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Multiplicities of Eigenvalues and Tree-Width of Graphs

Yves Colin de Verdière

Institut Fourier, UMR 5582 (UJF–CNRS), B.P. 74, F-38402 St Martin d'Hères Cedex, France
E-mail: yves.colin-de-verdiere@ujf-grenoble.fr

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Using multiplicities of eigenvalues of elliptic self-adjoint differential operators on graphs and transversality, we construct some new invariants of graphs which are related to tree-width. © 1998 Academic Press

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Contents

1. INTRODUCTION AND RESULTS

In this paper, we present some extensions of our invariant $\mu(G)$ introduced in [4] for a finite graph $G=(V, E)$. (See also [5] for english translation.)

Definition 1. O_G is the set of real symmetric $V\times V$ matrices with entries $a_{i,j}$ such that $a_{i,j}<0$ if $\{i, j\}\in E$ and $a_{i,j}=0$ if $i\neq j$ and $\{i, j\}\notin E$.

An operator $A\in O_G$ has a non-degenerate first eigenvalue λ_1 (ground-state) if G is connected (Perron and Frobenius). The invariant $\mu(G)$ is defined using multiplicities of the second eigenvalue λ_2 for some real symmetric matrix $A\in O_G$. Moreover, $\mu(G)$ is related to the genus of G: $\mu(G)\leqslant 3$ if and only if G is planar and more generally $\mu(G)\leqslant 4\,\mathrm{genus}(G)+3$. Recently, Lovász and Schrijver [20] proved that linklessly

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embeddable graphs are characterized by $\mu(G) \leqslant 4$. See the book [8] as well as [7, 10] for surveys.

What kind of extensions of these properties hold for self-adjoint (complex) matrices related to G? Such Hermitian matrices are obtained if we discretize Schrödinger operators with magnetic fields using the method of finite elements. We will use the same language in the discrete (differential operators on graphs) and continuous cases because it was our starting point and we want to insist on the similarities between the cases.

The eigenvalues of such operators can be very degenerate. Even for the Schrödinger operator H with a constant magnetic field $B > 0$ in the plane

$$H = -\left(\partial_x - \frac{iB}{2}\, y\right)^2 - \left(\partial_y + \frac{iB}{2}\, x\right)^2$$

the spectrum of H (whose elements are called Landau levels in physics) is the set of eigenvalues $\sigma(H) = \{E_n = (2n+1)|B| \mid n \in \mathbb{N}\}$ and the eigenspaces $F_n = \ker(H - E_n \,\mathrm{Id})$ are infinite dimensional (see [3, p. 756–772]).

The case of F_0 *works as follows: rewriting* $H = \Omega^* \Omega + B\,\mathrm{Id}$ *with* $\Omega = 2/i(\partial/\partial \bar{z} + z/4)$, *we see that* $H \geqslant B\,\mathrm{Id}$ *and* $F_0 = \ker \Omega$ *is the infinite dimensional space of* L^2 *functions* $\psi(z) = e^{-|z|^2/4} \varphi(z)$, *where* φ *is holomorphic.*

Therefore, we cannot expect an upper bound for multiplicities in terms of the genus of G; see [12].

The main idea of our paper is to compare G with a tree: if T is a tree and A is a self-adjoint elliptic operator on T (see Definition 2), a gauge transformation (conjugation by some diagonal unitary matrix) transforms A into an operator B in O_T. We can now apply the Perron–Frobenius theorem to B and show that the ground-state is non-degenerate. We will not use this argument, because we are aiming at a more general result (on arbitrary graphs and arbitrary eigenvalues).

Let us give more precise definitions and state the main results of the paper. If $G = (V, E)$ is a finite undirected graph, without loops or multiple edges, we write $N = |V|$ and we will often index the vertices from 1 to N. Let $n \geqslant 1$ be some integer and $\mathcal{H} = \mathcal{H}_{G,n} = \bigoplus_{i \in V} \mathbb{C}^n$ with the canonical Hilbert space structure. We will often consider elements of $\mathcal{H}$ as functions from V to $\mathbb{C}^n$ and use the notation $\varphi(i)$ for $\varphi \in \mathcal{H}$ and $i \in V$.

DEFINITION 2. An endomorphism A of $\mathcal{H}$ will be called an n-differential operator on G if $A = (a_{i,j})$, $(i, j) \in V \times V$, where the $a_{i,j}$'s are linear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ and $a_{i,j} = 0$ if $i \neq j$ and $\{i, j\} \notin E$. A is elliptic if the $a_{i,j}$'s ($\{i, j\} \in E$) are invertible, and self-adjoint if $\forall i, j$, $a^*_{i,j} = a_{j,i}$ (a^* denotes the adjoint of a).

Let us denote by $M_{G,n}$ the set (manifold) of all elliptic self-adjoint n-differential operators on G and $M_G = M_{G,1}$. We will denote by $R_G \subset M_G$

the matrices A of M_G with real coefficients and by $O_G \subset R_G$ the set of operators $A = (a_{i,j})$ which satisfy

$$\forall \{i, j\} \in E, \qquad a_{i,j} < 0.$$

For any $A \in M_{G,n}$, let us denote by $\lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_{nN}(A)$ the ordered set of its eigenvalues repeated according to their multiplicities and by $\sigma(A) = \{\lambda_j(A), j = 1, ..., nN\}$ the spectrum of A. If $\lambda \in \mathbb{R}$, let us denote by $d(\lambda, A)$ (or by $d(\lambda)$ if there is no ambiguity) the dimension of $\ker(A - \lambda \operatorname{Id})$.

The Perron–Frobenius theorem implies that $d(\lambda_1(A)) = 1$ if G is connected and $A \in O_G$. Van der Holst proved in [16] the following extension to graphs of Cheng's theorem for manifolds [2]: if G is the 1-skeleton of some triangulation of the 2-sphere S^2 and $A \in O_G$, then $d(\lambda_2(A), A) \leqslant 3$.

Robertson and Seymour introduced in [22, 23] the *tree-width* $tw(G)$ of a graph G (see definition in Section 6). We use a slightly different definition which is more convenient for us:

Definition 3. If G is a finite graph, $la(G)$ is the smallest integer n such that G is a minor of $T \times K_n$ where T is a tree and K_n is the clique (complete graph) with n vertices.

We have (see Section 6):

Proposition 1. *$tw(G)$ and $la(G)$ satisfy the inequalities*

$$la(G) - 1 \leqslant tw(G) \leqslant la(G).$$

There are graphs for which $tw(G) = la(G) - 1$: if $G = K_2 \times K_2 \times K_2$, $la(G) = 4$, and $tw(G) = 3$ (example given in [18]).

We prove the following results:

Theorem 1. *If $A \in M_{T,n}$, where T is a tree with all vertices of degree $\leqslant 3$, then $d(\lambda_1(A), A) \leqslant n$. Moreover, if $d(\lambda, A) \geqslant 2n + 1$, there exist $\{i, j\} \in E(T)$ and $\varphi \in \ker(A - \lambda \operatorname{Id})$ such that:*

(i) *$\varphi(i) = \varphi(j) = 0$, and*

(ii) *there exist $a \in V(T_1)$ and $b \in V(T_2)$ such that $\varphi(a) \neq 0$, $\varphi(b) \neq 0$, where T_1 and T_2 are the two connected components of T after deletion of the edge $\{i, j\}$.*

Let us recall [15]:

Definition 4. Let $x_0 \in X \cap Y$, where X, Y are smooth submanifolds of a manifold W. We say that X and Y intersect *transversally* at x_0 if the tangent spaces $T_{x_0}X$ and $T_{x_0}Y$ of X and Y at x_0 satisfy

$$T_{x_0}X + T_{x_0}Y = T_{x_0}W.$$

We introduce the important notion of a Z-stable eigenvalue:

Definition 5. Let Z be a submanifold of $\mathrm{Herm}(\mathbb{C}^V)$ or $\mathrm{Sym}(\mathbb{R}^V)$. An eigenvalue λ of $A \in Z$ is Z-stable if Z and $W_{l,\lambda}$ intersect transversally at A, where $l = d(\lambda, A)$ is the multiplicity of λ and $W_{l,\lambda} \subset \mathrm{Herm}(\mathbb{C}^V)$ or $\mathrm{Sym}(\mathbb{R}^V)$ is the manifold of all matrices B with

$$\dim \ker(B - \lambda\, \mathrm{Id}) = l.$$

Let us write $d_s(\lambda, A, Z) = d(\lambda, A)$ if $\lambda \in \sigma(A)$ is Z-stable and $d_s(\lambda, A, Z) = 0$ otherwise. We are mainly interested in the case $Z = M_G \subset \mathrm{Herm}(\mathbb{C}^V)$ or the case $Z = R_G \subset \mathrm{Sym}(\mathbb{R}^V)$.

It is possible to make the transversality condition more algebraic:

Definition 6. Let $G = (V, E)$ be a graph. For $i \in V$, we define $\varepsilon_i(x) = |x_i|^2$ (Hermitian form on $\mathbb{C}^V$ or quadratic form on $\mathbb{R}^V$). For $\{i, j\} \in E$, we define

$$\varepsilon'_{i,j}(x) = \Re(x_i \bar{x}_j), \qquad \varepsilon''_{i,j}(x) = \Im(x_i \bar{x}_j)$$

(in the case of Hermitian forms), where $\Re$ (resp. $\Im$) means real (resp. imaginary) part and

$$\varepsilon_{i,j}(x) = x_i x_j$$

(in the case of quadratic forms).

Proposition 2. *If $F = \ker(A - \lambda\, \mathrm{Id})$, the transversality condition is equivalent to the fact that the space of Hermitian forms (resp. quadratic forms) on F is generated over $\mathbb{R}$ by the restrictions to F of the $|V| + 2|E|$ forms ε_i, $i \in V$ and $\varepsilon'_{i,j}$, $\varepsilon''_{i,j}$, $\{i, j\} \in E$ (resp. of the $|V| + |E|$ forms ε_i, $i \in V$ and $\varepsilon_{i,j}$, $\{i, j\} \in E$).*

We have:

Theorem 2. *(1) $A \in M_G$ implies $d_s(\lambda_1(A), A, M_G) \leqslant la(G)$ and $d_s(\lambda, A, M_G) \leqslant 2la(G)$ $(\forall \lambda)$;*

(2) $A \in R_G$ implies $d_s(\lambda_1(A), A, R_G) \leqslant la(G)$ and $d_s(\lambda, A, R_G) \leqslant 2la(G)$ $(\forall \lambda)$.

These inequalities hold also for $A \in O_G$.

Theorem 2 can be reformulated by introducing the following invariants of graphs:

DEFINITION 7. Let us define

$$\nu_k^{\mathbb{R}}(G) = \max\{d_s(\lambda_k(A), A, R_G) \mid A \in R_G, \lambda_{k-1}(A) < \lambda_k(A)\},$$
$$\nu_k^{\mathbb{C}}(G)) = \max\{d_s(\lambda_k(A), A, M_G) \mid A \in M_G, \lambda_{k-1}(A) < \lambda_k(A)\}.$$

Remark. $\mu(G)$ can be defined as

$$\mu(G) = \max\{d_s(\lambda_2(A), A, O_G) \mid A \in O_G\},$$

and we have

$$\mu(G) \leqslant \nu_2^{\mathbb{R}}(G).$$

Remark. $\nu_k^K(G) = \nu_{N-k}^K(G)$ because if A is optimal for one case, $-A$ is optimal for the other case.

One of the main results of our paper is the following:

THEOREM 3. *The invariants ν_k^K satisfy*

$$\nu_k^K(G') \leqslant \nu_k^K(G),$$

for every minor G' of G.

Remark. One has $\nu_2^{\mathbb{R}}(K_{1,3}) = 2$ while $\nu_2^{\mathbb{C}}(K_{1,3}) = 1$.

Proof. The first equality results from $\mu(K_{1,3}) = 2 \leqslant \nu_2^{\mathbb{R}}(K_{1,3})$ and the easy fact that $\nu_l^K(G) = |V(G)| - 1$ if and only if G is a clique.

The second one is proved as follows: suppose that 0 is the vertex of degree 3 and 1, 2, 3 are the vertices of degree 1 in $K_{1,3}$. Take $A \in M_{K_{1,3}}$ with $F = \ker A$ of dimension 2. Since for every $x \in F$ $x_0 = 0$, the forms ε_0, $\varepsilon'_{i,j}$, and $\varepsilon''_{i,j}$ ($\{i, j\} \in E$) vanish on F. Proposition 2 shows that transversality cannot occur because dim Herm$(F) = 4$. ∎

Is it true in general that

$$\nu_k^{\mathbb{R}}(G) \geqslant \nu_k^{\mathbb{C}}(G)?$$

With Definition 7, Theorem 2 can be reformulated:

THEOREM 4. *For $K = \mathbb{R}$ or $\mathbb{C}$, we have*:

(i) $\nu_1^K(G) \leqslant la(G)$,

(ii) $\nu_k^K(G) \leqslant 2la(G)$.

In particular, $\mu(G) \leqslant 2la(G)$.

It is not always true that $\nu_1^{\mathbb{C}}(G) = la(G)$. If V_8 is the Möbius ladder, it is proved in [18] that $\nu_1^{\mathbb{C}}(V_8) < la(V_8)$.

The following characterization of forests is an easy corollary of the previous statements:

THEOREM 5. *The following statements are equivalent*:

(i) *G is a forest*,

(ii) $\nu_1^{\mathbb{R}}(G) = 1$,

(iii) $\nu_1^{\mathbb{C}}(G) = 1$.

Proof. By Theorem 3 and $\nu_1^K(K_3) = 2$, (ii) or (iii) implies (i). By Theorem 4, (i) implies (ii) and (iii). ∎

H. van der Holst gives in [17] (resp. [18]) a characterization of the class of graph G which satisfies $\nu_1^{\mathbb{C}}(G) \leqslant 2$ (resp. $\leqslant 3$).

It is interesting to observe that the new invariants $\nu_k^K(G)$ introduced above are not at all related to planarity:

THEOREM 6. *There exists a sequence G_n of* planar graphs (*described in Section* 7) *such that*

$$\nu_1^{\mathbb{R}}(G_n) = \nu_1^{\mathbb{C}}(G_n) = n,$$

and

$$la(G_n) = n.$$

Note. This paper is a complete revision, including new results (in particular, concerning eigenvalues λ_k with $k > 1$) and new proofs, of preprint [9].

2. A CRUCIAL LEMMA

LEMMA 1. *Let $G = (V, E)$ be a finite connected graph and $\{1, 2\} \in E(G)$ such that $G' = (V, E \setminus \{1, 2\})$ is disconnected. Let $A \in M_{G,n}$, $F = \ker A$, and let r: $F \to \mathbb{C}^n \oplus \mathbb{C}^n$ be given by*

$$r(\varphi) = (\varphi(1), \varphi(2)).$$

Then $\dim r(F) \leqslant n$.

Proof. The proof is based on a *discrete Green's formula* (the continuous Green's formula states that $\int_{\partial D} f\,\partial g/\partial n - g\,\partial f/\partial n = 0$ for harmonic functions f, g on a bounded smooth domain $D \subset \mathbb{R}^n$). Let $V_1 \subset V(G')$ be the vertices of the connected component containing the vertex 1. For $\varphi, \psi \in F$, let φ_1, ψ_1 denote the truncated functions defined by $\varphi_1(i) = \varphi(i)$, $\psi_1(i) = \psi(i)$ if $i \in V_1$ and $\varphi_1(i) = 0$, $\psi_1(i) = 0$ if $i \notin V_1$. We compute now the right-hand side of the equality

$$0 = \langle A\varphi_1 \mid \psi_1 \rangle - \langle \varphi_1 \mid A\psi_1 \rangle,$$

using the fact that only the values at vertex 1 contribute to the scalar products (if $i \neq 1$, $A\varphi_1(i) = 0$ or $\psi_1(i) = 0$). Let us compute $A\varphi_1(1)$ using the fact that $A\varphi(1) = 0$; we get

$$A\varphi_1(1) = -a_{1,2}(\varphi(2)).$$

Hence, the expression to be evaluated reduces to

$$0 = \langle a_{1,2}(\varphi(2)) \mid \psi(1) \rangle - \langle \varphi(1) \mid a_{1,2}(\psi(2)) \rangle,$$

which we call the discrete Green's formula.

Let us write $B = a_{1,2}$ and denote by ω the Hermitian form on $\mathbb{C}^n \oplus \mathbb{C}^n$ given by

$$\omega((x_1, x_2), (y_1, y_2)) = \sqrt{-1}\,(\langle Bx_2 \mid y_1 \rangle - \langle x_1 \mid By_2 \rangle).$$

It is easy to see that ω is non-degenerate. A subspace $K \subset \mathbb{C}^n$ is ω-isotropic if ω vanishes identically on $K \times K$. Any isotropic subspace K has complex dimension at most n because it is included in his orthogonal K^o with respect to ω and $\dim(K^o) = 2n - \dim(K)$. In particular, this is true for $K = r(F)$. ▮

Let us state without proof the following modification of the above result:

LEMMA 2. *Let $A \in M_G$, $V_1 \subset V$ and let $E_0 = \{e_j = \{a_j, b_j\} \in E, j = 1, ..., n\}$ be the set of all edges $e = \{a, b\}$ of G such that $a \in V_1$ and $b \notin V_1$; set $F = \ker A$, $V_0 = \{a_j, b_j, j = 1, ..., n\}$ (we have $\# V_0 \leqslant 2n$).*

If $r: F \to \mathbb{C}^{V_0}$ is the restriction to V_0, then $\dim r(F) \leqslant n$.

3. PROOF OF THEOREM 1

Let T be a finite tree with vertices of degree $\leqslant 3$. For each edge $\{i, j\} \in E(T)$, let us denote by $T_{i,j}$ and $T_{j,i}$ the two subtrees of T obtained

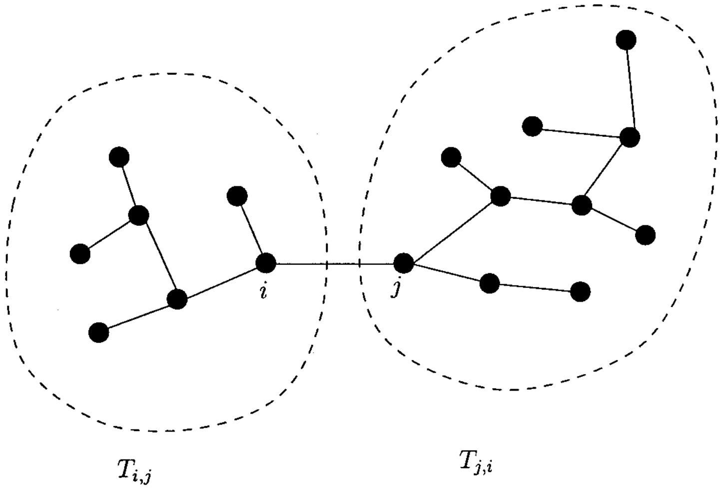

FIGURE 1

by deleting the edge $\{i, j\}$ and such that $i \in V(T_{i,j})$ and $j \in V(T_{j,i})$. (See Fig. 1.)

Let $n \geqslant 1$ be some integer and $A \in M_{T,n}$. Let us write $F = \ker A$. For each $\{i, j\} \in E(T)$, let us denote by $F_{i,j} \subset F$ the vector space of functions with support in $V(T_{i,j})$ which vanish at i. For $r = r_{i,j}: F \to \mathbb{C}^n \oplus \mathbb{C}^n$ defined by $r(\varphi) = (\varphi(i), \varphi(j))$ it is easy to check that

$$\ker r_{i,j} = F_{i,j} \oplus F_{j,i}.$$

We have then:

Lemma 3. *If* $\dim F > n$, *there exists* $\{i, j\} \in E(T)$ *such that*

(i) $F_{i,j}$ *is not reduced to* 0.

(ii) degree$(i) = 3$.

(iii) *For any neighbour* $\alpha \neq j$ *of* i, *we have an injective map* $\varepsilon_\alpha: F_{i,j} \to \mathbb{C}^n$ *defined by* $\varphi \to \varphi(\alpha)$. *In particular*, $1 \leqslant \dim F_{i,j} \leqslant n$.

Corollary 1. *If* $\dim F > 2n$, *there exists* $\{i, j\} \in E(T)$ *such that*

$$\dim F_{i,j} \geqslant 1, \qquad \dim F_{j,i} \geqslant 1.$$

Corollary 2. *If* $A \geqslant 0$ (*i.e., the Hermitian form associated to* A *is non-negative*), $\dim F \leqslant n$.

Proof of Corollary 1. Choose $\{i, j\}$ according to Lemma 3 and set $F_0 = \ker(r_{i,j})$. We have: $\dim F_0 \geqslant n+1$ (Lemma 1), and $F_{j,i}$ is not reduced to 0 because $F_0 = F_{i,j} \oplus F_{j,i}$, and $\dim F_{i,j} \leqslant n$ by (iii) of Lemma 3. ▮

Proof of Corollary 2. If $\dim F > n$, by (i) of Lemma 3, there exists $\varphi \in F_{i,j} \backslash 0$, with the edge $\{i, j\}$ given by Lemma 3, and by (iii), $\varphi(\alpha) \neq 0$, where $\alpha \neq j$ is any neighbour of i. Define ψ by $\psi(k) = \varphi(k)$ for $k \in V(T_{\alpha,i})$ and $\psi(k) = 0$ otherwise. Then for ψ belongs to $\ker A$. In fact,

$$(A\psi \mid \psi) = 0,$$

because $A\psi$ vanishes where ψ is not zero.

Set $Q(f) = (Af \mid f)$ and let δ be the numerical function on $V(T)$ which is defined by $\delta(i) = 1$ and $\delta(k) = 0$ if $k \neq i$.

Evaluating $Q(\psi + \varepsilon\,\delta v)$ for $v \in \mathbb{C}^n$ and $\varepsilon > 0$, we find

$$Q(\psi + \varepsilon\,\delta v) = 2\varepsilon \Re(v \mid A_{i,\alpha}(\psi(\alpha))) + O(\varepsilon^2).$$

It is always possible to choose v such that

$$Q(\psi + \varepsilon\,\delta v) < 0$$

for $\varepsilon > 0$ small enough (because $A_{i,\alpha}$ is non-singular and $\psi(\alpha) \neq 0$). This gives a contradiction with the fact that $A \geqslant 0$. ▮

Proof of Lemma 3. Choose first an arbitrary edge $\{i_1, j_1\}$ of T. By Lemma 1, we may then assume that F_{i_1, j_1} is not trivial (otherwise permute i_1 and j_1). It is clear that i_1 is of degree 2 or 3.

Step 1. If its degree is 2 and if $\{\alpha, j_1\}$ is the set of neighbours of i_1, then $\varphi(\alpha) = 0\ \forall \varphi \in F_{i_1, j_1}$. Replace then the edge $\{i_1, j_1\}$ by the edge $\{\alpha, i_1\}$ and iterate.

Step 2. Suppose now that the degree of i is 3 and that $\{j_1, \alpha, \beta\}$ is the set of neighbours of i_1. If the map $\varphi \to \varphi(\alpha)$ is not injective on F_{i_1, j_1}, then either F_{α, i_1} or F_{β, i_1} is not trivial; assume that the space F_{α, i_1} is not trivial, set $i_2 = \alpha$, $j_2 = i_1$, and go to Step 1.

This process will stop and yield a solution. ▮

Theorem 1 is an easy reformulation of Corollary 1 with $\lambda = 0$, and Corollary 2 with $\lambda_1 = 0$.

4. LAGRANGIAN COMPACTIFICATION

4.1. *Introduction*

We want to stabilize the multiplicities of eigenvalues with respect to minors. Let us explain what this means.

Consider the graphs $W_n = P_{2n} \times K_2$ (P_l is the path with l vertices) and S_n with $V(S_n) = \{1, 2, ..., 2n+1\}$ and $E(S_n) = \{\{i, i+1\}, \{2j-1, 2j+1\} \mid i = 1, ..., 2n, j = 1, ..., n\}$. For any $A \in M_{W_n}$, dim ker $A \leqslant 2$: if $\varphi \in \ker A$ vanishes at the two vertices in $\{a\} \times V(K_2)$, where a is an end of the path, it is easy to see that φ vanishes identically.

There exists $A_0 \in R_{S_n}$ which is $\geqslant 0$ and has dim ker $A_0 = n+1$. Define the quadratic form q associated with A_0 by $q(x) = \sum_{j=1}^{n} (x_{2j-1} + x_{2j} + x_{2j+1})^2$, where the sum is over the n triangles $\{2j-1, 2j, 2j+1\}$ of S_n, which is a union of n triangles. It is easy to check that S_n is a minor of W_n. (See Fig. 2.)

In some sense, the small dimension of ker A, for $A \in M_{W_n}$, is not *stable* with respect to minors.

Let $j: N \to M$ be a smooth map with injective differential between manifolds and let $W \subset M$ be a submanifold of M. We say that j is *transversal* to W at $x_0 \in N$ if $j(x_0) \in W$ and

$$T_{j(x_0)} M = T_{j(x_0)} W + j'(x_0)(T_{x_0} N).$$

We want to use the basic property of *transversality*; see [15, p. 27]. Let j_ε (ε small in absolute value real number) be a smooth map from N to M which converges near x_0 to j in the C^1-topology (this means that, in some neighbourhood of x_0, j_ε converges uniformly to j and the first order derivatives of j_ε converge also uniformly to the first order derivatives of j). Then there exists $x_\varepsilon \in N$ such that j_ε is transversal to W at x_ε for ε small enough.

Here we are interested in the following situation. Let G be a finite graph and let $G' = D_{1,2}(G)$ obtained from G by deleting the edge $\{1, 2\}$. We have

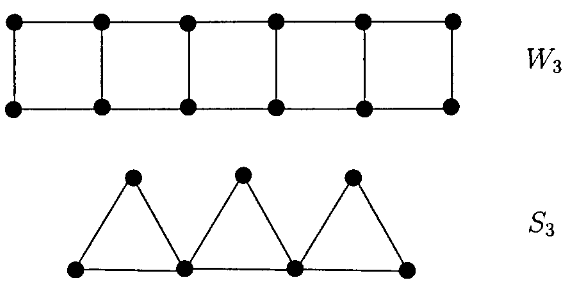

FIGURE 2

$V = V'$. Let $\lambda_k = 0$ be the kth eigenvalue having multiplicity l of $A \in M_{G'}$; we can think of this situation as follows: A belongs to the intersection of two submanifolds in $\mathrm{Herm}(\mathbb{C}^{V'})$: the manifold $j(M_{G'})$ (where j is the embedding of $M_{G'}$ into $\mathrm{Herm}(\mathbb{C}^{V'})$) and the manifold W_l of matrices whose kernel has dimension l. We can consider the maps $j_\varepsilon \colon M_{G'} \to \mathrm{Herm}(\mathbb{C}^{V'})$ defined by $j_\varepsilon(q) = j(q) + \varepsilon |x_1 - x_2|^2$. Then $d_s(0, A_0, M_{G'}) = l$ is equivalent to "j is transversal to W_l at A_0." As $j_\varepsilon(M_{G'}) \subset M_G$, the *basic property of transversality* shows that $\nu_k^{\mathbb{C}}(G') \leqslant \nu_k^{\mathbb{C}}(G)$.

If G' is obtained from G by contracting the edge $\{1, 2\}$ (we shall write $G' = C_{1,2}(G)$), we need to embed $M_{G'}$ and M_G as submanifolds into the same manifold: this is possible using appropriate Grassmann manifolds.

We will now describe the appropriate general tools necessary for stabilization; of course, the same kind of proofs as that in [4] applies, but we want to have a more *natural* setting, even if this seems to imply more geometric material! See also [6] and [11].

4.2. *Lagrangian Grassmann Manifolds and Quadratic Forms*

In the following, X is a real N-dimensional vector space. In fact, up to obvious changes, everything extends to the complex case. Proofs will be given only for the real case. For applications to graphs, X will be $\mathbb{R}^V$ (or $\mathbb{C}^V$).

Let us denote by Z the space $T^*(X) = X \oplus X^*$, where X^* is the dual of X. We endow Z with the canonical symplectic form ω defined by

$$\omega((x, \xi), (x', \xi')) = \xi(x') - \xi'(x),$$

and denote by $\mathscr{L}_X$ (or $\mathscr{L}$ if no ambiguity arises) the Grassmann manifold of Lagrangian subspaces in Z. Let us recall that a Lagrangian subspace of Z is a maximal subspace which is ω-isotropic (H is ω-isotropic means that $\omega(x, y) = 0$ for any $x, y \in H$): such subspaces are of dimension N and $\mathscr{L}_X$ is a real analytic compact manifold of dimension $N(N+1)/2$; cf. Duistermaat [14].

Remark. In the complex case, we need to consider the canonical Hermitian form $\omega_{\mathbb{C}}$ on $X \oplus X^*$, where X^* is the antidual of X, given by

$$\omega_{\mathbb{C}}((x, \xi), (x', \xi')) = \sqrt{-1}\,(\xi(x') - \bar{\xi}'(x)),$$

and the corresponding Grassmann manifold which is of dimension N^2.

Denote by $\mathscr{Q}(X)$ the vector space of all (real) quadratic forms on X (or all Hermitian forms on X in the complex case). Every quadratic form $q(x) = (Ax)(x)$ on X can be identified with the symmetric linear map A from X to X^* and this defines an embedding $J \colon \mathscr{Q}(X) \to \mathscr{L}_X$, where $J(q)$ is the graph of the linear map A.

We give the following:

DEFINITION 8. $\rho = (q, F)$ will be called a generalized quadratic form on X if F is a subspace of X and $q \in \mathcal{Q}(F)$.

To each generalized quadratic form $\rho = (q, F)$, we associate the Lagrangian space

$$J(\rho) = \{(x, \xi) \mid x \in F \quad \text{and} \quad \forall y \in F, C_q(x, y) = \xi(y)\},$$

where C_q is the symmetric bilinear form associated with q (i.e., $(B_q(x))(y) = C_q(x, y)$). In other words, if $B_q\colon F \to F^*$ is the linear map associated with q

$$J(\rho) = \{(x, \xi) \in F \times X^* \mid \xi|_F = B_q x\}.$$

Conversely, if L is a *Lagrangian subspace*, we associate with it a generalized quadratic form $K(L) := \rho = (q, F)$, where F is the projection of L onto X and $\forall x, y \in F$, $C_q(x, y) = \xi(y)$, where $(x, \xi) \in L$. The fact that $\xi(y)$ is independent of the choice of $(x, \xi) \in L$ comes from the fact that L is a Lagrangian: if (x, ξ) and (x, ξ') are in L, then $(0, \xi - \xi') \in L$ and, for $(y, \eta) \in L$,

$$0 = \omega((0, \xi - \xi'), (y, \eta)) = \xi(y) - \xi'(y).$$

Using $\omega((x, \xi), (y, \eta)) = 0$, it is clear that C_q is symmetric.

It is easy to check that J and K are inverse maps. In this way, we have a bijection of $\mathcal{L}$ with the set of all generalized quadratic forms. Since $\mathcal{L}$ is a compact manifold, we have also a compactification of $\mathcal{Q}(X)$. The corresponding topology on generalized quadratic forms will be called the *Lagrangian topology*.

Given a Lagrangian space L_0, it is possible to identify the tangent space of $\mathcal{L}$ at L_0 with the space $\mathcal{Q}(L_0)$ in the following way: there exists $L_1 \in \mathcal{L}$ (in fact, all elements of an open dense set in $\mathcal{L}$ work) such that $Z = L_0 \oplus L_1$ and ω identifies L_1 with the dual of L_0. Lagrangian spaces L which are close enough to L_0 can be considered as graphs of linear maps from L_0 to L_1 and these map are symmetric once L_1 is identified, using ω, with the dual L_0^* of L_0. In this way, we get charts of $\mathcal{L}$ near L_0.

The following proposition is proved in Duistermaat [14]:

PROPOSITION 3. *All these charts give rise to the same identification of the tangent space at L_0 with $\mathcal{Q}(L_0)$.*

4.3. *Some Examples of Singular Limits*

We consider a family of symmetric operators from X to X^* of the type

$$A(\varepsilon) = A_0 + \frac{1}{\varepsilon} A_1, \qquad \varepsilon \neq 0.$$

PROPOSITION 4. *In the manifold $\mathcal{L}$, the graph of $A(\varepsilon)$ has a limit for $\varepsilon \to 0$ which is the generalized quadratic form $\Phi(A_0) = (q, F)$, where $F = \ker A_1$ and q is the restriction to F of the quadratic form associated with A_0.*

Moreover the maps Φ_ε from $\mathrm{Sym}(X)$ *to $\mathcal{L}$ defined by $\Phi_\varepsilon(A_0) = J(A_0 + A_1/\varepsilon)$ converge in the C^1 topology to Φ.*

Proof. Let us consider the decompositions $X = U \oplus V$, with $U = \ker A_1$ and $A_1(V) \subset V^*$, and $X^* = U^* \oplus V^*$. We describe then the graph of $A(\varepsilon)$ in the following way. For $u \in U$, $v \in V$, let us write $A(\varepsilon)(u, v) = (\xi, \eta)$ with $\xi \in U^*$ and $\eta \in V^*$. Then we have

$$\xi = B(u, v), \qquad \eta = C(u) + D(v) + \frac{1}{\varepsilon} Gv$$

(here B: $X \to U^*$, C: $U \to V^*$, D and G: $V \to V^*$ are linear maps, and G is non-singular), which may be rewritten as

$$\xi = B(u, v), \qquad (G + \varepsilon D)(v) = \varepsilon(\eta - C(u)).$$

For ε small, $G + \varepsilon D$ is close to G and hence invertible; from the second equation, we obtain, for ε small enough,

$$v = \varepsilon K(\varepsilon)(\eta, u),$$

where $K(\varepsilon)$: $V^* \oplus U \to V$ is linear, and we insert into the first one

$$\xi = M(\varepsilon)(\eta, u),$$

where $M(\varepsilon)$: $V^* \oplus U \to U^*$ is linear. This shows that the graphs L_ε of $A(\varepsilon)$ admit a limit L_0 as ε goes to 0. This limit is the graph of the map

$$(\eta, u) \to (v = 0, \xi = B(u, 0))$$

from $V^* \oplus U$ into $V \oplus U^*$.

It remains to check that $K(L_0)$ is the *generalized quadratic form* $\rho = (q, U)$, where q is the restriction to U of the quadratic form associated with A_0. This follows from the definition of B.

The C^1 convergence of Φ_ε to Φ comes from the fact that $D \to (G+\varepsilon D)^{-1}$ is C^1. ∎

More generally, we can prove the following result [13]:

PROPOSITION 5. *Any meromorphic map from some open set $\Omega \subset \mathbb{C}$ into* $\mathrm{Sym}(X)$ *extends to a holomorphic map into $\mathscr{L}_X$.*

4.4. *Stratification of the Lagrangian Grassmann Manifold*

Fix some Lagrangian space $L_0 \in \mathscr{L}$ and denote by W_l the set of all Lagrangian spaces L such that $\dim(L \cap L_0) = l$.

Choose the Lagrangian subspace $L_0 = X \oplus 0$ of $Z = X \oplus X^*$, and consider a generalized quadratic form $\rho = (q, F)$. The two statements $J(\rho) \in W_l$ and $\dim \ker q = l$ are equivalent. This definition of W_l is the natural extension to the generalized quadratic forms of the definition of $W_{l,0}$ given in Definition 5.

The following theorem is proved in Duistermaat [14]:

THEOREM 7. *W_l is a (non-closed) submanifold of $\mathscr{L}$ whose tangent space at L is the set of quadratic forms on L which vanish identically on $L \cap L_0$.*

Comments. This result is strongly related to the perturbation theory of degenerate eigenvalues. If $Z = X \oplus X^*$ and $L_0 = X \oplus 0$, for any $A \in \mathrm{Sym}(X)$ whose graph is L_A, we have

$$\dim \ker A = \dim(L_0 \cap L_A).$$

Moreover, if this dimension is $\geqslant 1$, eigenvalues close to 0 of $A_\varepsilon = A + \varepsilon B$ are very close to eigenvalues of the quadratic form associated to B restricted to $\ker A$.

5. MONOTONICITY FOR MINORS

In this section, we will show how to use the previous tools (transversality, Lagrangian Grassmann manifolds) in order to obtain bounds on multiplicities with respect to minors for operators associated to graphs.

5.1. *Minors*

We call G' a minor of G if G' is obtained from G by a sequence of the following three operations:

(D) deletion of an edge,

(C) contraction of an edge and identification of its two vertices,

(R) deletion of an isolated vertex.

It is possible to describe this in a more global way:

Let us give a partition $V = \bigcup_{\alpha \in W} V_\alpha$ of the vertex set of G into connected subsets. Then G' is a minor of G if $V(G')$ is a subset of W, and $E' = E(G')$ satisfies

$$(\{\alpha, \beta\} \in E') \Rightarrow (\exists i \in V_\alpha,\ j \in V_\beta,\ \{i, j\} \in E).$$

Given a property (P) of graphs which is hereditary with respect to minors (for instance, the existence of an embedding in a given surface), a deep and difficult result (Wagner's conjecture, proved by Robertson and Seymour in a series of papers in this journal) states that this property is characterized by a *finite* number of excluded minors: there exists a *finite* list of graphs such that (P) is equivalent to the property of *having no minor in this list*.

The simplest example is the characterization of forests by excluding triangles as minors. A further classical example is Kuratowski's characterization of planar graphs by excluding K_5 and $K_{3,3}$ as minors.

5.2. *Monotonicity*

We will now prove Theorem 3.

Proof. We will give the proof in the real case (i.e., $K = \mathbb{R}$). It is enough to show the result for $G' = D_{1,2}(G)$ or $G' = C_{1,2}(G)$, where $\{1, 2\}$ is an edge of G. The contraction of an edge is the more difficult case since then $\mathbb{R}^{V'}$ is not equal to $\mathbb{R}^V$. We will only give the proof for this case.

Let us denote by 0 the vertex of G' which is obtained by contracting the edge $\{1, 2\} \in E(G)$, by B the set of vertices of G which are adjacent to 1, but not to 2, by C the set of vertices which are adjacent to 2, but not to 1, and by D the set of vertices which are adjacent to 1 and 2. (See Fig. 3).

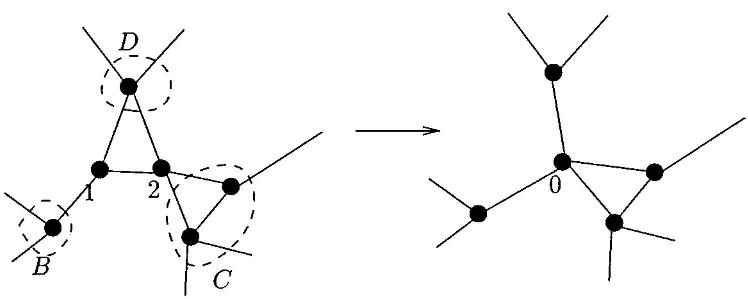

FIGURE 3

Put $\mathscr{L}_V = \mathscr{L}_{\mathbb{R}^V}$. We will define maps j_ε, $\varepsilon > 0$, from $R_{G'}$ to $R_G \subset \mathrm{Sym}(\mathbb{R}^V) \subset \mathscr{L}_V$. Let us denote by k_0 the embedding of $\mathrm{Sym}(\mathbb{R}^{V'})$ into $\mathscr{L}_V$, which associates to the quadratic form q on $\mathbb{R}^{V'}$, the Lagrangian space $J(\rho)$, where $\rho = (Q, F_{1,2})$ on $\mathbb{R}^V$ is defined in the following way: its domain $F_{1,2}$ is the subspace defined by the equation $\{x_1 = x_2\}$ and Q is defined by transferring q to $F_{1,2}$, using the bijection $\varphi\colon \mathbb{R}^{V'} \to F_{1,2}$ given by $\varphi(x_0, x_3, ..., x_N) = (x_0, x_0, x_3, ..., x_N)$. We will then prove, using Proposition 4, that j_ε converges in the C^1 topology to j_0, the restriction of k_0 to $R_{G'}$.

Let $q(x_0, x_3, ..., x_N)$ be some quadratic form in $R_{G'}$. We associate to it $j_\varepsilon(q) = J(q_\varepsilon)$ in the following way: let us write

$$q(x_0, x_3, ..., x_N) = W_0 x_0^2 + \sum_{j \sim 0} c_{0,j}(x_j - x_0)^2 + r(x_3, ..., x_N).$$

We define

$$q_\varepsilon(x_1, ..., x_N) = \frac{1}{\varepsilon}(x_1 - x_2)^2 + W_0 x_1^2 + \sum_{j \in B} c_{0,j}(x_1 - x_j)^2 + \sum_{j \in C} c_{0,j}(x_2 - x_j)^2$$
$$+ \sum_{j \in D} \frac{c_{0,j}}{2}((x_1 - x_j)^2 + (x_2 - x_j)^2) + r(x_3, ..., x_N).$$

With this definition, $q_\varepsilon \in R_G$ and the restriction of q_ε to $F_{1,2}$ yields $q(x_1, x_3, ..., x_N)$. Proposition 4 shows that j_ε converges smoothly to j_0.

Let us denote by $W'_l \subset \mathrm{Sym}(\mathbb{R}^{V'})$ the set of matrices whose kernel is of dimension l. If $d_s(0, A_0, R_{G'}) = l$, by definition

$$\mathrm{Sym}(\mathbb{R}^{V'}) = T_{A_0} W'_l + T_{A_0} R_{G'}.$$

If $Z = k_0(\mathrm{Sym}(\mathbb{R}^{V'}))$ and $Y = j_0(R_{G'})$, and using the fact that $k_0(W'_l) = W_l \cap Z$, we observe that, writing $L_0 = j_0(A_0)$, $T_{L_0} Z = T_{L_0} Y + T_{L_0}(W_l \cap Z)$.

We have then:

Lemma 4. *Using the same notations as before, $j_0\colon R_{G'} \to \mathscr{L}_V$ is transversal to W_l at A_0: absolute (inside* $\mathrm{Sym}(\mathbb{R}^V)$*) and relative (inside* $\mathrm{Sym}(\mathbb{R}^{V'})$*) transversality coïncide.*

Proof. We begin with the observation

$$T_{L_0}(\mathscr{L}_V) = T_{L_0} W_l + T_{L_0} Z.$$

Indeed, using Proposition 3 and Theorem 7 and writing $H = L_0 \cap (\mathbb{R}^V \oplus 0) = \ker A_0$, we have $T_{L_0}(\mathscr{L}_V) = \mathscr{Q}(L_0)$, $T_{L_0} W_l = \{q \in \mathscr{Q}(L_0)\; q|_H = 0\}$ and $T_{L_0}(Z) = \{S \circ \pi\}$, where π is the projection from L_0 to $D_{1,2}$ and $S \in \mathscr{Q}(D_{1,2})$. The observation follows then from the fact that $H \subset D_{1,2}$.

We use now the fact that

$$T_{L_0}(Z) = T_{L_0}(W_l \cap Z) + j'_0(T_{A_0}(R_{G'})),$$

to conclude that

$$T_{L_0}(\mathcal{L}_V) = T_{L_0}(W_l) + j'_0(T_{A_0}(R_{G'})). \quad \blacksquare$$

Let q_0 be the quadratic form associated with $A_0 \in R_{G'}$ such that $\lambda_k(A_0) = 0$ and

$$d_s(0, A_0, R_{G'}) = l.$$

We have seen that j_ε converges smoothly to j_0. By the basic property of transversality, for $\varepsilon > 0$ small enough, there exists some $A_\varepsilon \in j_\varepsilon(R_{G'})$ such that $\lambda_k(A_\varepsilon) = 0$ and $d_s(0, A_\varepsilon, j_\varepsilon(R_{G'})) = l$. Then, because $j_\varepsilon(R_{G'}) \subset R_G$, $d_s(0, A_\varepsilon, R_G) = 1$. This completes the proof of Theorem 3.

5.3. *Proof of Theorem* 2

First, define the product $K = G \times H$ of two graphs G and H by

$$V(K) = V(G) \times V(H),$$

and

$$\{(g_1, h_1), (g_2, h_2)\} \in E(K) \quad \text{if and only if} \quad g_1 = g_2$$
$$\text{and} \quad \{h_1, h_2\} \in E(H) \quad \text{or} \quad h_1 = h_2 \quad \text{and} \quad \{g_1, g_2\} \in E(G).$$

Let us now prove the second part of Theorem 2 (i.e., the real case):

Proof. Let G be a graph such that $la(G) = n$; then, there exists some tree T with vertices of degree $\leqslant 3$ such that G is a minor of $T \times K_n$. Hence

$$\nu_k^{\mathbb{R}}(G) \leqslant \nu_k^{\mathbb{R}}(T \times K_n).$$

We will use the natural identification of $\mathbb{C}^{V(T \times K_n)}$ with the space of maps from $V(T)$ into $\mathbb{C}^n$. Using this identification, every scalar elliptic self-adjoint operator A on $T \times K_n$ becomes an elliptic self-adjoint n-differential operator on T.

$k = 1$. In this case, by Theorem 1, the multiplicity of the ground state of $T \times K_n$ is always $\leqslant n$.

k arbitrary. If $\nu_k^{\mathbb{R}}(T \times K_n) > 2n$, there exists $A \in R_{T \times K_n}$ such that $d_s(\lambda_k, A, R_{T \times K_n}) > 2n$. Applying Theorem 1 (and using the notations there), let us denote by φ_i, $i = 1, 2$, the restrictions of φ to $V(T_i)$ extended by 0 outside T_i. Then $\varphi_i \in \ker(A - \lambda_k)$, for any $\alpha \in V(T \times K_n)$, $\varepsilon_\alpha(\varphi_1, \varphi_2) = 0$ (we identify here ε_α with the associated bilinear form) because supports are disjoint, and for any $\{\alpha, \beta\} \in E(T \times K_n)$, $\varepsilon_{\alpha,\beta}(\varphi_1, \varphi_2) = 0$, because there is no edge $\{\alpha, \beta\}$ for which $\varphi_1(\alpha)\,\varphi_2(\beta) \neq 0$. This shows that transversality does not hold. ∎

6. TREE-WIDTHS

In [22], N. Robertson and P. Seymour give the following definition for the *tree-width* $tw(G)$ of a graph G: we define a *tree-like decomposition* of G as a pair $(T, \mathscr{X})$, where T is a tree and where $\mathscr{X} = \{X_t \mid t \in V(T)\}$ is a family of subsets of $V(G)$ indexed by $t \in V(T)$, such that the following conditions hold:

(6.1) $V(G) = \bigcup_{t \in V(T)} X_t$,

(6.2) $\forall e = \{a, b\} \in E(G)$, $\exists t$ such that $a, b \in X_t$,

(6.3) $\forall x, y \in V(T)$, $\forall z \in]x, y[$, $X_x \cap X_y \subset X_z$.

Here $]x, y[$ denotes the set of interior vertices of the unique path between x and y. We define then the width of $(T, \mathscr{X})$ by

$$w(T, \mathscr{X}) = \max |X_t| - 1,$$

and

$$tw(G) = \min w(T, \mathscr{X}),$$

where the min is taken over all tree-like decompositions of G.

On the other hand, we defined (see Definition 3) some closely related invariant $la(G)$. Recall that $la(G)$ is the smallest natural integer N such that G is a minor of some product $T \times K_N$, where T is a tree and K_N is the clique with N vertices. We want to prove Proposition 1; i.e., we have, for any graph G,

$$tw(G) \leqslant la(G) \leqslant tw(G) + 1.$$

Proof. $tw(G) \leqslant la(G)$. This was proved first by H. van der Holst in his thesis and is reproduced from [17, p. 91] with the kind permission of the author.

If G is a clique sum of G_1 and G_2 then

$$tw(G) = \max\{tw(G_1), tw(G_2)\}.$$

Let us call a *k-clique tree* any graph G of the form $T \times K_k$ where T is a tree. Since each k-clique tree $G = T \times K_k$ can be obtained from clique sums of $K_2 \times K_k$ and since $tw(K_2 \times K_k) = k$, each k-clique tree G has $tw(G) = k$. So if G is a minor of a k-clique tree then $tw(G) \leqslant k$. ▮

Proof. ($la(G) \leqslant tw(G) + 1$). Let $(T, \mathscr{X})$, $\mathscr{X} = \{X_t \mid t \in T\}$, with $tw(G) = w(T, \mathscr{X}) = N - 1$, be a tree-like decomposition of G.

Let G' be the graph whose vertices are the pairs (t, x) with $t \in V(T)$, $x \in X_t$, and whose edges are of the form $\{(t, x), (t, x')\}$ with $\{x, x'\} \in E$ and of the form $\{(t, x), (t', x)\}$, where $\{t, t'\}$ is an edge of T and $x \in X_t \cap X_{t'}$.

Then G is a minor of G': contract the edges of the form $\{(t, x), (t', x)\}$ and use the fact that $A_x = \{t \mid x \in X_t\}$ induces a connected subgraph of T (a reformulation of property (6.3) of a tree-like decomposition) to embed the resulting vertex set in $V(G)$. This vertex set is actually $V(G)$ by (6.1) and all edges of G are present by (6.2).

The graph G' is also a minor of $T \times K_N$: to see this, it is enough to construct an injective map

$$j\colon V(G') \to V(T) \times \{1, ..., N\}$$

which satisfies

(1) $j(t, x) = (t, n(t, x))$,

(2) for any $x \in X_t \cap X_{t'}$, $n(t, x) = n(t', x)$.

We construct j starting from some root α of T: we choose an arbitrary numbering of X_α and propagate it along the edges of T using the condition (2). ▮

7. THE GRAPHS G_n

Here, we will give an explicit family of planar graphs $G_n = (V_n, E_n)$ such that:

(i) $v_1^K(G_n) = n$ for $K = \mathbb{R}$ and for $K = \mathbb{C}$,

(ii) $la(G_n) = n$.

Remark. I do not know a proof of $la(G_n) \geqslant n$ without spectral methods!

G_n is the 1-skeleton of the regular subdivision of an equilateral triangle into $(n-1)^2$ small equilateral triangles. Each edge of the big triangle is divided into $n - 1$ edges belonging to some small triangles. We may describe vertices of G_n by their Cartesian coordinates in the basis (e, f) of $\mathbb{R}^2 = \mathbb{C}$ ($e = (1, 0)$, $f = (\frac{1}{2}, \sqrt{\frac{3}{2}})$) (see Fig. 4):

$$V_n = \{S_{m,k} = me + kf \mid 0 \leqslant m \leqslant n-1, 0 \leqslant k \leqslant n-1, m+k \leqslant, -1\}.$$

It is easy to check that G_n is a minor of $P_{2n-2} \times P_n$, where P_k is the path with k vertices; this shows that $la(G_n) \leqslant n$. We will prove that $\nu_1^{\mathbb{C}}(G_n) = n$; the same kind of proof works for $K = \mathbb{R}$. By Theorem 4, it shows that $la(G_n) \geqslant n$.

First, for any $A \in M_{G_n}$, $\dim(\ker(A)) \leqslant n$: otherwise there exists a non-zero function in $\ker(A)$ which vanishes on the n vertices $S_{0,0}, S_{1,0}, ..., S_{n-1,0}$. It is clear that such a function φ vanishes identically because we can compute (using $A\varphi = 0$) by induction on k its values on the vertices $S_{\cdot,k}$ from its values on the vertices $S_{\cdot,k'}$, $k' < k$.

For the converse, we exhibit an element $A \in M_{G_n}$. The simplest one has real coefficients,

$$A\varphi(z) = \sum_{z' \sim z} \varphi(z') + \frac{d(z)}{2} \varphi(z),$$

where $d(z)$ is the degree of z ($d(z) = 2$, 4, or 6 depending on the position of z). It is easier to define A by its quadratic form $q_A(x) = \langle Ax \mid x \rangle$.

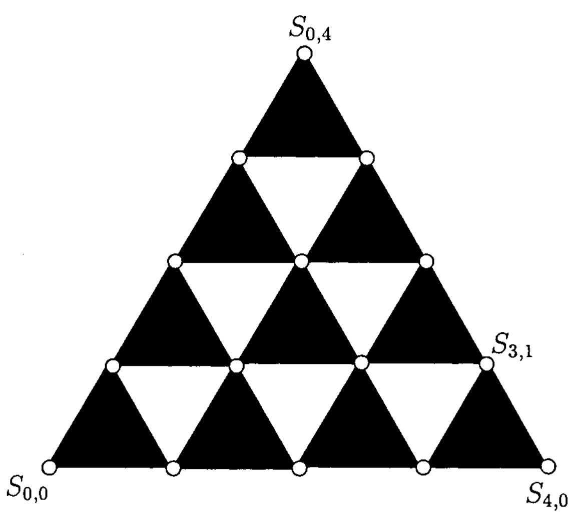

FIG. 4. The graph G_5.

Call a triangle of G_n *black* if it is of the form $(z, z+e, z+f)$. Then we have

$$q_A(x) = \sum_{\{i,\, j,\, k\} \in B} (x_i + x_j + x_k)^2,$$

where B is the set of black triangles. It is easy to check the following facts:

(i) $A \in R_{G_n}$ because each edge of G_n is (in a unique way) an edge of some black triangle.

(ii) A is non-negative.

(iii) The dimension of the kernel F of A is n because q_A is written as a sum of $|V(G_n)| - n$ (number of black triangles) squares of independent linear forms.

More precisely, there exist functions

$$\varphi_l \qquad (l = 0, ..., n-1),$$

where $\varphi_l(S_{i,0}) = \delta_{i,l}$, which form a basis of F.
What remains to do is to check transversality.

Proof. First, we need:

Lemma 5. *The support of φ_l consists of the $S_{m,k}$ in V_n which satisfy*

$$l - k \leqslant m \leqslant l.$$

The lemma follows from the relations

$$\varphi_l(S_{m,k}) = -(\varphi_l(S_{m,k-1}) + \varphi_l(S_{m+1,k-1})),$$

which are very close to the relations between binomial coefficients and can be solved explicitly:

$$\varphi_l(S_{m,k}) = (-1)^k \binom{k}{m+k-l}.$$

We will use Proposition 2. Let us introduce some notations. F is identified with $\mathbb{C}^{\{0,1,...,n-1\}}$ using the basis φ_l. We denote by $H = \mathrm{Herm}(F)$ the set of Hermitian forms on F and introduce a filtration

$$H_0 \subset H_1 \subset \cdots \subset H_{n-1} = H$$

in the following way: H_l is the set of Hermitian matrices whose entries $h_{i,j}$ vanish for $|i-j| > 1$.

We introduce the space $Q \subset \mathrm{Herm}(\mathbb{C}^{V_n})$, which is generated by the n^2 independent forms (using the notations of Definition 6)

$$\varepsilon_{m,0} = \varepsilon_{S_{m,0}}, \qquad m = 0, ..., n-1,$$

$$\varepsilon'_{m,k} = \varepsilon'_{z,z-f}, \qquad \varepsilon''_{m,k} = \varepsilon''_{z,z-f},$$

for $z = S_{m,k}$, $k \geqslant 1$.

We introduce the filtration $Q_0 \subset \cdots \subset Q_{n-1} = Q$, where Q_0 is generated by the $\varepsilon_{m,0}$, and Q_l, for $l \geqslant 1$, is generated by Q_0 and the $\varepsilon'_{m,k}$ and $\varepsilon''_{m,k}$ with $k \leqslant l$.

It is enough to prove that, if $\rho: Q \to H$ is the restriction to F, ρ is an isomorphism. In fact, ρ is compatible with the filtrations

$$\rho(Q_l) \subset H_l.$$

For example, we have

$$\rho(\varepsilon'_{m,k})(\varphi_i, \varphi_j) = \tfrac{1}{2}(\varphi_i(S_{m,k})\, \varphi_j(S_{m,k-1}) + \varphi_i(S_{m,k-1})\, \varphi_j(S_{m,k})),$$

which vanishes if $|i-j| > k$ by Lemma 5.

We shall check that

$$\rho_l \colon \frac{Q_l}{Q_{l-1}} \to \frac{H_l}{H_{l-1}}$$

is an isomorphism for $l \geqslant 0$ (setting $Q_{-1} = H_{-1} = 0$). Both spaces have the same dimension (n if $l = 0$ and $2(n-l)$ if $l \geqslant 1$).

Let us compute

$$B = \rho(\varepsilon'_{m,k}) \left(\sum x_i \varphi_i, \sum x_i \varphi_i \right);$$

we find

$$B = \Re \left(\sum x_i \bar{x}_j \varphi_i(S_{m,k})\, \varphi_j(S_{m,k-1}) \right),$$

and the product $\varphi_i(S_{m,k})\, \varphi_j(S_{m,k-1})$ vanishes if $|i-j| > k$ and, if $|i-j| = k$, it vanishes too except for $j = m$, $i = m+k$. This shows that ρ_l ($l > 0$) has a diagonal non-singular matrix with respect to the basis

$$\varepsilon'_{m,l} \bmod Q_{l-1}, \qquad \varepsilon''_{m,l} \bmod Q_{l-1}$$

for Q_l/Q_{l-1} and the basis of H_l/H_{l-1} consisting of elementary Hermitian matrices with non-zero entries at places where $|i-j| = l$. ∎

Remark. We started with a (slightly) more complicated example which is gauge equivalent to this one. Let us define a *holomorphic function* on $V(G_n)$ by the condition that the image of any direct black triangle be a direct equilateral triangle. Define $B \in M_{G_n}$ by the associated Hermitian form

$$q_B(\varphi) = \sum_z |\varphi(z+f) - \varphi(z) - e^{i\pi/3}(\varphi(z+e) - \varphi(z))|^2,$$

where the summation is on the $z = S_{m,k}$ with $m+k < n-1$. Then the kernel of B is the space of holomorphic functions on G_n and B is unitarily equivalent to A by the gauge transformation

$$\varphi(S_{m,k}) = e^{(2m+k)(2i\pi)/3}\varphi_1(S_{m,k}),$$

i.e., $q_B(\varphi) = q_A(\varphi_1)$.

8. QUESTIONS

Here is a selection of open questions which were presented at a CWI seminar.

1. *Computability questions.* Find algorithms computing $\mu(G)$ and $v_k^K(G)$ for a given graph G. Theoretically, there exist algorithms because everything can be expressed in terms of intersections of algebraic manifolds. Of course, it would be nice to have a computer program which computes these numbers.

2. *Maximizing the gap.* Let us come back to the real case. For many purposes it is interesting to have matrices A in O_Γ with a large gap ($\text{gap}(A) = \lambda_2 - \lambda_1$). The problem is to find an appropriate normalization condition which insures that the problem is well posed. Moreover, it seems reasonable to think that the multiplicity of $\lambda_2(A)$ is the largest possible if A maximizes the gap. Compare with [21] for the continuous case.

3. *$v_k^K(G)$ and $tw(G)$.* From general results by Robertson and Seymour, there exist functions $F_k^K\colon \mathbb{N} \to \mathbb{N}$ such that

$$tw(G) \leqslant F_k^K(v_k^K(G))$$

holds for planar graphs G. The question is to find some explicit functions $F_k^K\colon \mathbb{N} \to \mathbb{N}$, in other words to find explicit upper bounds for $tw(G)$ in terms of $v_k^K(G)$.

4. *Higher dimensional complexes.* The question is to extend the invariants considered in this paper to higher dimensional complexes, and find the relationship with Hodge–de Rham Laplace operators on forms.

5. *Chromatic number.* This problem is the most exciting: prove or disprove

$$\chi(G) \leqslant \mu(G) + 1,$$

where $\chi(G)$ is the chromatic number of G. This would imply the 4-color theorem and is weaker than the Hadwiger conjecture.

6. *Prescribing-spectras.* Describe all possible spectra for $A \in O_G$ or $A \in R_G$ or $A \in M_G$. For special graphs like trees this problem is not yet solved. It is solved for paths and for cycles.

For the cycle on N vertices C_N, we have the following set of inequalities for any $A \in O_{C_N}$:

$$\lambda_1 < \lambda_2 \leqslant \lambda_3 < \lambda_4 \leqslant \lambda_5 < \cdots.$$

It is known that for any graph G with N vertices and any subset $\sigma = \{\lambda_1 < \lambda_2 < \cdots < \lambda_N\}$ of $\mathbb{R}$, there exists $A \in O_G$ such that

$$\text{Spectrum}(A) = \sigma.$$

There is a general question: is it always true that the restrictions on possible spectra are given by restrictions on the multiplicities of eigenvalues? More precisely, if there exists $A_0 \in O_G$ whose spectrum is $\{\lambda_1 < \lambda_2 < \cdots < \lambda_N\}$ with multiplicity$(\lambda_i) = n_i$, $1 \leqslant i \leqslant N$, does there exist for any given $\mu_1 < \cdots < \mu_N$ some $A \in O_G$ whose spectrum is $\{\mu_1 < \cdots < \mu_N\}$ with multiplicity$(\mu_i) = n_i$, $1 \leqslant i \leqslant N$?

7. *A. Schrijver's question.* Is it always true that

$$\mu(G) = \min_{G \text{ minor of } G'} m(G'),$$

where $m(G')$ is the maximal multiplicity of the second eigenvalue for $A \in O_{G'}$?

This is true, for example, for paths, outerplanar graphs, and planar graphs: if G is planar and not outerplanar, $\mu(G) = 3$ and G is a minor of a triangulation G' of S^2 for which $m(G') = 3$ by H. van der Holst's result [16].

The question is the same for $\nu_k^K(G)$.

There is an additional question suggested by a referee: if G has enough connectivity, will the transversality conditions for $\mu(G)$ be fulfilled automatically? Again, this is true for paths (1-connectivity), for outerplanar graphs

(2-connectivity), and for planar graphs (3-connectivity). It can be shown that this is also true for $A \in M_G$ if G is 1-connected or 2-connected.

8. *Bounds on multiplicities using fluxes.* Given some $A \in M_G$, we may define the flux of the magnetic field through each cycle of G as a number in $\mathbb{R}/2\pi\mathbb{Z}$: if $\gamma = (a_1, a_2, ..., a_N)$ with $\forall i$ $(1 \leqslant i \leqslant N)$, $\{a_i, a_{i+1}\} \in E(G)$ $(a_{N+1} = a_1)$, the flux of the magnetic field associated with A is the argument of the product $\prod_{i=1}^{N} A_{i,i+1}$.

Question: is there any upper bound on dim(ker A) for $A \in M_G$ in terms of information on the flux?

For this problem, it is interesting to compare with the paper of Lieb of and Loss [19].

9. *Critical graphs.* Find critical graphs for $\mu(G)$ and $\nu_k^K(G)$; G is *critical* for ν if every strict minor G' of G satisfies $\nu(G') < \nu(G)$.

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