Scattering and correlations

Yves Colin de Verdière *

September 7, 2010

Introduction

Let us consider the propagation of scalar waves with the speed $v > 0$ given by the wave equation $u_{tt} - v^2 \Delta u = 0$ outside a compact domain D in the Euclidean space $\mathbb{R}^d$. Let us put $\Omega = \mathbb{R}^d \setminus D$. We can assume for example Neumann boundary conditions. We will denote by Δ_Ω the (self-adjoint) Laplace operator with the boundary conditions. So our stationary wave equation is the Helmholtz equation

$$v^2 \Delta_\Omega f + \omega^2 f = 0 \tag{1}$$

with the boundary conditions. We consider a bounded interval $I = [\omega_-^2, \omega_+^2] \subset]0, +\infty[$ and the Hilbert subspace $\mathcal{H}_I$ of $L^2(\Omega)$ which is the image of the spectral projector P_I of our operator $-v^2 \Delta_\Omega$.

Let us compute the integral kernel $\Pi_I(x, y)$ of P_I defined by:

$$P_I f(x) = \int_\Omega \Pi_I(x, y) f(y) |dy|$$

into 2 different ways:

1. From general spectral theory
2. From scattering theory.

Taking the derivatives of $\Pi_I(x, y)$ w.r. to ω_+, we get a simple general and exact relation between the correlation of scattered waves and the Green's function confirming the calculations from *Sanchez-Sesma and al. [March 2006]* in the case where D is a disk.

*Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France); yves.colin-de-verdiere@ujf-grenoble.fr

1 $\Pi_I(x,y)$ from spectral theory

Using the resolvent kernel (Green's function) $G(\omega,x,y) = [(\omega^2+v^2\Delta_\Omega)^{-1}](x,y)$ for $\mathrm{Im}\omega > 0$ and the Stone formula, we have:

$$\Pi_I(x,y) = -\frac{2}{\pi}\mathrm{Im}\left(\int_{\omega_-}^{\omega_+} G(\omega+i0,x,y)\omega d\omega\right)$$

Taking the derivative w.r. to ω_+ of $\Pi_{[\omega_-^2,\omega_+^2]}(x,y)$, we get

$$\frac{d}{d\omega}\Pi_{[\omega_-,\omega^2]}(x,y) = -\frac{2\omega}{\pi}\mathrm{Im}(G(\omega+i0,x,y)) \ . \tag{2}$$

2 Short review of scattering theory

They are many references for scattering theory: for example *Reed-Simon, Methods of modern Math. Phys. III; Ramm, Scattering by obstacles.*

Let us define for $\mathbf{k}\in\mathbb{R}^d$ the plane wave

$$e_0(x,\mathbf{k}) = e^{i<\mathbf{k}|x>} \ .$$

We are looking for solutions

$$e(x,\mathbf{k}) = e_0(x,\mathbf{k}) + e^s(x,\mathbf{k})$$

of the Helmholtz equation (1) in Ω where e^s, the scattered wave satisfies the so-called Sommerfeld radiation condition[1]:

$$e^s(x,\mathbf{k}) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}\left(e^\infty(\frac{x}{|x|},\mathbf{k}) + O(\frac{1}{|x|})\right), \ x\to\infty \ .$$

The complex function $e^\infty(\hat{x},\mathbf{k})$ is usually called the *scattering amplitude.*

It is known that the previous problem admits an unique solution. In more physical terms, $e(x,\mathbf{k})$ is the wave generated by the full scattering process from the plane wave $e_0(x,\mathbf{k})$. Moreover we have a generalised Fourier transform:

$$f(x) = (2\pi)^{-d}\int_{\mathbb{R}^d}\hat{f}(\mathbf{k})e(x,\mathbf{k})|d\mathbf{k}|$$

with

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^d}\overline{e(y,\mathbf{k})}f(y)|dy| \ .$$

From the previous generalised Fourier transform, we can get the kernel of any function $\Phi(-v^2\Delta_\Omega)$ as follows:

$$[\Phi(-v^2\Delta_\Omega)](x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d}\Phi(v^2k^2)e(x,\mathbf{k})\overline{e(y,\mathbf{k})}|d\mathbf{k}| \ . \tag{3}$$

[1]As often, we denote $k := |\mathbf{k}|$ and $\hat{\mathbf{k}} := \mathbf{k}/k$

3 $\Pi_I(x,y)$ from scattering theory

Using Equation (3) with $\Phi = 1_I$ the characteristic functions of some bounded interval $I = [\omega_-^2, \omega^2]$, we get:

$$\Pi_I(x,y) = (2\pi)^{-d} \int_{\omega_- \leq vk \leq \omega} e(x,\mathbf{k})\overline{e(y,\mathbf{k})}|d\mathbf{k}| \ .$$

Using polar coordinates and defining $|d\sigma|$ as the usual measure on the unit $(d-1)-$dimensional sphere, we get:

$$\Pi_I(x,y) = (2\pi)^{-d} \int_{\omega_- \leq vk \leq \omega} \int_{\mathbf{k}^2 = k^2} e(x,\mathbf{k})\overline{e(y,\mathbf{k})}k^{d-1}dk|d\sigma| \ .$$

We will denote by σ_{d-1} the total volume of the unit sphere in $\mathbb{R}^d$: $\sigma_0 = 2,\ \sigma_1 = 2\pi,\ \sigma_2 = 4\pi, \cdots$.

Taking the same derivative as before, we get:

$$\frac{d}{d\omega}\Pi_{[\omega_-^2,\omega^2]}(x,y) = (2\pi)^{-d}\frac{\omega^{d-1}}{v^d}\int_{vk=\omega} e(x,\mathbf{k})\overline{e(y,\mathbf{k})}|d\sigma| \ .$$

Let us look at $e(x,\mathbf{k})$ as a random wave with $k = \omega/v$ fixed. The point-point correlation of such a random wave $C_\omega^{\rm scatt}(x,y)$ is given by:

$$C_\omega^{\rm scatt}(x,y) = \frac{1}{\sigma_{d-1}}\int_{vk=\omega} e(x,\mathbf{k})\overline{e(y,\mathbf{k})}|d\sigma|.$$

Then we have:

$$\frac{d}{d\omega}\Pi_{[\omega_-^2,\omega^2]}(x,y) = (2\pi)^{-d}\frac{\omega^{d-1}\sigma_{d-1}}{v^d}C_\omega^{\rm scatt}(x,y) \ . \tag{4}$$

4 Correlation of scattered plane waves and Green's function: the scalar case

From Equations (2) and (4), we get:

$$(2\pi)^{-d}\frac{\omega^{d-1}\sigma_{d-1}}{v^d}C_\omega^{\rm scatt}(x,y) = -\frac{2\omega}{\pi}{\rm Im}(G(\omega+i0,x,y)) \ .$$

Hence

$$C_\omega^{\rm scatt}(x,y) = -\frac{2^{d+1}\pi^{d-1}v^d}{\sigma_{d-1}\omega^{d-2}}{\rm Im}(G(\omega+i0,x,y)) \ .$$

For later use, we put

$$\gamma_d = \frac{2^{d+1}\pi^{d-1}}{\sigma_{d-1}} \ . \tag{5}$$

5 The case of elastic waves

We will consider the elastic wave equation in the domain Ω:

$$\hat{H}\mathbf{u} - \omega^2 \mathbf{u} = 0,$$

with self-adjoint boundary conditions. We will assume that, *at large distances,* we have

$$\hat{H}\mathbf{u} = -a\ \Delta\mathbf{u} - b \text{ grad div}\mathbf{u}\ .$$

where a and b are constants:

$$a = \frac{\mu}{\rho},\ b = \frac{\lambda + \mu}{\rho}$$

with $\lambda,\ \mu$ the Lamé's coefficients and ρ the density of the medium. We will denote $v_P := \sqrt{a+b}$ (resp. $v_S := \sqrt{a}$) the speeds of the $P-$(resp. $S-$)waves near infinity.

5.1 The case $\Omega = \mathbb{R}^d$

We want to derive the spectral decomposition of $\hat{H}$ from the Fourier inversion formula. Let us choose, for $\mathbf{k} \neq 0$, by $\hat{\mathbf{k}}, \hat{\mathbf{k}}_1, \cdots, \hat{\mathbf{k}}_{d-1}$ an orthonormal basis of $\mathbb{R}^d$ with $\hat{\mathbf{k}} = \frac{\mathbf{k}}{k}$ such that these vectors depends in a measurable way of $\mathbf{k}$. Let us introduce $P_P^{\mathbf{k}} = \hat{\mathbf{k}}\hat{\mathbf{k}}^\star$ the orthogonal projector onto $\hat{\mathbf{k}}$ and $P_S^{\mathbf{k}} = \sum_{j=1}^{d-1} \hat{\mathbf{k}}_j \hat{\mathbf{k}}_j^\star$ so that $P_P + P_S = \text{Id}$. Those projectors correspond respectively to the polarisations of $P-$ and $S-$waves.

We have

$$\Pi_I(x,y) = (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega \left(v_P^{-d} \int_{v_P k = \omega} e^{i\mathbf{k}(x-y)} P_P^{\mathbf{k}} d\sigma + \right.$$
$$\left. v_S^{-d} \int_{v_S k = \omega} e^{i\mathbf{k}(x-y)} P_S^{\mathbf{k}} d\sigma \right)\ .$$

using the plane waves

$$e_P^O(x,\mathbf{k}) = e^{i\mathbf{k}x}\hat{\mathbf{k}}$$

and

$$e_{S,j}^O(x,\mathbf{k}) = e^{i\mathbf{k}x}\hat{\mathbf{k}}_j$$

we get the formula[2]:

$$\Pi_I(x,y) = (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega \left(v_P^{-d} \int_{v_P k = \omega} |e_P^O(x,\mathbf{k})\rangle\langle e_P^O(y,\mathbf{k})| d\sigma + \right.$$
$$\left. v_S^{-d} \sum_{j=1}^{d-1} \int_{v_S k = \omega} |e_{S,j}^O(x,\mathbf{k})\rangle\langle e_{S,j}^O(y,\mathbf{k})| d\sigma \right)\ .$$

[2] We use the "bra-ket" notation of quantum mechanics: $|e\rangle\langle f|$ is the operator $x \to \langle f|x\rangle e$ where the brackets are linear w.r. to the second entry and anti-linear w.r. to the first one

5.2 Scattered plane waves

There exists scattered plane waves

$$e_P(x, \mathbf{k}) = e_P^O(x, \mathbf{k}) + e_P^s(x, \mathbf{k})$$

$$e_{S,j}(x, \mathbf{k}) = e_{S,j}^O(x, \mathbf{k}) + e_{S,j}^s(x, \mathbf{k})$$

satisfying the Sommerfeld condition and from which we can deduce the spectral decomposition of $\hat{H}$.

5.3 Correlations of scattered plane waves and Green's function

Following the same path as for scalar waves, we get an identity which holds now for the full Green's tensor $\mathrm{Im}\mathbf{G}(\omega + iO, x, y)$:

$$\mathrm{Im}\mathbf{G}(\omega + iO, x, y) = -\gamma_d^{-1}\omega^{d-2}\left(\frac{1}{\sigma_{d-1}v_P^d}\int_{v_P k=\omega} |e_P(x, \mathbf{k})\rangle\langle e_P(y, \mathbf{k})| d\sigma + \frac{1}{\sigma_{d-1}v_S^d}\sum_{j=1}^{d-1}\int_{v_S k=\omega} |e_{S,j}(x, \mathbf{k})\rangle\langle e_{S,j}(y, \mathbf{k})| d\sigma\right) ,$$

with γ_d defined by Equation (5).

This formula expresses the fact that the correlation of scattered plane waves randomised with the appropriate weights (v_P^{-d} versus v_S^{-d}) is proportional to the Green's tensor. Let us insist on the fact that this true everywhere in Ω even in the domain where a and b are not constants.