x_\infty,\rho),g_\infty)$. By hypothesis, there is a (sub)sequence $y_k\in B(x_k,\rho_0)$ such that $d(x_k,y_k)\to\rho_0$ and $\bar{R}_k(y_k,0)\to\infty)$. Let $z_k\in[x_ky_k]$ the point closest from y_k such that $\bar{R}_k(y_k,0)=2$. Thus $[z_ky_k]$ is covered by canonical neighborhoods. The rays $[x_ky_k]$ converge to a ray $[x_\infty y_\infty)$ in the metric completion $\bar{Z}$ of Z and z_k converge to $z_\infty\in[x_\infty y_\infty)$. Then one can show that the metric is almost cylindric around

$[z_\infty y_\infty)$ and that the tangent cone $C_{y_\infty}Z$ based at y_∞ is a non flat metric cone. On the other hand, around a point $z \in C_{y_\infty}Z$ such that $d(z, y_\infty) = 1$, one can prove existence of a flow on a backward intervall. A crucial fact is that $Rd^2(.y_\infty$ remains bounded in $(0, \infty)$. But then we get a contradiction by a local version of the Strong Maximum Principle of Hamilton. Indeed, writing $\frac{\partial}{\partial t}Rm = \Delta Rm + Q(Rm) > 0$ and considering a plane of zero curvature, we get negative curvature backward - a contradiction. Thus $\rho_0 = \infty$. □

step 3 By the arguments above, there exists a subsequence of $(M_k, \bar{g}_k(0), (x_k, 0))$ which converge in the pointed Gromov-Hausdorff topology to a complete smooth manifold $(M_\infty, g_\infty, x_\infty)$ of non negative curvature.

Claim: g_∞ has bounded curvature and the convergence extends backward in time.

A proof by contradiction. Suppose there is sequence $y_j \in M_\infty$ such that $R_\infty(y_j) \to \infty$. Note that $d(x_\infty, y_j) \to \infty$. Any y_j is limit of a sequence of points in $(M_k, \bar{g}_k(0))$ with curvature $\approx R_\infty(y_j) > 2$, thus has a canonical neighborthood, which must be an ε-neck. Moreover, the radius of these necks is going to zero as $d(x_\infty, y_j) \to \infty$. But any complete manifold of non negative curvature has an exhaustion by compact convex sets C_s, $s > 0$, where $C_{s_1} \subset C_{s_2}$ if $s_1 \le s_2$. Moreover, there is a one lipschitz map from C_{s_2} onto C_{s_1} (see [S77],[**?**] and [G97]). One get a contradiction with the decreasing radius of the neck. Now using the bounded curvature of the limit and the estimates (2), one that for some $\tau < 0$, curvature on $B(x_k, \rho) \times [\tau, 0]$ is bounded by a constant independant of ρ, for any large k. Thus we obtain pointed convergence of $\bar{g}_k(t)$ on $M_k \times]\tau, 0]$ to a Ricci flow on $M_\infty \times]\tau, 0]$. □

Suppose τ is minimal for this property.

step 4 *Claim:* $\tau = -\infty$. A proof by contradiction. We suppose that the maximum of the scalar curvature of $(M_\infty, g_\infty(t)$ converge to ∞ as $t \to \tau$. From the trace Harnack Inequality [S05] (5.5), we get for any $\tau < t < 0$, $\frac{\partial}{\partial t}R_\infty(x, t) \ge -\frac{R_\infty(x,t)}{t-\tau}$. Thus, integrating from t to 0,

$$R_\infty(x, t) \le R_\infty(x, 0)\frac{-\tau}{t - \tau} \le Q\frac{-\tau}{t - \tau}$$

where Q is the maximum of the scalar curvature on $(M_\infty, g_\infty(0))$. The same bound holds for the Ricci curvature. By a standard argument, for any $g(t)$-geodesic γ,

$$\int_\gamma Ric_{g(t)}(\dot{\gamma}, \dot{\gamma})ds \le const.\sqrt{Q\frac{-\tau}{t - \tau}}$$

where the constant does not depend of γ,t. By integration, one deduces there exists $C > 0$ such that

$$|d_{g_\infty(t)}(x,y) - d_{g_\infty(0)}(x,y)| \leq C.$$

Recall that the minimum of the scalar curvature is increasing, thus $R_\infty(.,t) = 1$ for some point. If M_∞ compact, $g_\infty(0)$ has bounded diameter. As the diameter of $g_\infty(t))$ remains bounded, the arguments of step 2 apply using this point as a base point. Now suppose M_∞ non compact. Sketch of the proof : argue by contradiction. find for t close to τ a necklike part U with small radius, separating two points x, y far from U. Here small means smaller thant the lower bound on the injectivity radius at time 0. As the curvature is positive, distance decreases thus the radius of U at time 0 is smaller than the injectivity radius and cannot separate x and y which remains far by 2 - a contradiction. $\square$.

corollary 2.8. *Given a small $\varepsilon > 0$,$\kappa > 0$ and Φ, there exists $r = r(\varepsilon, \kappa, \Phi) > 0$ with the following property. If $g(t)$, $0 \leq t \leq T$, is a solution to the Ricci flow on a closed 3-dimensional manifold M, which has Φ-almost non negative curvature and is κ-noncollapsed at scales $< r$, then for any point (x,t) with $t \geq 1$ and $Q = R(x,t) \geq \frac{1}{r^2}$, has a open neighborhood B as in 2.4.*

proof: Fix a small $\varepsilon'(\varepsilon/2) > 0$ such that $\frac{1}{\varepsilon'} \leq 2C_1(\varepsilon/2)$ and $\varepsilon' < C_2(\varepsilon/2)^{-1}$, where $C_{1,2}(\varepsilon/2)$ are constants from 2.4. Define $r := r_0(\varepsilon', \kappa, \varphi)$ given by 2.7. Thus if $Q = R(x,t) \geq r^{-2}$, $t \geq 1$, then $P(x, t, \frac{1}{\sqrt{\varepsilon' Q}}, -\frac{1}{\varepsilon' Q})$ is ε'-close after parabolic scaling by factor Q to a parabolic neighborhood $P(\bar{x}, 0, \frac{1}{\sqrt{\varepsilon'}}, -\frac{1}{\varepsilon'})$ in a κ-solution. Here $R(\bar{x}, 0) = 1$. Apply the theorem 2.4 at $(\bar{x}, 0)$, with data $\varepsilon/2$. There exists $s \in [\frac{1}{C_1}, C_1]$ and B, $B(\bar{x}, 0, s) \subset B \subset B(\bar{x}, 0, 2s)$ with $\varepsilon/2$ almost canonical geometry. Pullback $B \subset B(\bar{x}, 0, 2C_1) \subset P(\bar{x}, 0, \frac{1}{\sqrt{\varepsilon'}})$ by the previous ε'-approximation into $P(x, t, \frac{1}{\sqrt{\varepsilon' Q}}, -\frac{1}{\varepsilon' Q})$. Canonical geometry holds with respect to ε for some constant $C'_{1,2}(\varepsilon)$. In particular estimates 2 holds (change η) and positivity of curvature is preserved in c) d). Moreover, neighborhood c) or d) cover M. $\square$.

3 Lecture 3: The flow at a singular time and the surgery procedure

In this section we consider $\mathcal{M} = (M \times [0, T), g(t))$ a smooth solution of the Ricci flow, where M is connected, such that the curvatures of $g(t)$ are not bounded as $t \to T < \infty$. We suppose that the flow satisfies the following assumptions.

Assumptions

1) there exists $\kappa, \rho_0 > 0$ such that $g(t)$ is κ-noncollapsed at scales below ρ_0.

2) $g(t)$ has Φ-almost nonnegative curvature for some function Φ. (say Φ-pinching assumption).

3) For a small $\varepsilon > 0$, there exists $r > 0$ such that if $R(x,t) \geq r^{-2}$, (x,t) has a canonical neighborhood. One says that the solution satisfies the (r, ε)-neighborhood assumption.

*R*emark: Given $\mathcal{M}$, theorem [P02]I.4.1 provides $\kappa, \rho_0 > 0$ in 1). The Hamilton-Ivey theorem provides a universal Φ_0 for normalized initial $g(0)$. By scaling, this gives a function φ for $\mathcal{M}$. Given $\varepsilon, \kappa, \Phi$, corollary 2.8 provides r such that any (x,t) with $R(x,t) \geq r^{-2}$, $t \geq 1$, has a canonical neighborhood. This applies to a scaling of $\mathcal{M}$ such that $T > 1$ and curvature $\leq r^{-2}$ on $[0,1]$. Scale back to $\mathcal{M}$.

Perelman describes $g(t)$, $t \to T$, as follows. Recall that the minimum of $R(.,t)$ is nondecreasing.

definition 3.1. *Let*

$$\Omega = \{x \in M, R(x,.) < C(x)\},$$

the set of points where $R(x,t)$ remains bounded.

1^{st} **case:** $\Omega = \emptyset$**.**

In this case, $R(x,t) \to \infty$ at each point. Precisely,

lemma 3.2. *for any $C > 0$, there exists $t_0 \in [0,T)$ such that*

$$R(x,t) > C, \qquad \forall x \in M, \forall t \in [t_0, T).$$

proof: suppose $C > r^{-2}$. At points where $R \geq r^{-2}$, the (r, ε)-neighborhood assumption gives the estimates $|\frac{\partial}{\partial t} R| < \eta R^2$. Thus if $R(x,t_1) \leq C$ and $R(x,t_2) \geq 2C$, integration gives $|t_2 - t_1| \geq \frac{1}{2\eta C}$. The curvature needs a definite time to double. By hypothesis, $R(x,t_i) \to \infty$ for some subsequence t_i thus $R(x,t) \geq C$ for $t \geq T - \frac{1}{2\eta C} := t_0$. □

Take t_0 such that $R(x,t_0) \geq r^{-2}$ for all $x \in M$. Thus each point has a canonical neighborhood which is an ε-neck, an ε-neck or a closed manifold of positive curvature. If the latter appears at least one time, by [**?**] $g(t)$ shrinks to a point as a round metric and M is diffeomophic to a finite quotient of S^3. Suppose there is only ε-necks or ε-caps. As the curvature is bounded at t_0, ε-necks and ε-caps have volume bounded below > 0. Thus one can cover M by a finite number of them. The only possibilities are

- only ε-necks $\Longrightarrow M = \mathbb{S}^2 \times \mathbb{S}^1$.
- 2 ε-caps + ε-necks $\Longrightarrow M = \mathbb{S}^3, \mathbb{RP}^3$ or $\mathbb{RP}^3 \# \mathbb{RP}^3 = \mathbb{S}^2 \times_{\mathbb{Z}_2} \mathbb{S}^1$.

where the $\mathbb{Z}_2$-action identifies $(x,\theta) \sim (-x,-\theta)$. Thus

$$\Omega = \emptyset \Longrightarrow M \in \{\mathbb{S}^3/\Gamma, \mathbb{RP}^3 \# \mathbb{RP}^3, \mathbb{S}^2 \times \mathbb{S}^1\}.$$

2^{nd} **case:** $\Omega \neq \emptyset$**.**

lemma 3.3. *Ω is open in $(M, g(0))$.*

Proof let $x \in \Omega$. By definition, there exists $C(x)$ ($\geq r^{-2}$) such that $R(x,t) \leq C(x)$, $\forall t < T$.

Claim: there exists $a > 0$ such that $R(.,t) \leq 2C(x)$ on $B(x,t,a)$ forall $t < T$.

Fix some $t < T$ and $y \in M$ such that $R(y,t) \geq 2C(x)$. Let $x_0 \in \overline{xy}$, the $g(t)$ segment, closest from y such that $R(x_0,t) = C(x)$. Integrating estimates (2) on x_0y one find $d_{g(t)}(x_0,y) \leq \frac{1}{\eta\sqrt{2C(x)}} := a$.

By the Φ-pinching assumption, $|\frac{\partial}{\partial t} g(t)| = |-2Ric_{g(t)}| \leq C.g(t)$ on $B(x,t,a)$, where $C = 2(n-1)C(x)$ is independant of t, although the ball vary with the metric.

Claim: there exists $t_0 \in [0,T)$ such that $\forall t \in [t_0,T)$, $B(x,t_0,\frac{a}{2}) \subset B(x,t,a)$.

Indeed, choose t_0 such that $e^{C(T-t_0)} < 4$. Then for any $y \in B(x,t_0,\frac{a}{2})$, $d_{g(t)}(x,y) \leq 2 d_{g(t_0)}(x,y) < a$ for any $t \in [t_0,T)$.

Thus we have $R(.,t) \leq 2C(x)$ on $B(x,t_0,\frac{a}{2})$ hence $B(x,t_0,\frac{a}{2}) \subset \Omega$. Now all curvatures are bounded on $M \times [0,t_0]$ thus $g(0)$ and $g(t_0)$ are bilipschitz equivalent. $\Box$.

lemma 3.4. *$g(t)$ extends smoothly to Ω as $t \to T$.*

Proof: let $x \in \Omega$. By previous claims and Φ-pinching assumption, there exists $t_0 < T$ and $a > 0$ such that $Rm_{g(t)}$ is uniformly bounded on $B(x,t_0,a) \subset \Omega$, forall $t \in [t_0,T)$. Hence all $g(t)$ are $(1+O(T-t_0))$-bilipschitz equivalent on $B(x,t_0,a) \times [t_0,T)$. Moreover, by Shi's estimates [S89], all covariant derivatives $D^k Rm(.,t)$ are uniformly bounded on $B(x,t_0,a) \times [t_0,T)$. Thus $g(t)$ converges on $B(x,t_0,a)$ to a smooth metric $g(T)$. On deduces convergence to $g(T)$ on Ω, uniform on compact sets. $\Box$.

Remark:

1. $R(x,T) \to \infty$ as $t \to \partial\Omega$ in $(M,g(0))$.
2. the metric $g(T)$ is locally complete but not globally à priori. According to Perelman, the diameter of a connected component of $(\Omega, g(T))$ is probably bounded.
3. As M is compact, $\Omega \neq M$ (else $g(T)$ has bounded curvature) thus Ω has no compact component (M is connected).
4. $vol(\Omega, g(T)) \leq \lim_{t\to T} vol(M, g(t) < \infty$. Indeed,
$$\frac{d}{dt} vol_{g(t)}(M) = \int -R_{g(t)} dv_{g(t)} \leq -R_{min}(0) vol_{g(t)}(M),$$
as $R_{min}(t)$ is increasing, thus $vol_{g(t)}(M) \leq vol_{g(0)}(M) e^{-R_{min}(0)t}$.
5. $g(T)$ is κ-noncollapsed at scale below ρ_0.
6. Φ-pinching holds at $g(T)$.
7. the (r',ε')-neighborhood assumption holds on $(\Omega, g(T))$ for slighter $r' < r$ and $\varepsilon' > \varepsilon$. We keep going use r and ε.

Fix a small $0 < \rho < r$ and define
$$\Omega_\rho := \left\{ x \in \Omega | R(x,T) \leq \frac{1}{\rho^2} \right\}$$

Claim: Ω_ρ is compact in $(\Omega, g(T))$. Indeed given ρ, by the estimates above there exists $t_0(\rho) \in [0, T)$,$a(t_0, \rho) > 0$ such that

$$R(y,t) \leq \frac{2}{\rho^2}, \forall (y,t) \in B(x, t_0, a) \times [t_0, T], \forall x \in \Omega_\rho.$$

Thus $g(T)$ and $g(t_0)$ are bilipschitz equivalent in a $g(t_0)$-neighborhood of Ω_ρ, and Ω_ρ is $g(t_0)$-closed in M thus compact. $d_{g(t_0)}(\Omega_\rho, M - \Omega) \geq a > 0$.

If $\Omega_\rho = \emptyset$, for t close to T any $(x,t) \in M \times \{t\}$ has a canonical neighborhood so the arguments of the 1^{st} case apply. Suppose $\Omega_\rho \neq \emptyset$. Recall that $\Omega \neq M$ has no compact component.

Structure of $\Omega - \Omega_\rho$

Let $x_0 \in \Omega - \Omega_\rho$. It has a canonical neighborhood U_0 which must be an ε-neck or an ε-cap by remark 3).

i) $x_0 \in U_0$ an ε-cap $= \mathbb{B}^3$ or $\mathbb{RP}^3$. Do the following procedure. Choose $x_1 \in \partial U_0$. If $x_1 \notin \Omega_\rho$ then $x_1 \in U_1$ an ε-neck, otherwise x_1 would be in a cap and in a compact component of Ω. Choose $x_2 \in \partial U_1 \cap U_0^c$ and iterate. Observe that the distance between two consecutive points x_i, x_{i+1} is approximatively $\frac{R(x_i,T)^{-1/2}}{\varepsilon}$ and the volume of each ε-neck is approximatively $\frac{R(x_i,T)^{-3/2}}{\varepsilon}$. As the volume of $g(T)$ is bounded,

- either there is some $x_n \in \Omega_\rho$.
- either the process does not terminate and $R(x_i, T) \to \infty$.

ii) $x_0 \in U_0$ an ε-neck. Define as above on the right (resp. left) of U_0 $x_i^+ \in \varepsilon$-neck U_i^+ (resp. $x_i^- \in \varepsilon$-neck U_i^-) as long as they not hit Ω_ρ or an ε-cap. By the compactness argument U_i^- and U_i^+ cannot close up Thus on each side,

- either there is some $x_n \in \Omega_\rho$.
- either there is some x_n in an ε-cap.
- either the process does not terminate and $R(x_i, T) \to \infty$.

and there is at most one ε-cap. Perelman introduces the following terminology.

definition 3.5. *A metric on $\mathbb{S} \times (-1, 1)$ such that each point is center of an ε-neck is called*

ε-tube *, if the curvatures says bounded, or*

ε-horn *, if the curvatures says bounded on one side, or*

double ε-horn *, if the curvatures are unbounded on each side.*

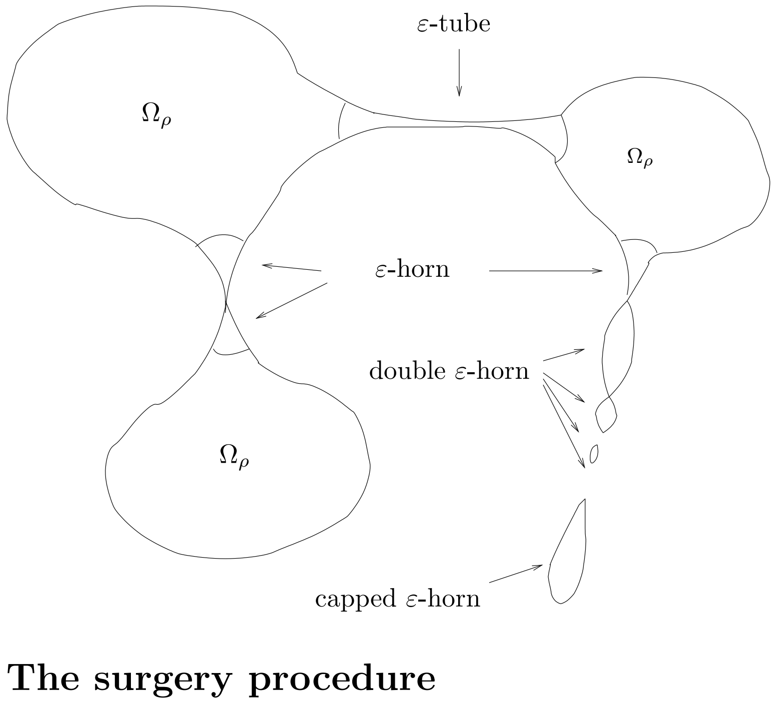
*A metric on $\mathbb{B}^3$ or $\mathbb{RP}^3 - \overline{\mathbb{B}^3}$ such that each point outside some compact subset is center of an ε-neck is called an **capped ε-horn** if the curvatures are unbounded.*

From the discussion abobe, we see that

- any component of Ω disjoint from Ω_ρ is a double ε-horn or a capped ε-horn.

- other components are ε-horn with one boundary in Ω_ρ, ε-tubes with one boundary in Ω_ρ, the other in Ω_ρ or an ε-cap. These components contains ε-necks of curvature ρ^{-2}, thus are finitely many.

The set Ω at time T



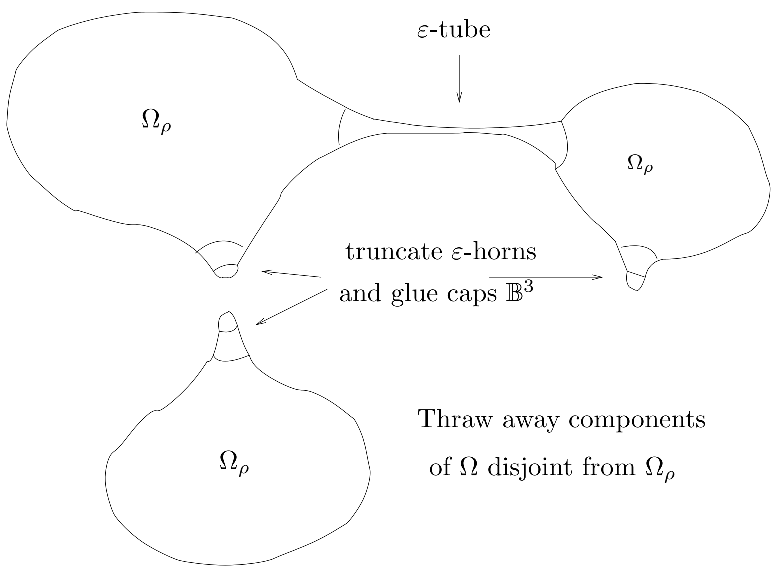
The surgery procedure

The surgery is the result of two things:

1. Thraw away all components of Ω disjoint from Ω_ρ.

2. Truncate each ε-horn along a 2-sphere of scalar curvature h^{-2} - a parameter that defined in the flow with δ-cutoff below - thraw away the component with unbounded curvature, and paste a ball $\mathbb{B}^3$ on the boundary 2- sphere.

The surgery



Denote by M_T^+ the (maybe nonconnected) manifold obtained. Let Ω^1, Ω^2,..., Ω_i the connected component of Ω disjoint from Ω_ρ. Then

$$M_T^+ = \bigcup_{j=1}^{i} \bar{\Omega^j}$$

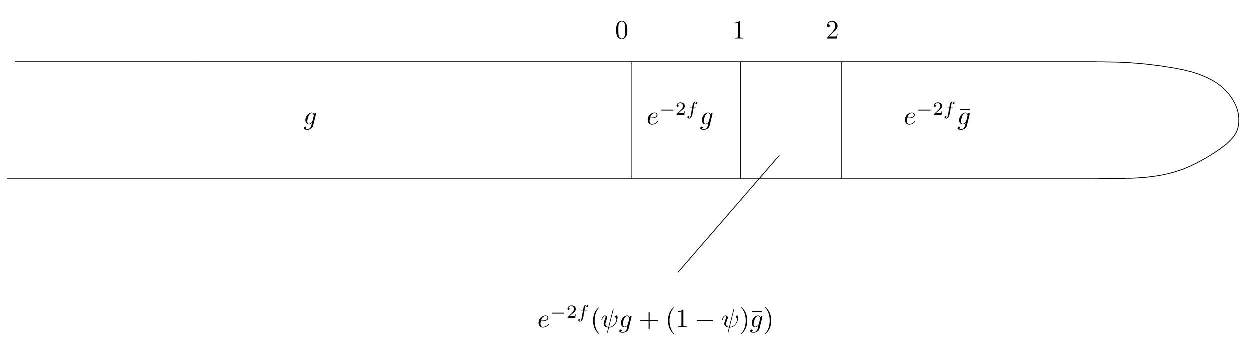
where $\bar{\Omega^i}$ is the one point compactification of Ω^i. To relate the topology of M and M_T^+ consider a time t_0 close enough to T such that each point $x \in \Omega - \Omega_\rho$ has curvature $R(x, t_0) \geq \frac{1}{\rho^2}$. Then one can cover $\Omega - \Omega_\rho$ with a finite number of ε-necks or ε-caps. Any double ε-horn is included in an ε-tube ending in Ω_ρ or in an ε-cap. Each capped ε-horn comes from an ε-cap and ε-tube ending in Ω_ρ. The ε is diffeomorphic to $\mathbb{B}^3$ of $\mathbb{RP}^3 - \bar{B}B$ thus troncate the tube is the inverse of the connected sum with $\mathbb{B}^3$ or $\mathbb{RP}^3$. Any ε-horn comes from an ε-tube with one boundary in $\Omega^j \subset \Omega_\rho$, the other in a cap or $\Omega^k \subset \Omega_\rho$. The cap is as above. If the second boundary is in $\Omega^k \neq \Omega^j$ the truncation is the inverse of the connected sum of $\bar{\Omega^j} \# \bar{\Omega^k}$. If $\Omega^k = \Omega^j$, then the truncation is the inverse of the connected sum with $\mathbb{S}^2 \times \mathbb{S}^1$. Thus M is diffeomorphic = to the connected sum of the $\bar{\Omega^j}$ with a finite number of $\mathbb{S}^3$, $\mathbb{RP}^{3^3}$ and $\mathbb{S}^2 \times \mathbb{S}^1$.

Ricci flow with δ-cutoff

The metric on the added cap is defined by interpolation of the metric of the truncated horn with a metric on a standard cap. The gluing must preserve the assumptions true at time T - the κ-noncollapsing, the Φ-pinching and the (r, ε)-neighborhood. It will be possible if the metric ε-neck in the horn is sufficiently close to the standard one.

definition 3.6. *A* ***standard cap*** *is a metric $\bar{g}$ defined on $\mathbb{R}^3$ with the following properties. It is rotationally symetric with positive curvature, spheric in a neighborhood of 0 and cylindrical like $\mathbb{S}^2 \times \mathbb{R}$ at distance ≥ 5 from 0 - the scalar curvature is one in those parts. It is defined by a $dr^2 + \psi(r)^2 d_{\mathbb{S}^2}$ with, say, $\psi(r) = \sin(r) \leq \frac{\pi}{4}$, $\psi(r) = 1$ for $r \geq 2$.*

We fix such a metric on $\mathbb{R}^3$ that will be the standard cap. For any small $0\delta < \delta_0$, we define an interpolation between any δ-neck and an almost standard cap - a standard cap slightly deformed by a conformal transformation. Consider a cylinder $\mathbb{S}^2 \times (-5, 5)$ with a metric g which is δ-close to the cylinder part of the standard cap $\bar{g}$. We define a new metric $\tilde{g}$ on $\mathbb{S}^2 \times (-5, 5)$, as on the picture.



$$\tilde{g} = \begin{cases} g & \text{if } z \leq 0 \\ e^{-2f} g & \text{if } z \in [0,1] \\ e^{-2f}(\psi g + (1-\psi)\bar{g}) & \text{if } z \in [1,2] \\ e^{-2f}\bar{g} \text{ if } z & \text{if } \geq 2 \end{cases}$$

where, if z is the radial cordonate of $\mathbb{S}^2 \times (-5, 5)$,

- $\psi(z)$ is a smooth positive function, such that $f(z) = 1$ if $z \leq 1$ and $f(z) = 0$ if $z \geq 2$.

- $f(z)$ is defined by

$$f(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ e^{-\frac{p}{z}} & \text{if } z \geq 0 \end{cases}$$

The aim of the conformal transformation is to give strictly positive curvature to $e^{-2f}g$ when $z \geq 1$ and in the same time preserving the φ-pinching assumption on $\mathbb{S}^2 \times [0,1]$. The idea is to have $f''(z) >> f'(z)$ when nonzero. This is possible if the parameter $p > 0$ is sufficiently large. The formula for the curvatures of g and $e - 2fg$ are

$$\tilde{K}_{ij} = e^{2f}K_{ij} + e^{2f}(\partial_j f \partial_j f + \partial_i f \partial_i f) - |df|^2 e^{2f} + e^{2f}(Hessf(\partial_j, \partial_j) + Hessf(\partial_i, \partial_i))$$

where $\partial_i, \partial j$ are orthonormal vectors for g. One can shows that for p large enough, the minimum of the sectional curvatures and the scalar curvature of $e{-}2f\tilde{g}$ is increasing with z. Thus the φ-pinching is preserved on $[0,1]$. Moreover, there exists $\mu > 0$, $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, the sectional curvatures of $e{-}2fg$ are $> \mu$ for $z \geq 1$. Now if δ_0 is small enough the sectional curvatures of $e^{-2f}(\psi g + (1-\psi)\bar{g})$ are strictly positive by continuity.

Now to choose the δ-neck where the surgery is done, we use the following: Fix a small $\delta > 0$, $\rho = \delta r$.

lemma 3.7. *[P03]II.4.3 There exists a radius h, $0 < h < \delta\rho$, depending only on ε, δ,ρ, φ such that for each point x in a ε-horn of $(\Omega, g(T))$, if $R(x,T) = Q \geq h^{-2}$, the parabolic neighborhood $P(x, t, \frac{1}{\delta Q^{1/2}}, -Q^2)$ is a strong δ-neck.*

The surgery with δ-cutoff is defined as follows. Fix a small $\delta > 0$ and $\rho = \delta r$, define h as above. in each ε-horn of $(\Omega, g(T))$, find a δ-neck of scalar curvature h^{-2} and paste an almost standard cap as above.

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