Mathematical models for passive imaging I: general background.

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Abstract

Passive imaging is a new technics which has been proved to be very efficient, for example in seismology: the correlation of the noisy fields between different points is strongly related to the Green function of the wave propagation. The aim of this paper is to provide a mathematical context for this approach and to show, in particular, how the methods of semi-classical analysis can be be used in order to find the asymptotic behaviour of the correlations.

Introduction

Passive imaging is a way to solve inverse problems: it has been succesfull in seismology and acoustics [2, 3, 11, 15, 16, 20, 21, 23]. The method is as follows: let us assume that we have a medium X (a smooth manifold) and a smooth, deterministic (no randomness in it) linear wave equation in X. We hope to recover (part of) the geometry of X from the wave propagation. We assume that there is somewhere in X a source of noise $\mathbf{f}(x,t)$ which is a stationary random field. This source generates, by the wave propagation, a field $\mathbf{u}(x,t) = (u^\alpha(x,t))_{\alpha=1,\cdots,N}$ which people do record on long time intervalls. We want to get some information on the propagation of waves from B to A in X from the correlation matrix[1]

$$C_{A,B}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{u}(A,t)\otimes \mathbf{u}(B,t-\tau)^\star dt$$

(equivalently

$$C^{\alpha\beta}_{A,B}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{u}^\alpha(A,t)\overline{\mathbf{u}^\beta(B,t-\tau)}dt \ )$$

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[1]For every matrix (a_{ij}), we write $(a_{ij})^\star := (\overline{a_{ji}})$.

which can be computed numerically from the fields recorded at A and B. It turns out that $C_{A,B}(\tau)$ is closely related to the deterministic Green's function $G(A, B, \tau)$ of the wave equation in X. It means that one can hope to recover, using Fourier analysis, the propagation speeds of waves between A and B as a function of the frequency, or, in other words, the so-called dispersion relation.

If the wave dynamics is time reversal symmetric, the correlation admits also a symmetry by change of τ into $-\tau$; this observation has been used for clock synchronization in the ocean, see [17].

The goal of this paper is to give precise formulae for $C_{A,B}(\tau)$ in the high frequency limit assuming a rapide decay of correlations of the source $\mathbf{f}$. More precisely, we have 2 small parameters, one of them entering into the correlation distance of the source noise, the other one in the high frequency propagation. The fact that both are of the same order of magnitude is crucial for the method.

Let us also mention on the technical side that, rather than using mode decompositions, we prefer to work directly with the dynamics; in other words, we need really a *time dependent* rather than a *stationary approach.* Mode decompositions are often usefull, but they are of no much help for general operators with no particular symmetry.

For clarity, we will first discuss the non-physical case of a first order wave equation like the Schrödinger equation, then the case of a more usual wave equations (acoustics, elasticity).

The main result expresses, for $\tau > 0$, $C_{A,B}(\tau)$ as the Schwartz kernel of $\Omega(\tau)\circ\Pi$ where Π is a suitable pseudo-differential operator (a ΨDO), whose principal symbol can be explicitely computed, and $\Omega(\tau)$ is the (semi-)group of the (damped) wave propagation. It implies that we can recover the dispersion relation, i.e. the classical dynamics, from the knowledge of all two-points correlations.

In order to make the paper readable by a large set of people, we have tried to make it self-contained by including sections on pseudo-differential operators and on random fields.

In Section 1, we start with a quite general setting and discuss a general formula for the correlation (Equation (4)).

Section 2 is devoted to exact formulae in case of an homogeneous white noise.

In Section 3, we discuss the important property of time reversal symmetry which plays a prominent part in the applications and is also usefull as a numerical test.

In Section 4, we introduce a large family of anisotropic random fields and show the relation between their power spectra and the Wigner measures.

Section 5 contains the main result expressing the correlation in the case of a Schrödinger wave equation.

Section 6 does the same in case of a wave equation.

The short Section 7 is a problem section.

Section 8 is a about a quite independent issue relative to correlations of scattered waves.

Finally, there is a very short introduction to pseudo-differential operators in Section 9.

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1 A general formula for the correlation

1.1 The model

We will first consider the following damped wave equation:

$$\frac{d\mathbf{u}}{dt} + \hat{H}\mathbf{u} = \mathbf{f} \tag{1}$$

- X is a *smooth manifold* of dimension d with a smooth measure $|dx|$
- $\mathbf{u}(x,t),\ x \in X,\ t \in \mathbb{R}$ is *the field* (scalar or vector valued) with values in $\mathbb{C}^N$ (or $\mathbb{R}^N$).
- The linear operator $\hat{H}$ is the *Hamiltonian,* acting on $L^2(X,\mathbb{C}^N)$. It satisfies some attenuation property: if we define the semi-group $\Omega(t) = \exp(-t\hat{H}), t \geq 0$, there exists $k > 0$, so that we have the estimate $\|\Omega(t)\| = 0(e^{-kt})$.
- The *source* $\mathbf{f}$ is a *stationary ergodic random field* on $X \times \mathbb{R}$ with values in $\mathbb{C}^N$ (or $\mathbb{R}^N$) whose matrix valued correlation kernel is given by

  $$\mathbb{E}(\mathbf{f}(x,s) \otimes \mathbf{f}^\star(y,s')) = K(x,y,s-s')\ . \tag{2}$$

  We will usually assume that $K(x,y,t)$ vanishes for large t, say $|t| \geq t_0 > 0$.

1.2 Examples

1.2.1 Schrödinger equation

Let X be a smooth Riemannian manifold with Laplace-Beltrami operator Δ. Let us give $a : X \to \mathbb{R}$ a non negative function, V a smooth real valued function on X, $\hbar$ a non negative constant, and $\hat{K} = -\hbar^2\Delta + V(x)$, and take:

$$\frac{\hbar}{i}(u_t + a(x)u) + \hat{K}u = g\ .$$

It is a particular case of Equation (1) where $\hat{H} = \frac{i}{\hbar}\hat{K} + a(x)$ and $f = \frac{i}{\hbar}g$. Let us note for future use that, if $\hbar \to 0$, the principal symbol of our equation is $\omega + \|\xi\|^2 + V(x)$ and $a(x)$ is only a sub-principal term entering into the transport equation, but not in the classical dynamics.

1.2.2 Wave equations

Let us start with

$$u_{tt} + 2au_t - \Delta u = f,\ a > 0 \tag{3}$$

(with Δ the Laplace-Beltrami operator of a Riemannian metric on X) which corresponds to Equation (1) with

$$\mathbf{u} = \begin{pmatrix} u \\ u_t \end{pmatrix},\ \mathbf{f} := \begin{pmatrix} 0 \\ f \end{pmatrix} .$$

and

$$\hat{H} = \begin{pmatrix} 0 & -\mathrm{Id} \\ -\Delta & 2a \end{pmatrix} .$$

1.2.3 Pseudo-differential equations

We can assume that the dynamics is generated by a ΨDO (see Section 9). Our equation looks then like:

$$\frac{\varepsilon}{i}\mathbf{u}_t + \hat{H}_\varepsilon \mathbf{u} = \mathbf{f}$$

with

$$\hat{H}_\varepsilon = \mathrm{Op}_\varepsilon(H_0 + \varepsilon H_1) .$$

This allows to include

- An effective surface Hamiltonian associated to stratified media (included in the H_0 term) [5]. They are usually ΨDO's with a non trivial dispersion relation
- Frequency dependent damping included in H_1: this is usually the case for seismic waves.

1.3 The correlation

Definition 1 *Let us define, for $t \geq 0$, $\Omega(t) := \exp(-t\hat{H})$ and the* propagator P *by the formula:*

$$(\Omega(t)\mathbf{v})(x) = \int_X P(t,x,y)\mathbf{v}(y)|dy| .$$

The propagator P satisfies

$$\int_X P(t,x,y)P(s,y,z)|dy| = P(t+s,x,z)$$

which comes from: $\Omega(t+s) = \Omega(t) \circ \Omega(s)$. The causal solution of Equation (1) is then given by

$$\mathbf{u}(x,t) = \int_0^\infty ds \int_X P(s,x,y)\mathbf{f}(t-s,y)|dy| .$$

We define

$$C_{A,B}(\tau) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbf{u}(A,t)\otimes \mathbf{u}(B,t-\tau)^\star dt \ .$$

Ergodicity allows to replace time average by ensemble average and we get by a simple calculation:

Theorem 1 *If P is defined as in Definition 1 and K by Equation (2), we have, for $\tau > 0$:*

$$C_{A,B}(\tau) = \int_0^\infty ds \int_{-\infty}^{s+\tau} d\sigma \int_{X\times X} |dx||dy| P(s+\tau-\sigma, A, x) K(x,y,\sigma) P^\star(s,B,y) \tag{4}$$

and $C_{A,B}(-\tau) = C_{B,A}(\tau)^\star$.

We get, for $\tau > t_0$, the formula[2]*:*

$$C_{A,B}(\tau) = [\Omega(\tau)\Pi](A,B) \tag{5}$$

with

$$\Pi = \int_0^\infty \Omega(s)\mathcal{L}\Omega^\star(s)ds \tag{6}$$

and

$$\mathcal{L} = \int_{|t|\leq t_0} \Omega(-t)\hat{K}(t)dt \ .$$

If we assume that $K(x,y,\sigma) = L(x,y)\delta(\sigma)$ we get the simpler formula

$$\mathcal{L} = \hat{L} \ .$$

2 Exact formulae for white noises

2.1 Vector valued white noise

Let us assume that $L(x,y) = \delta(x-y)\mathrm{Id}$ meaning that $\mathbf{f}$ is vector valued white noise on $X\times\mathbb{R}$. We get, for $\tau > 0$,

$$\Pi = \int_0^\infty \Omega(s)\Omega^\star(s)ds$$

and, assuming $\hat{H} = i\hat{A} + k$, with $\hat{A}$ self-adjoint, the following simple formula:

$$C_{A,B}(\tau) = \frac{1}{2k}P_0(\tau, A, B) \ ,$$

which is an exact relation between the correlation and the propagator P_0 of the wave equation without attenuation ($k=0$).

[2] If R is an operator we will denote by $[R](x,y)$ its Schwartz kernel; $\hat{L}$ is the integral operator whose kernel is $L(x,y)$

2.2 Twisted white noise

This section was motivated by a question of Philippe Roux.

Definition 2 *A* twisted white noise *is a random field given by* $\mathbf{f} = L_0\mathbf{w}$ *with* $\mathbf{w}$ *a white noise as defined in Section 2.1 and* $L_0 \in \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)$*. Its correlation is* $\delta(x-y)\delta(s-s')K_0$ *with* $K_0 = L_0 L_0^\star$*.*

We have then

$$C_{A,B}(\tau) = [\Omega(\tau)\Pi](A,B)$$

with

$$\Pi = \int_0^\infty \Omega(s)K_0\Omega^\star(s)ds \ .$$

In the particular case of the scalar wave equation with a constant damping a and the dynamics described in Section 2.3 by Equation (7), we get using a gauge transform where

$$\Omega(t) = e^{-at}\begin{pmatrix} e^{itP} & 0 \\ 0 & e^{-itP} \end{pmatrix}$$

$$\Pi = \frac{1}{2a}K_{0,\text{diag}} + R$$

where $K_{0,\text{diag}}$ is the diagonal part of K_0 and R is going to vanish in the high frequency limit.

2.3 Wave equations

Let us take the case of a scalar wave equation with constant damping; the closest results were derived in [22, 13].

We will consider the wave equation (3) with

- $a > 0$ is a constant damping coefficient
- Δ a Riemannian laplacian in some Riemannian manifold X, possibly with boundary:

  $$\Delta = g^{ij}(x)\partial_{ij} + b_i(x)\partial_i$$

  which is self-adjoint with respect to $|dx|$ and appropriate boundary conditions; in fact we could replace the Laplacian by any self-adjoint operator on X!
- $f = f(x,t)$ the source of the noise which will be assumed to be a scalar white noise (homogeneous diffuse field):

  $$\mathbb{E}(f(x,s)f(y,s')) = \delta(s-s')\delta(x-y)$$

Let us compute the "causal solution", i.e. the solution given by $u = \mathbf{G}f$ with $\mathbf{G}$ linear and satisfying $u(.,t) = 0$ if $f(.,s)$ vanishes for $s \leq t$.

We introduce the vector

$$\mathbf{u} = \begin{pmatrix} u \\ \partial_t u + au \end{pmatrix}$$

which satisfies:

$$\partial_t \mathbf{u} + a\mathbf{u} + \hat{H}\mathbf{u} = \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{7}$$

with

$$\hat{H} = -\begin{pmatrix} 0 & \mathrm{Id} \\ \Delta + a^2 & 0 \end{pmatrix}$$

In order to get a readable expression, it is convenient to introduce $P = \sqrt{-(\Delta + a^2)}$. We get then easily:

$$u(x,t) = \int_0^\infty ds \int_X e^{-as} \left[\frac{\sin sP}{P} \right] (x,y) f(y, t-s) |dy|$$

where $\sin sP$ is defined from the spectral decomposition of P (any choice of the square root gives the same result for $\frac{\sin sP}{P}$). The meaning of the brackets is "Schwartz kernel of". We define

$$G_a(t,x,y) = Y(t) \left[e^{-at} \frac{\sin tP}{P} \right] (x,y)$$

with Y the Heaviside function. We will call G_a the *(causal) Green function.* We can rewrite:

$$u(x,t) = \int_{\mathbb{R}} ds \int_X G_a(t-s, x, y) f(y,s) |dy| \ .$$

Let us assume now that f is an homogeneous white noise and compute the correlation $C_{A,B}(\tau) := \lim_{T\to\infty} \frac{1}{T} \int_0^T u(A,t) u(B, t-\tau) dt$.

We get quite easily, using

$$\sin\alpha \sin\beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta)) \ :$$

$$C_{A,B}(\tau) = e^{-a|\tau|} \left[\frac{\cos \tau P}{4aP^2} - \frac{a \cos \tau P - P \sin |\tau| P}{4P^2(P^2 + A^2)} \right] (A,B)$$

or:

$$C_{A,B}(\tau) = \frac{e^{-a|\tau|}}{4(P^2 + a^2)} \left[\frac{\cos \tau P}{a} + \frac{\sin |\tau| P}{P} \right] (A,B) \tag{8}$$

Some comments:

- *Low frequency filtering:* from Equation (8), we see for each eigenmode $\Delta u_j = \omega_j^2 u_j$ a prefactor, which in the limit of large τ is $\approx e^{|\tau|(r_j - a)}$ for $\omega_j^2 = a^2 - r_j^2 < a^2$. This acts as low frequency filter as observed in [13].

- *High frequency regime:* in the high frequency regime, we get a simplified expression:
$$C_{A,B}(\tau) \approx e^{-a|\tau|} \frac{\cos \tau P}{4aP^2}$$
of which, taking the derivative w.r.t. to τ, we get
$$\partial_\tau C_{A,B}(\tau) \approx -\frac{e^{-a|\tau|}}{4a} \left(\frac{\sin \tau P}{P} \right)$$
recovering the expression derived by many authors:
$$\partial_\tau C_{A,B}(\tau) \approx \frac{e^{-a|\tau|}}{4a} \left(-G(A, B, \tau) + G(A, B, -\tau) \right) \tag{9}$$

- If the *attenuation is small,* we get
$$C_{A,B}(\tau) \approx \frac{1}{4a} \frac{\cos \tau Q}{Q^2} \ ,$$
with $Q = \sqrt{-\Delta}$, and hence the relation (9).

3 Time reversal symmetry

Definition 3
- *A dispersion relation $\mathbb{D}(x, \xi, \omega) = 0$ is said to be time reversal symmetric (TRS) if it is invariant by $\alpha : (x, \xi) \to (x, -\xi)$.*
- *A linear wave equation (with no attenuation!) is said to be time reversal symmetric (TRS) if for any solution $\mathbf{u}(x, t)$ the field $\overline{\mathbf{u}}(x, -t)$ is also a solution.*

Example 3.1
- *Schrödinger equations without magnetic fields*
- *Acoustic and elastic wave equations.*

Lemma 1 *If $\mathbb{D}$ is TRS and $\gamma(t) = (x(t), \xi(t), \omega_0)$ is a solution of Hamilton's equations, $\alpha(\gamma(-t)) = (x(-t), -\xi(-t), \omega_0)$ too.*

We have the following:

Proposition 1 *The correlation satifies the following general identity:*

$$C_{A,B}(\tau) = C^{\star}_{B,A}(-\tau)$$

and, in case of a white noise and a time reversible wave dynamics modified by a constant attenuation (as in Section 2.1):

$$C_{A,B}(-\tau) = \overline{C_{A,B}(\tau)} \ . \tag{10}$$

Approximations of Equation (10) turn out to be important in applications to clocks synchronisation.

4 Random fields and pseudo-differential operators

4.1 Goal

Our aim in this section is to build quite general random fields with correlation distances given by a small parameter ε. It seems to be natural for that purpose to use $\varepsilon-$pseudo-differential operators. We will see how to compute the generalized power spectrum using Wigner measures.

4.2 White noises

Let $(\mathcal{H}, \langle .|. \rangle)$ be an Hilbert space. There exists a canonical *Gaussian random field* on it, called the *white noise* and denoted by $\mathbf{w}_{\mathcal{H}}$ (or simply $\mathbf{w}$ if there is no possible confusion). This random field is defined by the properties that:

- For all $\vec{e} \in \mathcal{H}$,
$$\mathbb{E}(\langle \mathbf{w}|\vec{e}\rangle) = 0$$
- For all $\vec{e}, \vec{f} \in \mathcal{H}$,
$$\mathbb{E}(\langle \mathbf{w}|\vec{e}\rangle \overline{\langle \mathbf{w}|\vec{f}\rangle}) = \langle \vec{e}|\vec{f}\rangle$$

Unfortunately, $\mathbf{w}$ is not a random vector in $\mathcal{H}$ unless $\dim \mathcal{H} < \infty$[3] , but only a random *Schwartz distribution.*

We have nevertheless the following usefull proposition:

Proposition 2 *If A is an Hilbert-Schmidt operator on $\mathcal{H}$, the random field $A\mathbf{w}$ is almost surely in $\mathcal{H}$.*

[3]If $\mathbf{w}$ were a vector in $\mathcal{H}$, we would have $\mathbf{w} = \sum \langle \mathbf{w}|\vec{e_j}\rangle \vec{e_j}$ for any orthonormal basis $(\vec{e_j})$ and we see that

$$\mathbb{E}(\|\mathbf{w}\|^2) = \sum \mathbb{E}(\langle \mathbf{w}|\vec{e_j}\rangle)^2 = \dim \mathcal{H} \ .$$

Proof.–

$\mathbb{E}(\langle A\mathbf{w}|A\mathbf{w}\rangle) = \mathbb{E}(\langle A^\star A\mathbf{w}|\mathbf{w}\rangle) = \text{Trace}(A^\star A)$ which is finite, by definition, exactly for Hilbert-Schmidt operators.

□

4.3 Examples

Example 4.1 Stationary noise on the real line: *let us take a random field on the real line which is given by the convolution product of the scalar white noise w with a fixed smooth compactly supported function F: $f = F \star w$. Then f is stationary: it means that the correlation kernel $K(t,t') = \mathbb{E}(f(t)\overline{f(t')})$ is a fonction of $t - t'$. On the level of Fourier transforms $\hat{f} = \hat{F}\hat{w}$, $\mathbb{E}(\hat{f}(\omega)\overline{\hat{f}(\omega')}) = |\hat{F}|^2(\omega)\delta(\omega-\omega')$ and the positive function $|\hat{F}|^2(\omega)$ is usually called the* power spectrum *of the stationary noise.*

Example 4.2 *If X is a d-dimensional bounded domain. Let us denote the Sobolev spaces on X by $H^s(X)$. If $P : L^2(X) \to H^s(X)$ with $s > d/2$, Pw is in $L^2(X)$.*

Example 4.3 Brownian motions: *if $X = \mathbb{R}$, w is the derivative of the Brownian motion: if $b(t) = \int_0^t w(s)ds$, $b : [0,+\infty[\to \mathbb{R}$ is the Brownian motion which is in $L^2([0,T])$ for all finite T.*

Example 4.4 *If X is a smooth compact manifold or domain and P is smoothing, meaning that P is given by an integral smooth kernel*

$$Pf(x) = \int_X [P](x,y)f(y)|dy| \ ,$$

$F = Pw$ is a random smooth function. Its correlation kernel

$$C(x,y) := \mathbb{E}(F(x)\overline{F(y)})$$

is given by:

$$[PP^\star](x,y) = \int_X [P](x,z)\overline{[P](y,z)}|dz| \ .$$

Example 4.5 Random vector fields: *let us consider $\mathcal{H} = L^2(X,\mathbb{R}^N)$. For example, in the case of elasticity, X is a 3D domain and $N = 3$. The field here are just fields of infinitesimal deformations (a vector field).*

4.4 Modelling the noise using pseudo-differential operators

The main goal of the present section is to build natural random fields which are non homogeneous with small distances of correlation of the order of $\varepsilon \to 0$. The noise is non homogeneous in X, but could also be non isotropic w.r. to directions.

4.4.1 Noises from pseudo-differential operators

It is therefore natural to take for noise on a manifold Z the image of an homogeneous white noise by a pseudo-differential operator N of smooth compactly supported symbol $n(z, \zeta)$. The correlation $C(z, z')$ will then be given as the Schwartz kernel of $NN^\star$ which is a ΨDO of principal symbol $|n|^2$. We have

$$C(z, z') \sim \varepsilon^{-d} \tilde{|n|^2} \left(z, \frac{z' - z}{\varepsilon} \right) ,$$

while $|n|^2(z, \zeta)$ is the "power spectrum" of the noise at the point z.

This construction gives smooth random fields which can be localized in some very small domains of the manifold Z, which are non isotropic and which have small distance of correlations. Moreover it will allow to use technics of microlocal analysis with the small parameter given by ε.

4.4.2 Power spectrum and Wigner measures

Definition 4 *If $f = (f_\varepsilon)$ is a suitable family of functions on Z, the* Wigner measures W_f^ϵ *of f are the signed measures on the phase space $T^\star Z$ defined by*

$$\int a dW_f^\epsilon := \langle \text{Op}_\varepsilon(a) f_\varepsilon | f_\varepsilon \rangle \ .$$

The measures dW_f^ε are the phase space densities of energy *of the functions f_ε.*

We now define:

Definition 5 *The* power spectrum *of the random field $f = (f_\varepsilon)$ is the phase space density P_f^ε defined by:*

$$P_f^\varepsilon = \mathbb{E}(W_f^\varepsilon) \ :$$

the power spectrum of a random field is its average Wigner measure.

Proposition 3 *The power spectrum P of $f_\varepsilon = \text{Op}_\varepsilon(n) w$, satisfies:*

$$P_f^\varepsilon \sim (2\pi\varepsilon)^{-d} |n|^2(x, \xi) |dx d\xi| \ .$$

Proof.–

Let us put $N = \text{Op}_\varepsilon(n)$, we have

$$\langle \text{Op}_\varepsilon(a) N w | N w \rangle = \langle N^\star \text{Op}_\varepsilon(a) N w | w \rangle$$

and $\mathbb{E}(\langle A w | w \rangle) = \text{trace}(A)$. We get

$$\mathbb{E}(\int a dW_f^\varepsilon) = \text{trace}(N^\star \text{Op}_\varepsilon(a) N)$$

which can be evaluated using the ΨDO calculus as

$$\mathbb{E}(\int a dW_f^\varepsilon) \sim (2\pi\varepsilon)^{-d} \int a |n|^2 dx d\xi \ .$$

□

4.4.3 Space-time noises

If $Z = X \times \mathbb{R}$ is the space-time, we will take our noise as before $f = Lw$; we will assume the noise *homogeneous in time,* the symbol l of L is assumed to be given by $l(x, \xi, \omega)$.

In this case, the correlation is given by:

$$K(x, y; t) = [LL^\star](x, y; 0, t) \tag{11}$$

which is the Schwartz kernel of a ΨDO of principal symbol $ll^\star(x, \xi; \omega)$.

4.5 Equipartition and polarizations

If $\mathbf{f}_\varepsilon : Z \to \mathbb{R}^q$ is a vector valued family of functions, the Wigner measure is matrix valued: if $\mathbf{a} : T^\star X \to \mathrm{Sym}(\mathbb{R}^q)$, we define

$$\int \mathbf{a} dW_{\mathbf{f}}^\varepsilon := \langle \mathrm{Op}_\varepsilon(\mathbf{a})\mathbf{f}_\varepsilon | \mathbf{f}_\varepsilon \rangle$$

If we have, for each point (z, ζ) of $T^\star Z$ a splitting $\mathbb{R}^q = \oplus E_j(z, \zeta)$ with projectors P_j, there exists canonical measures μ_j defined by

$$\int \mathbf{a} d\mu_j = \mathrm{Trace}\ (\mathbf{a} P_j) |dz d\zeta|\ .$$

If the E_j's are defined by the polarizations of an Hamiltonian $\hat{H}$, the *micro-canonical Liouville measures* are given by

$$\int \mathbf{a} dL_E := \sum_j \int_{\lambda_j \le E} \mathbf{a} d\mu_j\ .$$

A (random) state of energy E of $\hat{H}$ is said to equipartited if its (average) Wigner measure converges to the microcanonical measure. If P_E is the spectral projector of $\hat{H}$ over the modes of energies less than E, the random state $P_E w$ is equipartited.

5 High frequency limit of the correlation: Schrödinger equations

The main result is easier to derive in the case of a scalar field gouverned by a wave equation which gives the first order derivative of the field: it is a generalization of the Schrödinger equation.

5.1 Assumptions

Let us start with the semi-classical Schrödinger like equation

$$\frac{\varepsilon}{i}u_t + \hat{H}_\varepsilon u = \frac{\varepsilon}{i}f$$

where

- $\hat{H}_\varepsilon = \frac{\varepsilon}{i}\hat{H}$ is an $\varepsilon-$**pseudo-differential operator:**

  $$\hat{H}_\varepsilon := \mathrm{Op}_\varepsilon(H_0 + \varepsilon H_1)$$

  with

  - The **principal symbol** $H_0(x,\xi) : T^\star X \to \mathbb{R}$ which gives the classical ("rays") dynamics:

    $$\frac{dx_j}{dt} = \frac{\partial H_0}{\partial \xi_j},\ \frac{d\xi_j}{dt} = -\frac{\partial H_0}{\partial x_j},\ 1 \leq j \leq d\ .$$

    We will denote by ϕ_t the flow of the previous vector field.
  - For technical reasons (see Lemma 3), we assume that H_0 is **elliptic at infinity:** $\lim_{\xi\to\infty} H_0(x,\xi) = +\infty$. We define $\hat{H}_0 = \mathrm{Op}_\varepsilon(H_0)$ and the unitary group $U(t) = \exp(-it\hat{H}_0/\varepsilon)$. We define also $\hat{H}_1 = \mathrm{Op}_\varepsilon(H_1)$.
  - The **sub-principal symbol** $H_1(x,\xi)$ admits some positivity property which controles the attenuation: there exists $k > 0$, such that

    $$\Im H_1 \leq -k\ .$$

- The **random field** f is given by $f = \mathrm{Op}_\varepsilon(l(x,\xi,\omega))w$ with w the white noise on $X \times \mathbb{R}$ and with l smooth, compactly supported w.r.t. to (x,ξ) and whose Fourier transform w.r.t. to ω is compactly supported. The power spectrum of f is $(2\pi\varepsilon)^{-(d+1)}|l|^2(x,\xi,\omega)$.

The previous assumptions will be used everywhere inside Section 5.

5.2 Subprincipal symbols and attenuation

Lemma 2 *Under the assumptions of Section 5.1, we have:*

a) There exists $c > 0$ so that, for $|t| \leq c|\log(\varepsilon)|$, $\Omega(t) = U(t)Y(t)$ with $Y(t)$ a ΨDO of principal symbol $\exp(-i\int_0^t H_1(\phi_s(x,\xi))ds)$.

b) for all $t \geq 0$, the estimates

$$\|Y(t)\| = O(e^{-k't})$$

with $\|.\|$ the operator norm in $L^2(X)$ and $0 < k' < k$.

Proof.–

a) We start with $\Omega(t) = U(t)Y(t)$ and hence

$$Y'(t) + iU(-t)\hat{H}_1 U(t)Y(t) = 0 \ .$$

Using Egorov's Theorem with logarithmic times, as in Section 5.4, we get

$$Y'(t) + i\hat{H}_1(t)Y(t) = 0$$

with $\hat{H}_1(t)$ a ΨDO of principal symbol $iH_1(\phi_t(x,\xi))$. It is then enough to start with a formal expansion in ε of the symbol of $Y(t)$ and to work by induction on the powers of ε.

b) We have

$$\frac{d}{dt}\langle v(t)|v(t)\rangle = 2\Re\langle v(t)| - i\hat{H}_1(t)v(t)\rangle$$

and we use Gårding inequality (see [7]): if $a \geq 0$, $\mathrm{Op}_\varepsilon(a) \geq -C$ for any $C > 0$ and ε small enough.

□

5.3 Some lemmas

Lemma 3 *If $A = \mathrm{Op}_\varepsilon(a)$ with a compactly supported, the operator $B = \exp(it\hat{H})A$ is a ΨDO of principal symbol $b = \exp(itH_0)a$.*

Proof.–

Let us choose a function $\chi \in C_o^\infty(\mathbb{R},\mathbb{R})$ so that $\chi(H_0)$ is equal to 1 in some neighbourhood of the support of a. We have

$$B = \exp(it\hat{H}_0)\left(\chi(\hat{H}_0) + (1-\chi(\hat{H}_0))\right)C$$

with $C = Y(-\varepsilon t)A$ is a compactly supported ΨDO of principal symbol a. The previous expression of B splits into 2 terms $B = I + II$. The first one rewrites $I = \Phi(\hat{H}_0)C$ with $\Phi = e^{it.}\chi \in C_o^\infty$ which is a ΨDO of principal symbol $e^{itH_0}a$ thanks to the functional calculus of elliptic self-adjoint ΨDO's (see [7], Chap. 8). The second one is smoothing.

□

The following Lemma follows directly from the definitions:

Lemma 4 *If $K(x,y,t)$ is the correlation kernel of f, then $\hat{K}(\sigma_1)$, the operator whose kernel is $K(.,.,\varepsilon\sigma_1)$, is a ΨDO which vanishes for $|\sigma_1| \geq C$ and whose principal symbol is $(2\pi\varepsilon)^{-1}\int e^{i\sigma_1\omega}|l|^2(x,\xi,\omega)d\omega$.*

From Equation (4), we get, for $\tau > 0$ and $\varepsilon \leq C/\tau$:

$$C_{A,B}(\tau) = [\Omega(\tau) \int_0^\infty ds \Omega(s) \mathcal{L} \Omega^\star(s)](A,B) \tag{12}$$

with

$$\mathcal{L} = \varepsilon \int_{|\sigma_1| \leq C} \Omega(-\varepsilon\sigma_1) \hat{K}(\sigma_1) d\sigma_1 \ .$$

Lemma 5 *$\mathcal{L}$ is a ΨDO of principal symbol $|l|^2(x,\xi,-H_0(x,\xi))$.*

Proof.–

The result follows from Lemma 3 and the value of the symbol of $\hat{K}(\sigma_1)$ given in Lemma 4, by integrating w.r. to σ.

□

5.4 Applying Egorov Theorem

We can apply Egorov Theorem:

Theorem 2 (Egorov's Theorem) *If $A = \mathrm{Op}_\varepsilon(a)$, $U(-t)AU(t) = A_t$ where A_t is a ΨDO of principal symbol $a \circ \phi_t$ with ϕ_t is the Hamiltonian flow of H_0.*

We will need a *large time* estimation in the Egorov's Theorem. Such estimates are provided in the nice paper [1]: Egorov's Theorem still works under suitable hypothesis on H_0 for time bounded by $c|\log \varepsilon|$ where c is related to the Liapounov exponent of the classical dynamics. Such time is called *Eherenfest time* and will be denoted by $T_{\text{Ehrenfest}}$.

We get the main result:

Theorem 3 *With the assumptions of Section 5.1, the correlation is given, for $\tau > 0$, by*

$$C_{A,B}(\tau) = [\Omega(\tau) \circ \Pi)](A,B)$$

where $\Pi = \mathrm{Op}_\varepsilon(\pi) + R$ with:

$$\pi(x,\xi) = \int_{-c|\log\varepsilon|}^0 \exp\left(2\int_t^0 \Im(H_1)(\phi_s(x,\xi))ds\right) |l|^2 \left(\phi_t(x,\xi), -H_0(x,\xi)\right) dt \ .$$

and R the remainder term is "$O(\varepsilon^\alpha)$". More precisely, let us consider $C_{A,B}(\tau)$ as the Schwartz kernel of an operator $\hat{C}(\tau)$. This operator is Hilbert-Schmidt [4] with an Hilbert-Schmidt norm of the order of $\varepsilon^{-d/2}$. We have

$$\|\hat{R}\|_{\mathrm{H-S}} = O(\varepsilon^{\alpha - d/2}) \ .$$

[4] An Hilbert-Schmidt operator A is an operator whose Schwartz kernel $[A](x,y)$ is in $L^2(X \times X)$ and the Hilbert-Schmidt norm $\|A\|_{\mathrm{H-S}}$ of A is the L^2 norm of $[A]$.

Proof.–

We start from

$$\Pi = \int_0^\infty \Omega(s)\mathcal{L}\Omega^\star(s)ds \tag{13}$$

as given by Equations (4) and (12). We have $\Omega(s) = U(s)Y(s)$ with $Y(s)$ is a ΨDO of principal symbol $\exp(-i\int_0^s H_1(\phi_u(x,\xi))du)$ and

$$\Omega(s)\mathcal{L}\Omega^\star(s) = U(s)\left(Y(s)\mathcal{L}Y^\star(s)\right)U(-s)$$

to which we want to apply Egorov's Theorem. There is a technical problem due to the fact that Egorov's Theorem is only valid untill Ehrenfest times. We split the integral (13) into two parts $\Pi = \int_0^{T_{\rm Ehrenfest}} + \int_{T_{\rm Ehrenfest}}^\infty$. The first term is estimated using Egorov's Theorem for large times. An upper bound for second part follows from the decay estimate of $\|\Omega(t)\|$ given in Lemma 2 and the estimates $\|{\rm Op}_\varepsilon(a)\|_{\rm H-S} = O(\varepsilon^{-d/2})$.

□

Assuming still $\tau > 0$, we see that the correlation $C_{A,B}(\tau)$ is close to the kernel of a Fourier integral operator associated to the canonical transformation ϕ_τ. It is given as a sum over all classical trajectories γ from B to A in time τ of Cauchy data (B, ξ_B) with $H_0(B,\xi_B) = -\omega$ for which the backward trajectories crosses the support of the power spectrum $ll^\star(.,.,\omega)$. If γ is such a trajectory and B and A are non conjugated along it, this contribution is given by the well known Van Vleck formula[5] multiplied by $\pi(B,\xi_B)$.

Corollary 1 *Let K be the support of $l(x,\xi,-H_0(x,\xi))$ and K_∞ the smallest closed set of $T^\star X$ invariant by the Hamiltonian flow of H_0 and containing K. The Hamiltonian H_0 restricted to K_∞ can be recovered from the knowledge of $\hat{C}(\tau)$ for $0 < |\tau| \leq \tau_0$.*

In particular, if there exists (x,ξ) with $H_0(x,\xi) = E$ and $l(x,\xi,-E) \neq 0$ and if ϕ_t is ergodic on $H_0^{-1}(E)$, then we can recover the flow ϕ_t on $H_0^{-1}(E)$.

5.5 Remarks

We would like to extend the previous approach to the general case, i.e. to $N > 1$. There are 2 difficulties to overcome:

- One has to extend Lemma 5 to the case of systems
- Egorov Theorem is no more true, but remains true on average as in Lemma 6

[5]The Van Vleck formula expresses the propagator $P(\tau, A, B)$ as a sum of $p_\gamma = (2\pi i\varepsilon)^{-d/2} a_\gamma(\varepsilon)\exp(iS(\gamma)/\varepsilon)$ with $a_\gamma(\varepsilon)$ a formal power series in ε with a first term explicitely computable

6 High frequency limit of the correlation: wave equations

We want to derive results similar to those of Theorem 3 in the case of the scalar wave equation ($N = 2$) given as follows:

$$u_{tt} + 2au_t - \Delta u = f$$

where $a > 0$ is *constant* and $-\Delta$ is the Laplace-Beltrami operator on a smooth complete Riemannian manifold X.

We will assume that the source noise f is given by $f = Lw$ with w a white noise on $X \times \mathbb{R}$ and $L = \mathrm{Op}_\varepsilon(l)$ with $l = l(x, \xi)$ smooth, with compact support, and independant of ω. We will moreover assume time reversal symmetry of the noise, namely $l(x, -\xi) = l(x, \xi)$; it implies that the kernel $[L](x, y)$ is real valued.

6.1 Direct derivation

Let us introduce $P = \sqrt{-\Delta - a^2}$, the causal solution of Equation (3) is given, as already used in Section 2.3, by

$$u(x,t) = \int_0^\infty ds \int_X G(s,x,y)u(t-s,y)|dy|$$

with the Green function

$$G(t,x,y) = Y(t)e^{-at}\frac{\sin tP}{P} \ .$$

By a direct calculation and denoting by $\hat{K} = P^{-1}LL^\star P^{-1}$ with $L = \mathrm{Op}_\varepsilon(l)$, we get, for $\tau > 0$,

$$C_{A,B}(\tau) = \frac{1}{2}e^{-a\tau}\left[\Re\left(e^{-i\tau P}\int_{-\infty}^0 e^{2as}e^{isP}\hat{K}\left(e^{-isP} - e^{isP}\right)ds\right)\right](A,B) \ . \qquad (14)$$

This integral splits into 2 parts, the first one can be asymptotically computed using Egorov's Theorem as in the proof of Theorem 3, while the second is smaller by the

Lemma 6 *If $\hat{A}$ is a compactly supported ΨDO of order 0 and if*

$$J := \int_{-\infty}^0 e^{2as}e^{isP}\hat{A}e^{isP}ds \ ,$$

we have

$$\|J\|_{\mathrm{H-S}} = O(\varepsilon^{\gamma-\frac{d}{2}}) \ ,$$

with some non negative γ.

This is proved by cuting the integral into 2 pieces as in the proof of Theorem 3, using Egorov's Theorem and integrating by part.

The final result is:

Theorem 4 *With the previous assumptions, we have, for $\tau > 0$:*

$$C_{A,B}(\tau) = \frac{\varepsilon^2}{2} e^{-a\tau} \left[\cos \tau P \circ \Pi \right] (A, B) + R$$

with Π a ΨDO of principal symbol

$$\pi(x,\xi) = \|\xi\|^{-2} \int_{-\infty}^{0} e^{2as} |l|^2 (\phi_s(x,\xi)) ds \ ,$$

and $\|R\|_{\mathrm{H-S}} = O(\varepsilon^{\gamma+2-d/2})$ with $\gamma > 0$.

Remark 1 *The prefactor ε^2 is just here because we have to pass from the ΨDO's without small parameter P to an $\varepsilon-\Psi DO$: $P = \varepsilon^{-1} \mathrm{Op}_\varepsilon(\|\xi\|) + l.o.t.$.*

6.2 Using the general formalism

We can again start from Equation (6). We are reduced to calculate the correlation between $u(A,t)$ and $u(B,t)$ which is given by

$$C^{11}_{A,B}(\tau) = \left[\Omega(\tau) \Pi \right]^{11} (A, B) \ .$$

We should first put the wave equation as a first order semi-clasical equation. We put

$$\mathbf{u} = \begin{pmatrix} u \\ \varepsilon(u_t + au) \end{pmatrix} \ .$$

and

$$-i\varepsilon \hat{H} = \begin{pmatrix} -ia & 1 \\ i\varepsilon^2(\Delta + a^2) & -ia \end{pmatrix} \ .$$

We get, by definig $P = \sqrt{-(\Delta + a^2)}$,

$$\Omega(t) = e^{-at} \begin{pmatrix} \cos tP & \frac{\sin tP}{\varepsilon P} \\ -\varepsilon P \sin tP & \cos tP \end{pmatrix} \ .$$

Moreover, if $f = \mathrm{Op}_\varepsilon(m) w$,

$$\mathbf{f} = \begin{pmatrix} f \\ 0 \end{pmatrix} = \mathrm{Op}_\varepsilon(l) w$$

with

$$l = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} \ .$$

We see that the computation is less easy $\cdots$

7 What's left?

There are still several problem to discuss:

- The case of vector valued wave equations with several polarizations.
- The precise study of autocorrelations: as mentionned to me by U. Smilanski, the autocorrelations can be usefull in order to learn more about the source of the noise $\mathbf{f}$.
- The case of surface waves: effective Hamiltonains and the associated inverse spectral problems will be discussed in [5].
- The case of the source noise located on the boundary:

$$\begin{cases} u_{tt} + au_t - \Delta u = 0 \\ u_{\partial X} = f \end{cases}$$

  will be discussed in [6].

8 Random scattered waves

In this last independant section, we will revisit what was maybe the starting point of this story by Keiiti Aki in the fifties: he wanted to measure the speed of propagation of seismic plane waves by averaging over the incidence directions. It turns out that we get nice formulae even for non homogoeneous media.

8.1 Introduction

Let us consider the propagation of waves outside a compact domain D in the Euclidian space $\mathbb{R}^d$. Let us put $\Omega = \mathbb{R}^d \setminus D$. We can assume for example Neumann boundary conditions. We will denote by Δ_Ω the previous self-adjoint operator. So our stationary wave equation is the Helmoltz equation $\Delta_\Omega f + k^2 f = 0$ with the boundary conditions. We consider a bounded intervall $I = [E_-, E_+] \subset]0, +\infty[$ and the Hilbert subspace $\mathcal{H}_I$ of $L^2(\Omega)$ which is the image of the spectral projector P_I of our Laplace operator Δ_Ω.

Let us compute the integral kernel $\Pi_I(x, y)$ of P_I defined by:

$$P_I f(x) = \int_\Omega \Pi_I(x, y) f(y) |dy|$$

into 2 different ways:

1. From general spectral theory
2. From scattering theory.

Taking the derivatives of $\Pi_I(x,y)$ w.r. to E_+, we get a simple general and exact relation between the correlation of scattered waves and the Green's function confirming the calculations from [14] in the case where D is a disk.

8.2 $\Pi_I(x,y)$ from spectral theory

Using the resolvent kernel (Green's function) $G(k,x,y) = [(k^2+\Delta_\Omega)^{-1}](x,y)$ for $\Im k > 0$ and the Stone formula, we have:

$$\Pi_I(x,y) = \frac{2}{\pi}\Im\left(\int_{k_-}^{k_+} G(k+i0,x,y)kdk\right)$$

Taking the derivative w.r. to k_+ of $\Pi_{[E_-,k^2]}(x,y)$, we get

$$\frac{d}{dk}\Pi_{[E_-,k^2]}(x,y) = \frac{2k}{\pi}\Im(G(k+i0,x,y)) \ . \tag{15}$$

8.3 Short review of scattering theory

They are many references for scattering theory: for example [12].

Let us define the plane waves

$$e_0(x,\mathbf{k}) = e^{i<\mathbf{k}|x>} \ .$$

We are looking for solutions

$$e(x,\mathbf{k}) = e_0(x,\mathbf{k}) + e^s(x,\mathbf{k})$$

of the Helmoltz equation in Ω where e^s, the scattered wave satisfies the so-called Sommerfeld radiation condition:

$$e^s(x,\mathbf{k}) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}\left(e^\infty(\frac{x}{|x|},\mathbf{k}) + O(\frac{1}{|x|})\right), \ x\to\infty \ .$$

The complex function $e^\infty(\hat{x},\mathbf{k})$ is usually called the *scattering amplitude*.

It is known that the previous problem admits an unique solution. In more physical terms, $e(x,\mathbf{k})$ is the wave generated by the full scattering process from the plane wave $e_0(x,\mathbf{k})$. Moreover we have a generalized Fourier transform:

$$f(x) = (2\pi)^{-d}\int_{\mathbb{R}^d}\hat{f}(\mathbf{k})e(x,\mathbf{k})|d\mathbf{k}|$$

with

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^d}\overline{e(y,\mathbf{k})}f(y)|dy| \ .$$

From the previous generalized Fourier transform, we can get the kernel of any function $\Phi(-\Delta_\Omega)$ as follows:

$$[\Phi(-\Delta_\Omega)](x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d}\Phi(k^2)e(x,\mathbf{k})\overline{e(y,\mathbf{k})}|d\mathbf{k}| \ . \tag{16}$$

8.4 $\Pi_I(x,y)$ from scattering theory

Using Equation (16) with $\Phi = 1_I$ the characteristic functions of some bounded intervall I, we get:

$$\Pi_I(x,y) = (2\pi)^{-d} \int_{E_- \leq \mathbf{k}^2 \leq E_+} e(x,\mathbf{k})\overline{e(y,\mathbf{k})}|d\mathbf{k}| \ .$$

Using polar coordinates and defining $|d\sigma|$ as the usual measure on the unit $(d-1)-$dimensional sphere, we get:

$$\Pi_I(x,y) = (2\pi)^{-d} \int_{E_- \leq k^2 \leq E_+} \int_{\mathbf{k}^2=E} e(x,\mathbf{k})\overline{e(y,\mathbf{k})} k^{d-1} dk |d\sigma| \ .$$

We will denote by σ_{d-1} the total volume of the unit sphere in $\mathbb{R}^d$: $\sigma_0 = 2,\ \sigma_1 = 2\pi,\ \sigma_2 = 4\pi, \cdots$.

Taking the same derivative as before, we get:

$$\frac{d}{dk}\Pi_{[E_-,k^2]}(x,y) = \frac{k^{d-1}}{(2\pi)^d} \int_{\mathbf{k}^2=E} e(x,\mathbf{k})\overline{e(y,\mathbf{k})}|d\sigma| \ .$$

This integral can be interpreted, using the correlation $C_E^{\text{scatt}}(x,y)$ of random scattered waves of energy E defined by

$$C_E^{\text{scatt}}(x,y) = \frac{1}{\sigma_{d-1}} \int_{\mathbf{k}^2=E} e(x,\mathbf{k})\overline{e(y,\mathbf{k})}|d\sigma|,$$

as

$$\frac{d}{dk}\Pi_{[E_-,k^2]}(x,y) = \frac{k^{d-1}\sigma_{d-1}}{(2\pi)^d} C_E^{\text{scatt}}(x,y) \ . \tag{17}$$

8.5 Correlation of scattered plane waves and Green's function: the scalar case

From Equations (15) and (17), we get:

$$\frac{k^{d-1}\sigma_{d-1}}{(2\pi)^d} C_E^{\text{scatt}}(x,y) = \frac{2k}{\pi}\Im(G(k+i0,x,y)) \ .$$

Hence

$$C_E^{\text{scatt}}(x,y) = \frac{2^{d+1}\pi^{d-1}}{k^{d-2}\sigma_{d-1}}\Im(G(k+i0,x,y)) \ .$$

For later use, we put

$$\gamma_d(k) = \frac{2^{d+1}\pi^{d-1}}{k^{d-2}\sigma_{d-1}} \ . \tag{18}$$

8.6 The case of elastic waves

We will consider the vectorial stationary elastic wave equation in the domain Ω:

$$\hat{H}\mathbf{u} - \omega^2 \mathbf{u} = 0,$$

with symmetric boundary conditions, where

$$\hat{H}\mathbf{u} = -a\ \Delta \mathbf{u} - b \text{ grad div}\mathbf{u}\ .$$

where a and b are constant:

$$a = \frac{\mu}{\rho},\ b = \frac{\lambda + \mu}{\rho}$$

with $\lambda,\ \mu$ the Lamé's coefficients and ρ the density of the medium.

- *The case* $\Omega = \mathbb{R}^d$

  We want to derive the spectral decomposition of $\hat{H}$ from the Fourier inversion formula. Let us choose, for $\mathbf{k} \neq 0$, by $\hat{\mathbf{k}}, \hat{\mathbf{k}}_1, \cdots, \hat{\mathbf{k}}_{d-1}$ an orthonormal basis of $\mathbb{R}^d$ with $\hat{\mathbf{k}} = \frac{\mathbf{k}}{k}$ such that these vectors depends in a measurable way of $\mathbf{k}$. Let us introduce $P_P^{\mathbf{k}} = \hat{\mathbf{k}}\hat{\mathbf{k}}^\star$ the orthogonal projector onto $\hat{\mathbf{k}}$ and $P_S^{\mathbf{k}} = \sum_{j=1}^{d-1} \hat{\mathbf{k}}_j \hat{\mathbf{k}}_j^\star$ so that $P_P + P_S = \text{Id}$. Those projectors correspond respectively to the polarizations of $P-$ and $S-$waves.

  We have

  $$\Pi_I(x,y) = (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega \left((a+b)^{-d/2} \int_{k^2 = \omega^2/(a+b)^2} e^{i\mathbf{k}(x-y)} P_P^{\mathbf{k}} d\sigma + a^{-d/2} \int_{k^2 = \omega^2/a^2} e^{i\mathbf{k}(x-y)} P_S^{\mathbf{k}} d\sigma \right)\ .$$

  using the plane waves

  $$e_P^O(x,\mathbf{k}) = e^{i\mathbf{k}x}\hat{\mathbf{k}}$$

  and

  $$e_{S,j}^O(x,\mathbf{k}) = e^{i\mathbf{k}x}\hat{\mathbf{k}}_j$$

  we get the formula:

  $$\Pi_I(x,y) = (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega \left((a+b)^{-d/2} \int_{k^2 = \omega^2/(a+b)^2} e_P^O(x,\mathbf{k})(e_P^O(y,\mathbf{k}))^\star d\sigma + a^{-d/2} \sum_{j=1}^{d-1} \int_{k^2 = \omega^2/a^2} e_{S,j}^O(x,\mathbf{k})(e_{S,j}^O(y,\mathbf{k}))^\star d\sigma \right)\ .$$

- *Scattered plane waves*

  There exists scattered plane waves

  $$e_P(x,\mathbf{k}) = e_P^O(x,\mathbf{k}) + e_P^s(x,\mathbf{k})$$

  $$e_{S,j}(x,\mathbf{k}) = e_{S,j}^O(x,\mathbf{k}) + e_{S,j}^s(x,\mathbf{k})$$

  satisfying the Sommerfeld condition and from which we can deduce the spectral decomposition of $\hat{H}$.

- *Correlations of scattered plane waves and Green's function*

  Following the same path as for scalar waves, we get an identity which holds now for the full Green's tensor $\Im \mathbf{G}(\omega + iO, x, y)$:

  $$\Im \mathbf{G}(\omega + iO, x, y) = \gamma_d(\omega) \Big((a+b)^{-d/2} \int_{k^2=\omega^2/(a+b)^2} e_P(x, \mathbf{k})(e_P(y, \mathbf{k}))^\star d\sigma + a^{-d/2} \sum_{j=1}^{d-1} \int_{k^2=\omega^2/a^2} e_{S,j}(x, \mathbf{k})(e_{S,j}(y, \mathbf{k}))^\star d\sigma \Big) \ ,$$

  with $\gamma_d(\omega)$ defined by Equation (18).

  This formula expresses the fact that the correlation of scattered plane waves randomized with the appropriate weights is proportional to the Green's tensor.

9 Appendix: A short review about pseudo-differential operators

We will define pseudo-differential operator (ΨDO's) on $\mathbb{R}^d$. ΨDO's on manifold are defined locally by the same formulae. More details can be foud in [4, 7, 8, 19].

Definition 6 • *The space Σ_k is the space of smooth functions $p : T^\star\mathbb{R}^d \to \mathbb{C}$ which satisfies*

$$\forall \alpha, \beta,\ |D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{k-|\beta|} \ .$$

- *A symbol of order m and degree l is a family of functions*

  $$p_\varepsilon : T^\star\mathbb{R}^d \to \mathbb{C}$$

  which admits an asymptotic expansion

  $$p_\varepsilon \sim \sum_{j=0}^{\infty} \varepsilon^{j+m} p_j(x,\xi)$$

  with $p_j \in \Sigma^{l-j}$. We will denote this space by $S_{m,l}$.

Definition 7 *An ε-pseudo-differential operator P (a ΨDO) of order m and degree l on $\mathbb{R}^d$ is given locally by the kernel*

$$[P](z,z') = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} e^{i\langle z-z'|\zeta\rangle/\varepsilon} p_\varepsilon\left(\frac{z+z'}{2}, \zeta\right) |d\zeta|$$

where $p_\varepsilon(z,\zeta)$, the so-called (total) symbol of P, is in $S_{m,l}$.

We will denote $P = \mathrm{Op}_\varepsilon(p_\varepsilon)$.

The kernel of P is then given by:

$$[P](z, z') = \varepsilon^{-d}\tilde{p}\left(\frac{z + z'}{2}, \frac{z' - z}{\varepsilon}\right)$$

with $\tilde{p}$ the partial Fourier transform of $p_\varepsilon(z, \zeta)$ w.r. to ζ. Very often, one is only able to compute the symbol p_0 which is called the *principal symbol* of P.

The most basic fact about ΨDO's is the fact they can be composed: if $P = \mathrm{Op}_\varepsilon(p)$ and $Q = \mathrm{Op}_\varepsilon(q)$, we have $PQ = \mathrm{Op}_\varepsilon\left(pq + O(\varepsilon)\right)$.

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