A mathematical model for passive imaging in seismology

Yves Colin de Verdière
Institut Fourier (Grenoble)

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New methods in seismology

Seismic waves have been used since a long time in order to know the inner structure of the earth. Our knowledge of the global (large scale) structure is well known, but the fine structure of the crust (up to 50 km deep) is more difficult to know.

The classical method: people use waves created by an earthquake or an explosion. Those waves propagate inside the earth and propagation times allow to get some knowledge of the earth structure. This method has some intrinsic limitations:

- they are some non seismic areas

- the power generated by explosives is limited!

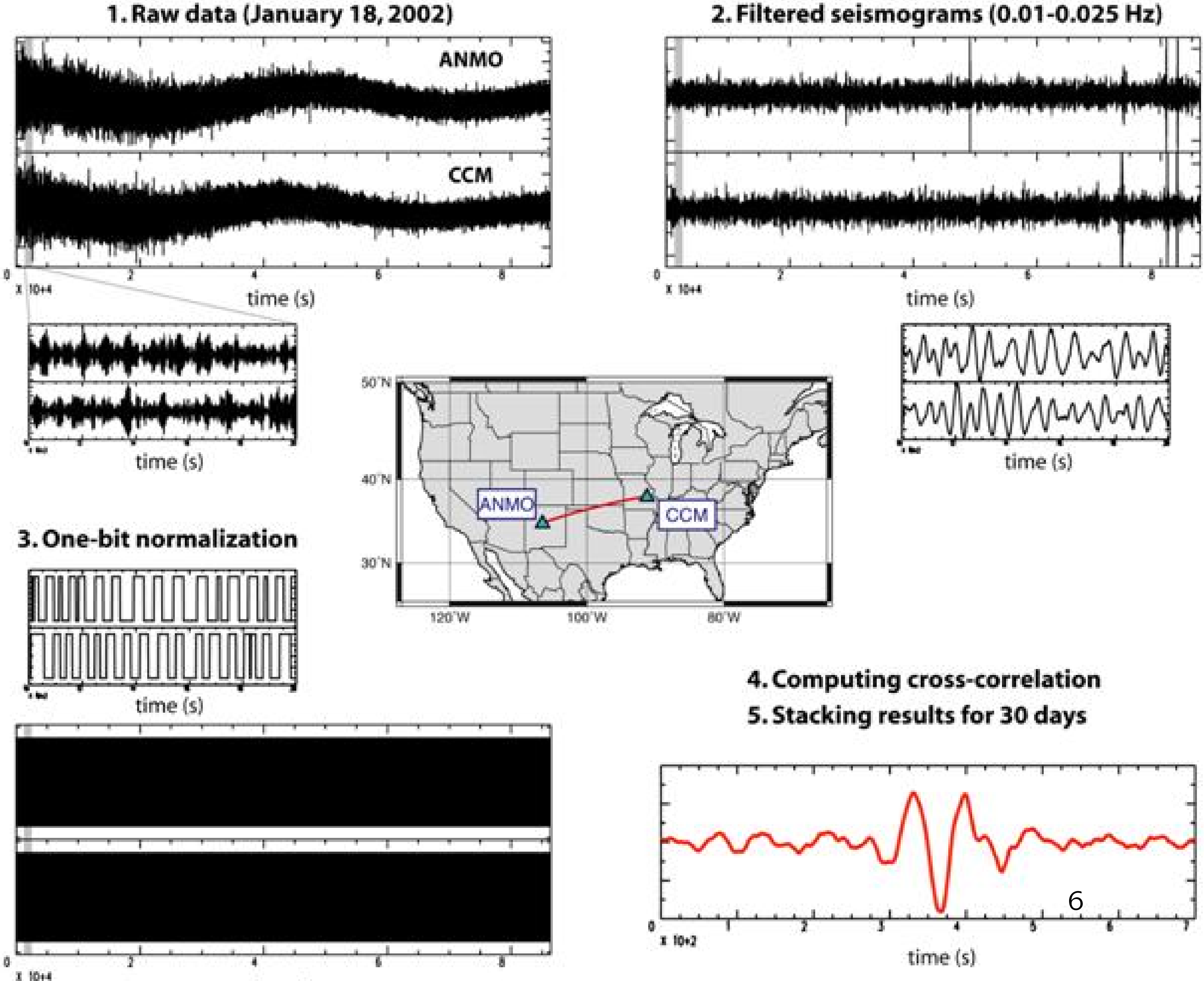
The new method (Michel Campillo (LGIT, Grenoble) and co-workers): they use the *seismic noise* which is recorded by seismographs. The noisy field at a single point contains no information, but noises at different points are correlated. People calculate the correlation functions of noises recorded during a long time (months) at the stations of a network.

They are many sources of noise, the most interesting one comes from the interaction of oceanic waves with the earth crust.

The key observation is that the correlation function $C_{A,B}(\tau)$ of the seismic waves at the points A and B is very similar to the signal observed at the point A when an earthquake occurs at the point B and is propagated during a time τ. Our goal is to give theoritical models in order to explain that...

The scheme of the reconstruction: from the correlations functions, one recover the geometric (classical) part of the Green function; from it, one can deduce the effective Hamiltonian of surface waves; then using an inverse spectral problem, we can recover the vertical structure of the (stratified) crust.

It works! Michel Campillo and co-workers succeeded to produce maps of the Californian underground with a rather good resolution.



1. Raw data (January 18, 2002)

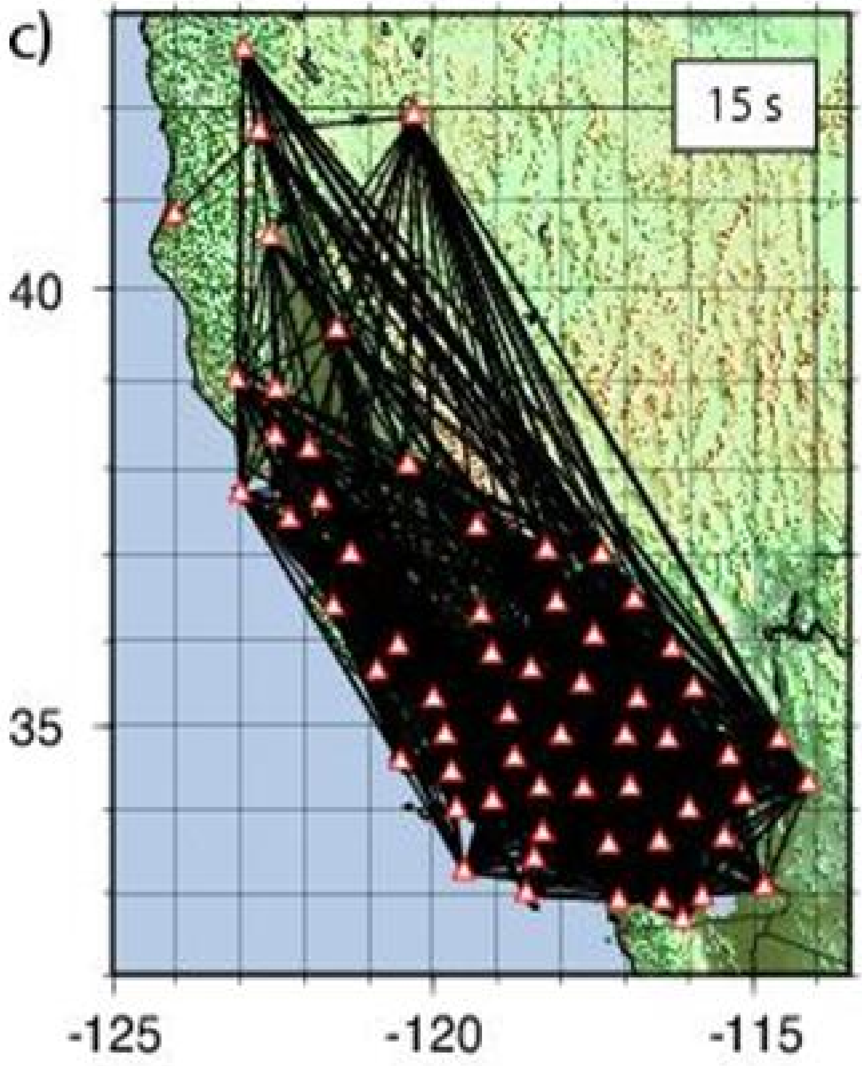
2. Filtered seismograms (0.01-0.025 Hz)

3. One-bit normalization

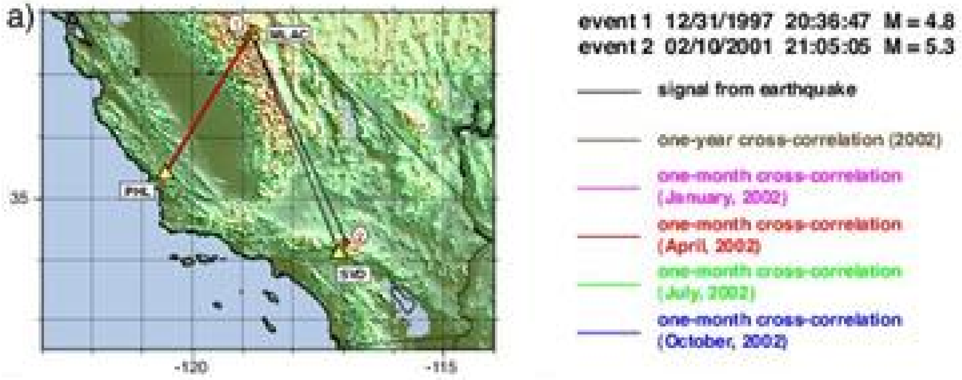
4. Computing cross-correlation

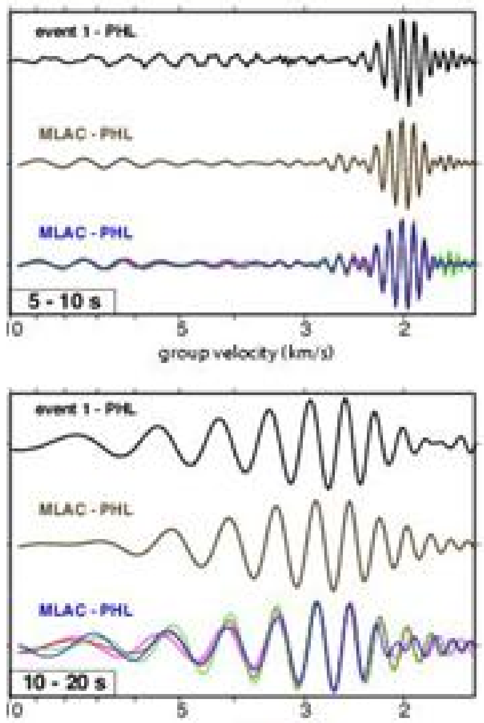
5. Stacking results for 30 days

Interstation paths for correlations of noise records (Rayleigh)

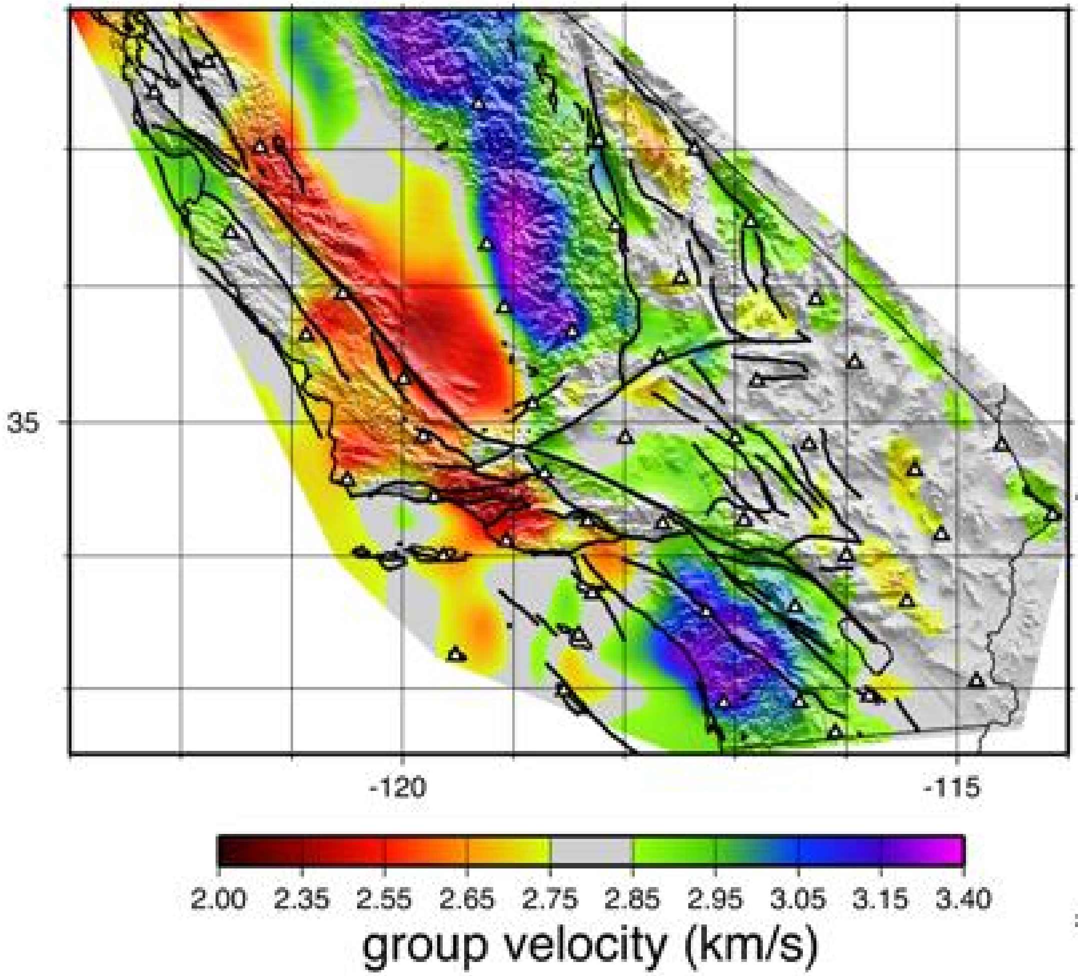


Comparison between earthquake records and reconstructed response



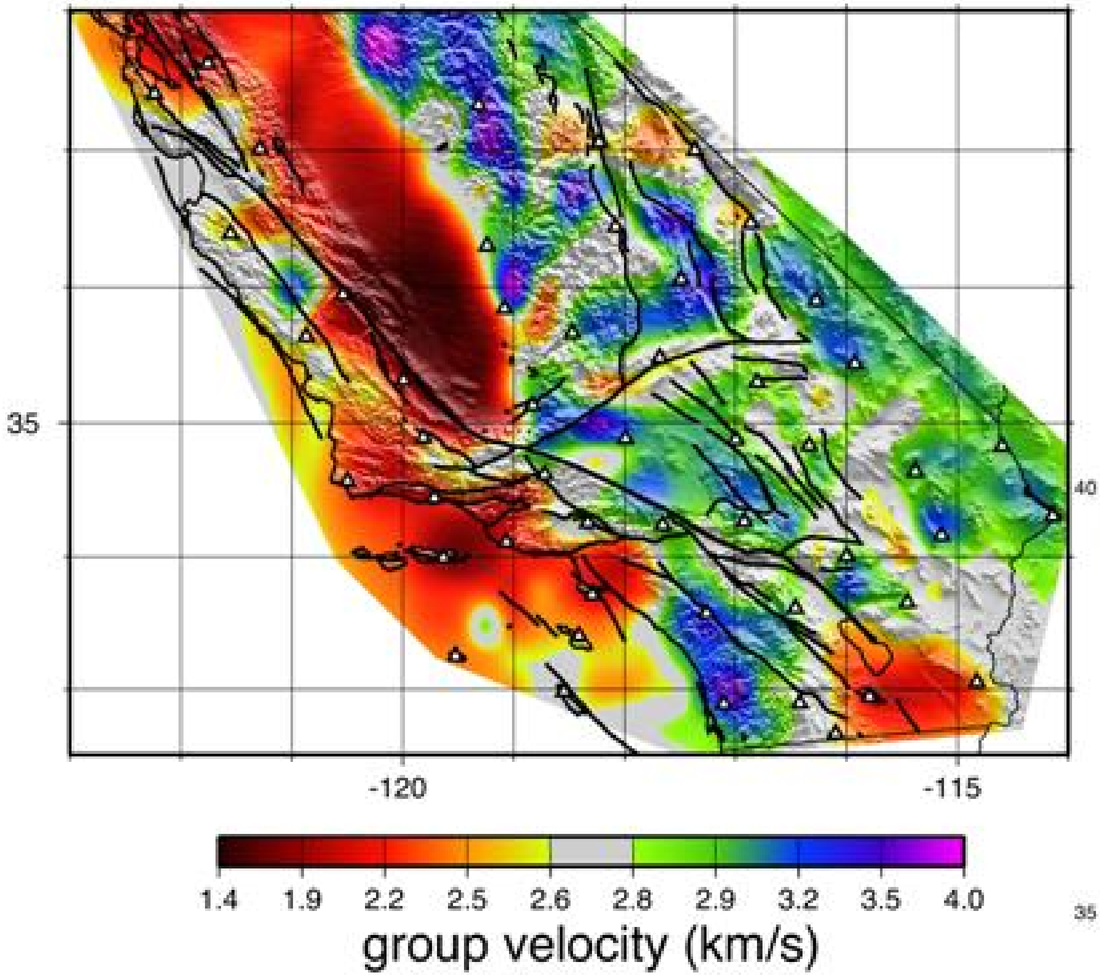


High resolution velocity map obtained from noise (Rayleigh 15 s ~ middle crust)



Main geological features

High resolution velocity map obtained from noise (Rayleigh 7.5 s)

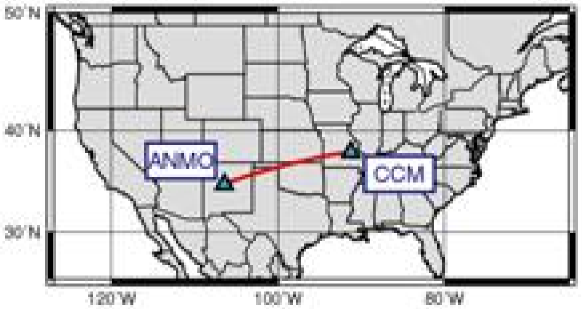


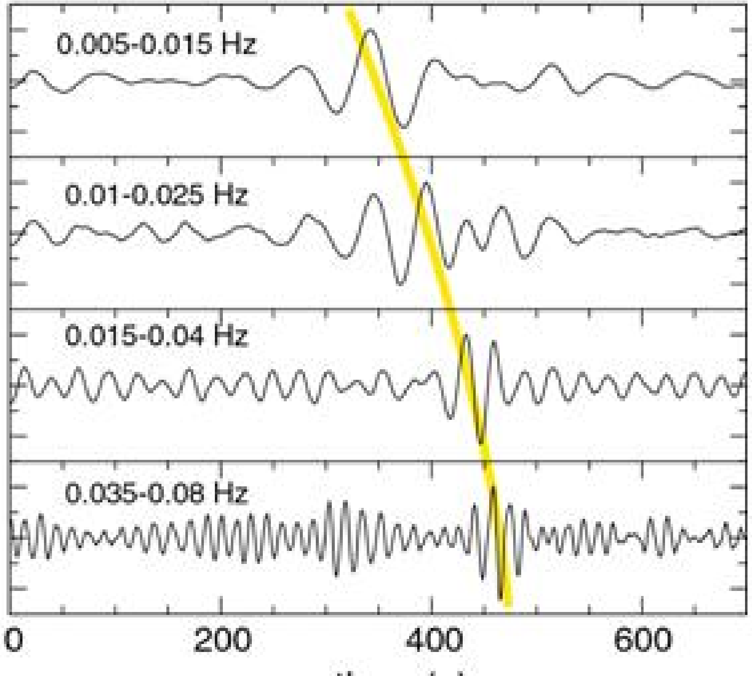
Main geological features

Cross-correlations of seismic noise: ANMO - CCM

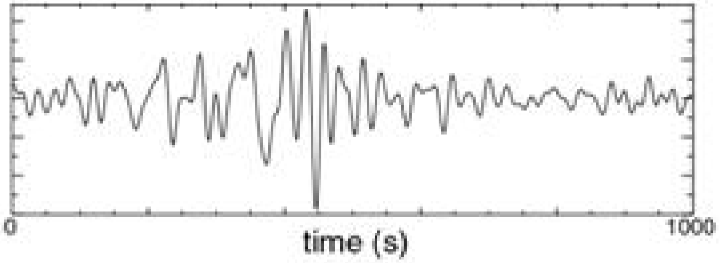
(from Shapiro and Campillo, GRL, 2004)

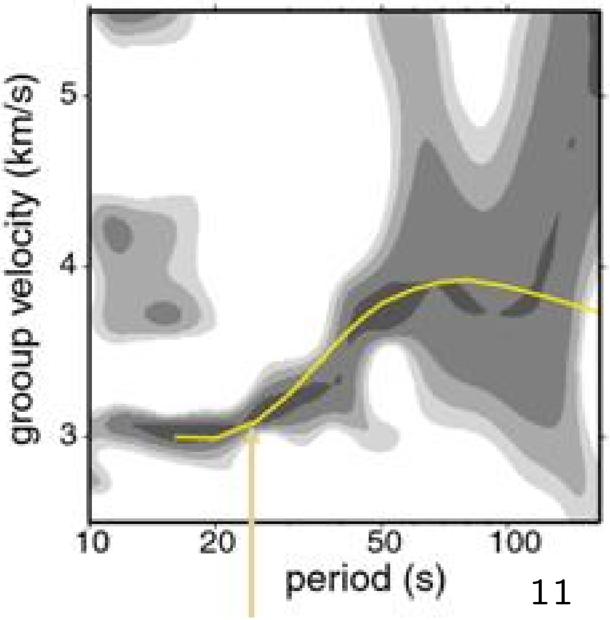
30 days of vertical motion





Dispersion analysis





global model by

Using the noise is not a completely new idea; it have been already used in order to know something of the interior of the sun using the “thermic noise”.

The goal of my lecture is to produce a mathematical model which explains this method, called *passive imaging*.

1. **A model wave equation**
2. **The case of a white noise**
3. **Modelling the source of the noise**
4. **The main result: calculus of the correlation function in the semi-classical regime**
5. **Effective Hamiltonians for surface waves in the case of a stratified medium**

1. A model wave equation

$$u_t + \widehat{H} u = f \tag{1}$$

- $u = u(x, t)$ the field

- $x \in X$, X a *smooth manifold* of dimension d with a smooth measure $|dx|$

- $\widehat{H}$ the generator of the free dynamics is acting on $L^2(X)$. It satisfies some attenuation property: if we define the semi-group $\Omega(t) = \exp(-t\widehat{H}), t \geq 0$, there exists $k > 0$, so that we have the estimate $\|\Omega(t)\| = 0(e^{-kt})$.

- $f(x,t)$, the source of the noise is a random field assumed to be *stationary in time* and *ergodic.* We will write

  $$K(s-s',x,y) := \mathbb{E}(f(x,s)\overline{f(y,s')})$$

  the *covariance* kernel of f. For simplicity and w.l.o.g., we will assume that $K(t,x,y) = L(x,y)\delta(t=0)$.

This simple model can be easily generalized to usual wave equation: just write it with vector valued fields as usual.

$$u_{tt} + a(x)u - \Delta u = f$$

$$\mathbf{u} = \begin{pmatrix} u \\ u_t \end{pmatrix}, \mathbf{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \widehat{H} = \begin{pmatrix} 0 & -1 \\ \Delta & a \end{pmatrix} ,$$

$$\mathbf{u}_t + \widehat{H}\mathbf{u} = \mathbf{f} \ .$$

The solution of Equation (1) with $f \equiv 0$, $u(t) = \Omega(t)(u(0))$, can be written as

$$\left(\Omega(t)u\right)(x) = \int_X P(t,x,y)u(y)dy \ .$$

$P(t,x,y)$, the Schwartz kernel of $\Omega(t)$, is called the **propagator**. It satisfies:

$$\int_X P(t,x,y)P(t',y,z)|dy| = P(t+t',x,z) \ .$$

Some usefull notations:

$[A](x,y)$ is the Schwartz kernel of the operator A.

$\widehat{a}$ is the operator of Schwartz kernel $a(x,y)$.

The **causal** solution of Equation (1) is:

$$u(x,t) = \int_0^\infty ds \int_X P(s,x,y) f(y,t-s) dy \tag{2}$$

The kernel $Y(s)P(s,x,y)$ is called the **Green function.**

The **correlation** of 2 complex fields $\varphi(t)$ and $\psi(t)$ is defined by:

$$C_{\varphi,\psi}(\tau) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \varphi(t)\overline{\psi(t-\tau)}dt \ .$$

The correlation of the fields at A and B is then given for $\tau > 0$, by

$$C_{A,B}(\tau) = \int_0^\infty ds \int_{X\times X} |dx||dy| P(s+\tau, A, x) L(x,y) \overline{P}(s, B, y) \quad (3)$$

and $C_{A,B}(-\tau) = \overline{C_{B,A}(\tau)}$.

We get the nicer formula:

$$\begin{array}{l} \text{for } \tau > 0,\ C_{A,B}(\tau) = [\Omega(\tau)\Pi](A,B) \\ \text{with } \Pi = \int_0^\infty \Omega(s)\hat{L}\Omega^\star(s)ds \end{array} \tag{4}$$

Recall that $\hat{L}$ is defined from:

$$L(x,y)\delta(t-t') = \mathbb{E}(f(x,t)\overline{f}(y,t'))\ .$$

2. The case of a white noise

A white noise of an Hilbert space $(\mathcal{H}, \langle .|.\rangle)$ is a random field f whose correlation satisfies: $\mathbb{E}(\langle f|v\rangle\overline{\langle f|w\rangle}) = \langle w|v\rangle$.

If we assume

- f a white noise
- $\widehat{H} = \widehat{H}_0 + k$ with $k > 0$ a constant and $\widehat{H}_0$ anti-self-adjoint with unitary propagator P_0,

we get, for $\tau > 0$, an exact formula: $C_{A,B}(\tau) = \frac{1}{2k} P_0(\tau, A, B)$.

3. Modelling the source noise

In our applications, the noise are far from being homogeneous and isotropic. We will introduce random fields f given by $f = Aw$ (w a white noise). The correlation kernel of f, $K(x,y) := \mathbb{E}\left(f(x)\overline{f}(y)\right)$, is the kernel $[AA^\star]$. If we introduce a small parameter ε and if we want correlations distances of the order of ε, it is natural to take for A an ε-pseudo-differential operator $A = \mathrm{Op}_\varepsilon(a)$ with the symbol $a(x,\xi)$ smooth and rapidly decaying w.r. to ξ.

$$[A](x,y) = (2\pi\varepsilon)^{-d} \int e^{i\langle x-y|\xi\rangle/\varepsilon} a(x,\xi)|d\xi| \ .$$

Using the ΨDO calculus, we get the correlation kernel of the random field f:

$$K_\varepsilon \sim (2\pi\varepsilon)^{-d} B\left(x, \frac{x-y}{\varepsilon}\right)$$

with B the ξ-Fourier transform of $|a|^2$.

The **power spectrum of f**, defined as the **expected Wigner measure**, is then given by

$$P_\varepsilon \sim (2\pi\varepsilon)^{-d} |a|^2(x,\xi)|dx d\xi| \ .$$

4. The main result: calculus of the correlation function in the semi-classical regime

We will assume that the wave operator $\widehat{H}$ is also a ΨDO with the same small parameter ε (high frequency regime): our wave equation is now $\frac{\varepsilon}{i}u_t + \widehat{K}u = \frac{\varepsilon}{i}f$ with $\widehat{K} = \mathrm{Op}(H_0 + \varepsilon H_1)$ and

- H_0 is real valued; ϕ_t is the flow of

$$X_{H_0} = \sum_j \frac{\partial H_0}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial H_0}{\partial x_j}\frac{\partial}{\partial \xi_j} \ .$$

- H_1 satisfies $\Im H_1 < 0$ (implies attenuation)

- The noise f is given by $f = \mathrm{Op}_\varepsilon(a)w$ with w a white noise and $a = a(x,\xi) \in C_o^\infty$.

Theorem 1 *The correlation is given, for $\tau > 0$, by*

$$C_{A,B}(\tau) = [\Omega(\tau) \circ \Pi)](A, B)$$

where $\Pi = \mathrm{Op}_\varepsilon(\pi) + R$ *with:*

$$\pi(x, \xi) = \int_{-c|\log \varepsilon|}^{0} \exp\left(2\int_t^0 \Im(H_1)(\phi_s(x,\xi))ds\right) |a|^2 \left(\phi_t(x,\xi), -H_0(x,\xi)\right) dt$$

and R *the remainder term is “$O(\varepsilon^\alpha)$” for some* $\alpha > 0$.

More precisely, let us consider $C_{A,B}(\tau)$ as the Schwartz kernel of an operator $\widehat{C}(\tau)$. This operator is Hilbert-Schmidt with an Hilbert-Schmidt norm of the order of $\varepsilon^{-d/2}$ and we have

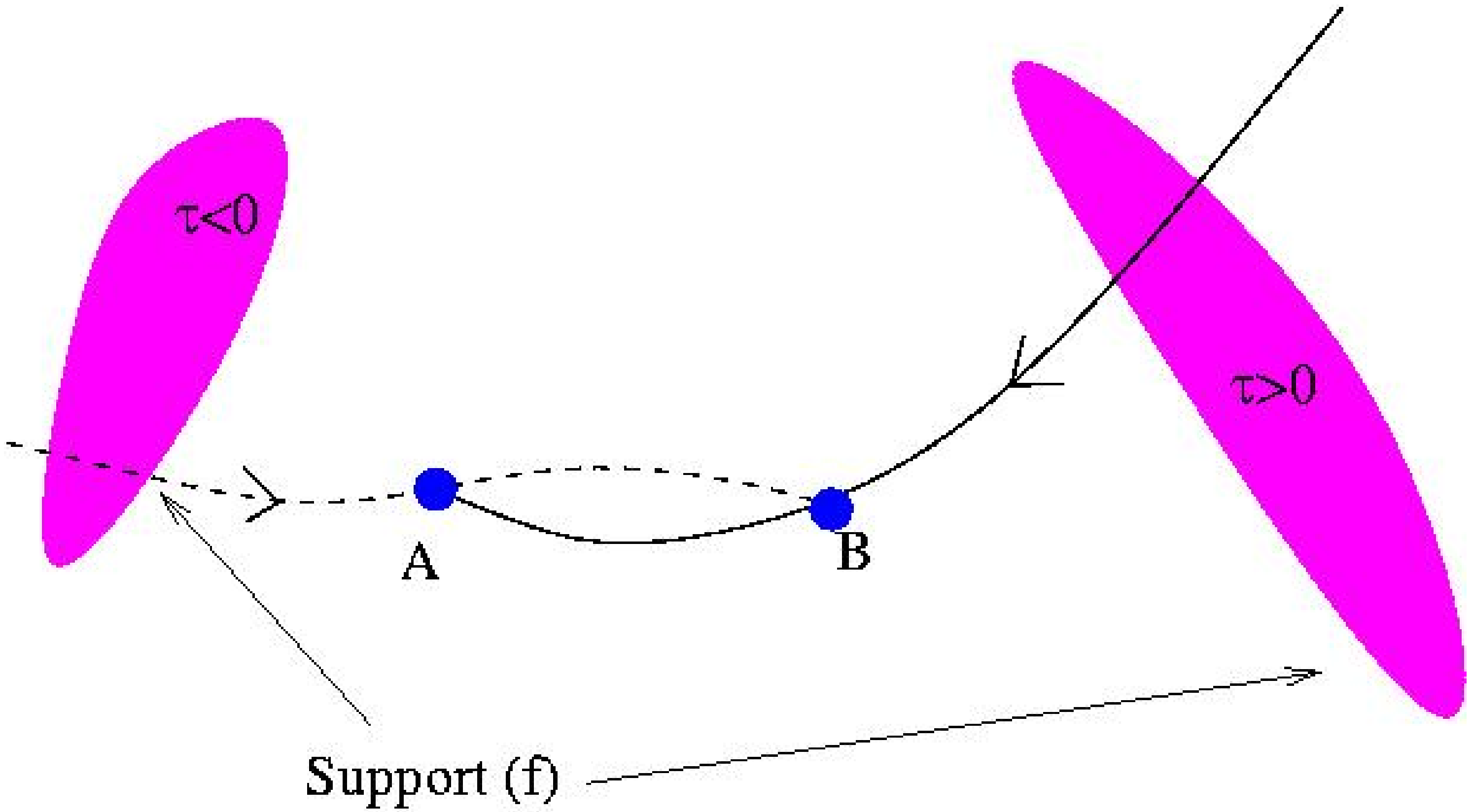
$$\|\widehat{R}\|_{\mathsf{H-S}} = O(\varepsilon^{\alpha-d/2}) \ .$$

In the technical jargon, the previous Theorem says that $\widehat{C_{\cdot,\cdot}(\tau)}$ is close to a Fourier integral operator F_τ associated to the flow ϕ_τ.

The principal symbol of F_τ at the point $(\phi_\tau(B,\eta_B); B,\eta_B)$ is the principal symbol of exp$(-it\widehat{K}/\varepsilon)$ at the same point multiplied by $\pi(y,\eta)$.

Generically, it is a finite sum of contributions of classical trajectories γ from B to A in time τ: $(A, \xi_A) = \varphi_\tau(B, \eta_B)$ which satisfy:

$$\exists (x, \xi) \in \text{Support}(a),\ t > 0,\ \text{so that } \varphi_t(x, \xi) = (B, \xi_B)$$


τ<0
τ>0
A
B
Support (f)

Using Theorem 1.

If we apply Theorem 1 to a dense array of recording stations, we can hope to recover the dispersion relation $\omega + H_0(x, \xi) = 0$ in some part of the phase space, because it allows to get the speed map as a function of frequencies.

Using van Vleck's formula

The previous technical statement can be reformulated in the case of dispersive wave; if A and B are not conjugate points along γ, the contribution of γ to $C_{A,B}(\tau)$ admits a WKB expression

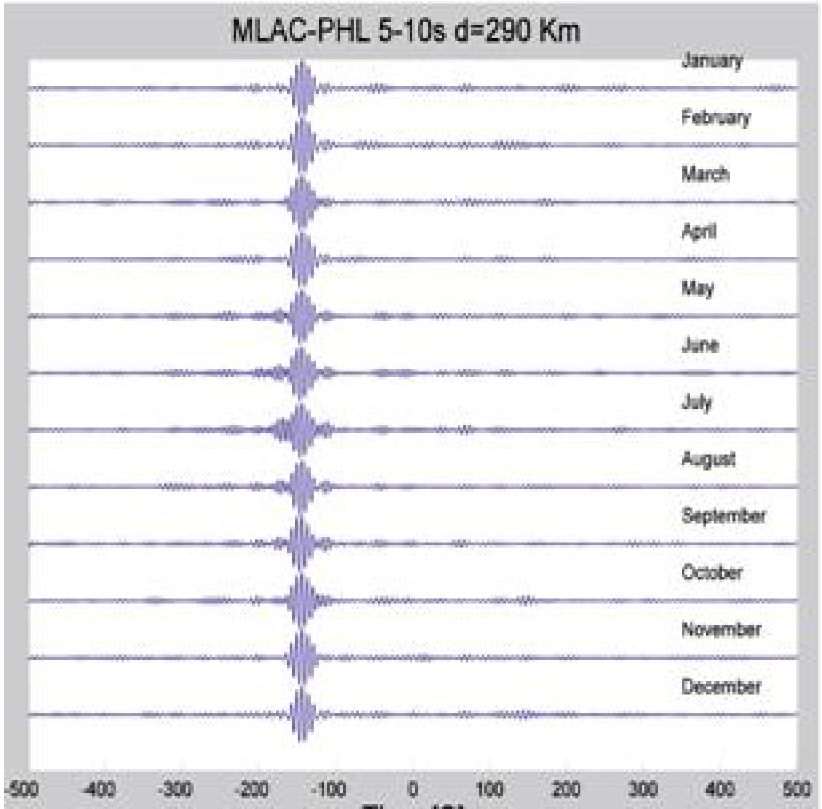
$$(2\pi i\varepsilon)^{-d/2}\pi(B,\xi_B)s(A,B)e^{iS(\gamma)/\varepsilon}\ .$$

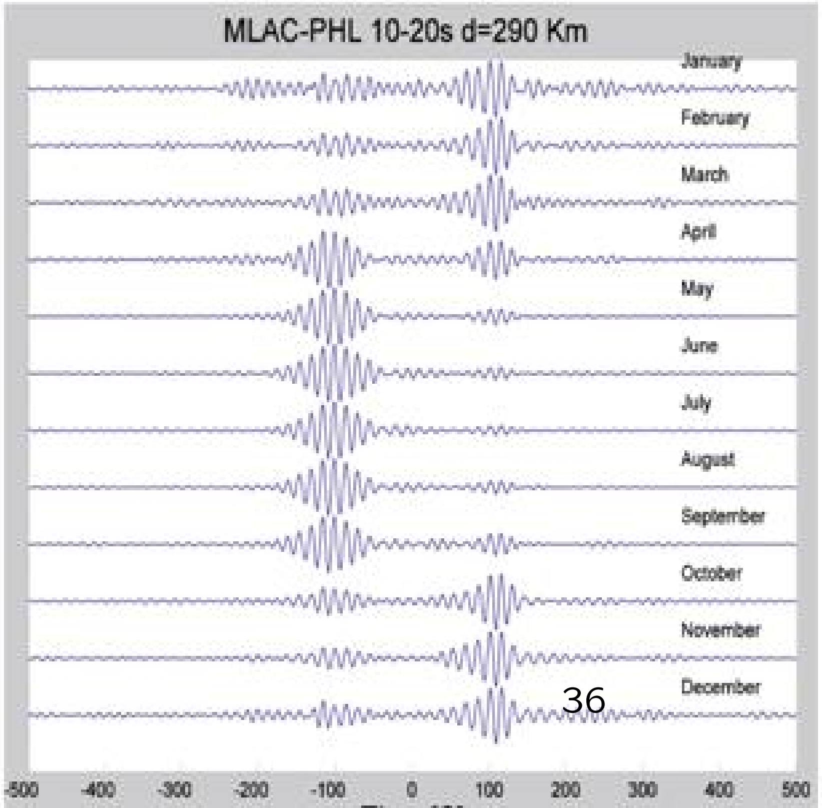
Time reversal symmetry

The wave equation $W(u) = 0$ is said to be time reversal symmetric if, for any solution $u(x, t)$, $\overline{u(x, -t)}$ is also a solution. For any classical ray $x(t)$, $x(-t)$ is also a ray.

It implies that $C_{A,B}(-\tau)$ is very similar (up to scaling) to $\overline{C_{A,B}(-\tau)}$. This can be checked as a test for the theory or for clock synchronisation.

Tracking the origin of the seismic noise





5. Effective Hamiltonians for surface waves in the case of a stratified medium

How to apply what I have said before to seismology?

There are several kinds of seismic waves: body waves (S- or P-waves) and surface waves.

The energy decay of body waves is much faster than the decay of surface waves. It implies that the most significant part of the correlation will come from surface waves.

They are several kinds of surface waves. The simplest one are due to a *wave guide effect* in the crust: the propagation speed is higher in the rocks than in the sediments layers.

Such waves are living deeper and deeper as the frequency decreases and are then going faster: there is a non trivial **dispersion relation.**

Assuming that we are able to recover this dispersion relation, how do we recover the vertical structure of the crust?

An example of an effective Hamiltonian

Let us consider the following wave equation in a stratified medium:

$$u_{tt} - V^2\left(x, y, \frac{z}{\varepsilon}\right) \Delta u = 0 \ ,$$

($V > 0$ and $V(x, y, Z) \equiv 1$ for $Z << 0$) in the domain

$$D = \{(x, y, z) | z \leq 0\},$$

with Neumann or Dirichlet boundary conditions.

In order to calculate the effective Hamiltonian for waves of frequency ω/ε, I start with the Ansatz

$$u(t,x,y,z) = e^{i(\omega t - x\xi - y.\eta)/\varepsilon} f(x,y,\frac{z}{\varepsilon})$$

where $f(x,y,Z)$ is rapidly decaying as $Z \to -\infty$.

Taking all terms in ε^{-2}, we get

$$-V^2(x,y,Z)\frac{\partial^2 f}{\partial Z^2}(x,y,Z)+V^2(x,y,Z)(\xi^2+\eta^2)f(x,y,Z) = \omega^2 f(x,y,Z)$$

with boundary conditions.

This equation admits L^2 solutions in z iff ω^2 is in the discrete spectrum of

$$L_{x,y,\xi,\eta} = -V^2(x,y,.)\frac{\partial^2}{\partial Z^2} + V^2(x,y,.)(\xi^2 + \eta^2)$$

acting on $L^2(\mathbb{R}^-)$ with boundary conditions.

The discrete spectrum can be non empty only if V takes values < 1.

We can write the **effective dispersion relation**:

ω^2 belongs to the discrete spectrum of $L_{x,y,\xi,\eta}$.

It remains to solve the following inverse problem:

find $V(x,y,Z)$ from the discrete spectra of the $L_{x,y,\xi,\eta}$'s.

It is easy to see from Weyl's asymptotic formula that it is possible to do it if V is monotonic in Z.

One more remark:

there are in this problem 3 small parameters:

- The correlation distance of the source noise (oceanic noise)
- The semi-classical parameter for the wave propagation (seismic waves)
- The small parameter associated to the stratification (the layered crust).

The fact that all 3 small parameters are of compatible sizes is crucial for the efficiency of the method.

Thanks for your attention...

More on

http://www-fourier.ujf-grenoble.fr/~ycolver/