An extension of the Duistermaat-Singer Theorem to the semi-classical Weyl algebra

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Abstract

Motivated by many recent works (by L. Charles, V. Guillemin, T. Paul, J. Sjöstrand, A. Uribe, San Vũ Ngọc, S. Zelditch and others) on the semi-classical Birkhoff normal forms, we investigate the structure of the group of automorphisms of the graded semi-classical Weyl algebra. The answer is quite similar to the Theorem of Duistermaat and Singer for the usual algebra of pseudo-differential operators where all automorphisms are given by conjugation by an elliptic Fourier Integral Operator (a FIO). Here what replaces general non-linear symplectic diffeomeorhisms is just linear complex symplectic maps, because everything is localized at a single point[1].

1 The result

Let $W = W_0 \oplus W_1 \oplus \cdots$ be the semi-classical graded Weyl algebra (see Section 2 for a definition) on $\mathbb{R}^{2d}$. Let us define $X_j := W_j \oplus W_{j+1} \oplus \cdots$. We want to prove the:

Theorem 1.1 *There exists an exact sequence of groups*

$$0 \to \mathcal{I} \to_1 \mathrm{Aut}(W) \to_2 \mathrm{Sympl}_{\mathbb{C}}(2d) \to 0$$

where

- $\mathrm{Sympl}_{\mathbb{C}}(2d)$ *is the group of linear symplectic transformations of* $\mathbb{C}^{2d} = \mathbb{R}^{2d} \otimes \mathbb{C}$

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[1]Thanks to Louis Boutet de Monvel for his comments and suggestions, in particular the proofs of Lemmas 6.1 and 6.2. As he said, the main result is not surprising!

- $\mathrm{Aut}(W)$ *is the group of automorphisms* Φ *of the semi-classical graded*[2] *Weyl algebra preserving* $\hbar$
- $\mathcal{I}$ *is the group of "inner" automorphisms* Φ_S *of the form* $\Phi_S = \exp(i\mathrm{ad}S/\hbar)$, *i.e.* $\Phi_S(w) = \exp(iS/\hbar) \star w \star \exp(-iS/\hbar)$ *as a formal power series, with* $S \in X_3$
- *The arrow* $\to_2$ *is just given from the action of the automorphism* Φ *on* $W_1 = \left(\mathbb{R}^{2d}\right)^\star \otimes \mathbb{C}$ *modulo* X_2

The proof follows [3] and also the semi-classical version of it by H. Christianson [2]. This result is implicitely stated in Fedosov's book [4] in Chapter 5, but it could be nice to have an direct proof in a simpler context. The result is a consequence of Lemmas 4.1, 6.1 and 6.2.

2 The Weyl algebra

The elements of the "Weyl algebra" are the formal power series in $\hbar$ and (x, ξ)

$$W = \oplus_{n=0}^{\infty} W_n$$

where W_n is the space of complex valued homogeneous polynomials in $z = (x, \xi)$ and $\hbar$ of total degree n where the degree of $\hbar^j z^\alpha$ is $2j + |\alpha|$. The Moyal $\star-$product

$$a \star b := \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\hbar}{2i}\right)^j a \left(\sum_{p=1}^{d} \overleftarrow{\partial}_{\xi_p} \vec{\partial}_{x_p} - \overleftarrow{\partial}_{x_p} \vec{\partial}_{\xi_p}\right)^j b = ab + \frac{\hbar}{2i}\{a, b\} + \cdots$$

(where $\{a, b\}$ is the Poisson bracket of a and b) gives to W the structure of a graded algebra: we have $W_m \star W_n \subset W_{m+n}$ and hence, for the brackets, $\frac{i}{\hbar}[W_m, W_n]^\star \subset W_{m+n-2}$.

The previous grading of W is obtained by looking at the action of W on the (graded) vector space $\mathcal{S}$ of *symplectic spinors* (see [5]): if $F \equiv \sum_{j=0}^{\infty} \hbar^j F_j(X)$ with $F_j \in \mathcal{S}(\mathbb{R})$, we define $f_\hbar(x) = \hbar^{-d/2} F(x/\hbar)$ whose micro-support is the origin. W acts on $\mathcal{S}$ in a graded way as differential operators of infinite degree: if $w \in W$, $w.f = \mathrm{OP}_\hbar(w)(f)$.

3 A remark

We assumed in Theorem 1.1 that $\hbar$ is fixed by the automorphism. If not, the symplectic group has to be replaced by the homogeneous symplectic group: the group of the linear automorphisms M of $(\mathbb{R}^{2d}, \omega)$ which satisfies $M^\star\omega = c\omega$. We then have to take into account a multiplication of $\hbar$ by c. For $c = -1$, it is a semi-classical version of the transmission property of Louis Boutet de Monvel.

[2] "graded" means that $\Phi(W_n) \subset W_n \oplus W_{n+1} \oplus \cdots$

4 Surjectivity of the arrow $\to_2$

Lemma 4.1 *The arrow $\to_2$ is surjective.*

Proof.–

Let us give $\chi \in \mathrm{Sympl}_{\mathbb{C}}(2d)$. The map $a \to a \circ \chi$ is an automorphism of the Weyl algebra: the Moyal formula is given only in terms of the Poisson bracket.

□

5 The principal symbols

Let Φ be an automorphism of W. Then Φ induces a linear automorphism Φ_n of W_n: if $w = w_n + r$ with $w_n \in W_n$ and $r \in X_{n+1}$ and $\Phi(w) = w'_n + r'$ with $w'_n \in W_n$ and $r' \in X_{n+1}$, $\Phi_n(w_n) := w'_n$ is independent of r. The polynomial w_n is the *principal symbol* of $w \in X_n$. We have $\Phi_{m+n}(w_m \star w_n) = \Phi_m(w_m) \star \Phi_n(w_n)$. Hence Φ_n is determined by Φ_1 because the algebra W is generated by W_1 and $\hbar$. The linear map Φ_1 is an automorphism of the complexified dual of $\mathbb{R}^{2d}$. Let us show that it preserves the *Poisson bracket* and hence is the adjoint of a linear symplectic mapping of $\mathbb{C}^{2d}$. We have:

$$\Phi([w, w']^\star) = [\Phi(w), \Phi(w')]^\star \ .$$

By looking at principal symbols, for $w, w' \in X_1$, we get

$$\{\Phi_1(w), \Phi_1(w')\} = \{w, w'\} \ .$$

6 Inner automorphisms

The kernel of $\to_2$ is the group of automorphisms Φ which satisfy $\Phi_n = \mathrm{Id}$ for all n, i.e. for any $w_n \in W_n$

$$\Phi(w_n) = w_n \mod X_{n+1} \ .$$

The following fact is certainly well known:

Lemma 6.1 *If $\Phi \in \ker(\to_2)$, $\Phi = \exp(D)$ where $D : W_n \to X_{n+1}$ is a derivation of W.*

Proof [following a suggestion of Louis Boutet de Monvel]– We define Φ^s for $s \in \mathbb{Z}$. Let $\Phi^s_{p,n} : W_n \to W_{n+p}$ be the degree $(n+p)$ component of $(\Phi^s)_n : W_n \to X_n$. Then $\Phi^s_{p,n}$ is polynomial w.r. to s. This allows to extend Φ^s to $s \in \mathbb{R}$ as a 1-parameter group of automorphims. We put $D = \frac{d}{ds}(\Phi^s)_{|s=0}$. We have $\Phi^s = \exp(sD)$. We deduce that D is a derivation. □

We need to show the:

Lemma 6.2 *Every derivation D of W sending W_1 into X_2 is an inner derivation of the form*

$$Dw = \frac{i}{\hbar}[S, w]$$

with $S \in X_3$.

Proof [following a suggestion of Louis Boutet de Monvel]–

Let (ζ_k) the basis of W_1 dual to the canonical basis (z_k) for the star bracket, i.e. satisfying $[\zeta_k, z_l] = \frac{\hbar}{i}\delta_{k,l}$. We have $[\zeta_k, w] = \frac{\hbar}{i}\frac{\partial w}{\partial z_k}$. Put $y_k = D\zeta_k \in X_2$. As the brackets $[\zeta_k, \zeta_l]$ are constants, we have: $[D\zeta_k, \zeta_l] + [\zeta_k, D\zeta_l] = 0$, or $\partial y_k/\partial z_l = \partial y_l/\partial z_k$. There exists an unique S vanishing at 0 so that:

$$[S, \zeta_k] = -\frac{\partial S}{\partial z_k} = \frac{\hbar}{i} y_k \ .$$

Hence $D = (i/\hbar)[S, .]$. Because $y_k \in X_2$, S is in X_3. □

7 An homomorphism from the group G of elliptic FIO's whose associated canonical transformation fixes the origine into $\mathrm{Aut}(W)$

Each $w \in W$ is the Taylor expansion of a Weyl symbol $a \equiv \sum_{j=0}^{\infty} \hbar^j a_j(x, \xi) \in S^0$ of a pseudo-differential operator $\hat{a}$. Let us give an elliptic Fourier Integral Operator U associated to a canonical transformation χ fixing the origin. The map $\hat{a} \to U^{-1}\hat{a}U$ induces a map F from S^0 into S^0 which is an automorphism of algebra (for the Moyal product).

Lemma 7.1 *The Taylor expansion of $F(a)$ only depends on the Taylor expansion of a.*

This is clear from the explicit computation and the stationary phase expansions.

As a consequence, F induces an automorphism F_0 of the Weyl algebra graded by powers of $\hbar$.

Lemma 7.2 *F_0 is an automorphism of the algebra W graded as in Section 2.*

Proof.–

We have to check the $F_0(W_n) \subset X_n$. Because F_0 preserves the $\star-$product, it is enough to check that $F_0(W_1) \subset X_1$. It only means that the (usual) principal symbol of $F(a)$ vanishes at the origin if a does. It is consequence of Egorov Theorem.

□

Summarizing, we have constructed a group morphism α from the group G of elliptic FIO's whose associated canonical transformation fixes the origin in the group $\mathrm{Aut}(W)$.

Definition 7.1 *An automorphism* Φ *of* W *is said to be* real *(*$\Phi \in \mathrm{Aut}_{\mathbb{R}}(W)$*) if the mapping* $\Phi \bmod(\hbar W)$ *is real.*

Theorem 7.1 *The image of the group* G *by the homomorphism* α *is* $\mathrm{Aut}_{\mathbb{R}}(W)$*. In particular, any* $\Phi \in \mathrm{Aut}_{\mathbb{R}}(W)$ *can be "extended" to a semi-classical Fourier Integral Operator.*

Proof.–

The image of α is in the sub-group $\mathrm{Aut}_{\mathbb{R}}(w)$ because the canonical transformation χ is real.

We have still to prove that the image of α is $\mathrm{Aut}_{\mathbb{R}}(W)$. Using the Theorem 1.1 and the metaplectic representation, it is enough to check that the automorphism $\exp(i\mathrm{ad}S/\hbar)$ comes from an FIO. Let H be a symbol whose Taylor expansion is S (the principal symbol of H is a real valued Hamiltonian). The OIF $U = \exp\left(i\hat{H}/\hbar\right)$ will do the job.

□

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