Modes and quasi-modes on surfaces: variation on an idea of Andrew Hassell

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1 Introduction

This paper is inspired from the nice idea of A. Hassell in [5]. From the classical paper of V. Arnol'd [1], we know that quasi-modes are not always close to exact modes. We will show that, for *almost all* Riemannian metrics on closed surfaces with an elliptic generic closed geodesic γ, there exists exact modes located on γ. Similar problems in the integrable case are discussed in several papers of J. Toth and S. Zelditch (see [8]).

2 Quasi-modes associated to an elliptic generic closed geodesic

2.1 Babich-Lazutkin and Ralston quasi-modes

Definition 2.1 *A periodic geodesic γ on a Riemannian surface (X, g) is said to be* elliptic generic *if the eigenvalues of the linearized Poincaré map of γ are of modulus 1 and are not roots of the unity.*

Theorem 2.1 (Babich-Lazutkin [2], Ralston [6, 7]) *If γ is an elliptic generic closed geodesic of period $T > 0$ on a closed Riemannian surface (X, g), there exists a sequence of* quasi-modes *$(u_m)_{m\in\mathbb{N}}$ of $L^2(X, dx_g)$ norm equal to 1 which satisfies*

- $\|(\Delta_g - \lambda_m)u_m\|_{L^2(X,dx_g)} = O(m^{-\infty})$
- *There exists α so that*[1]

$$\lambda_m = \left(\frac{2\pi m + \alpha}{T}\right)^2 + O(1)$$

- *For any compact K disjoint of γ, $\int_K |u_m|^2 = O(m^{-\infty})$.*

Corollary 2.1 *There exists a sub-sequence $(\mu_{j_m})_{m\in\mathbb{N}}$ of the spectrum $(\mu_j)_{j\in\mathbb{N}}$ of the Laplace operator so that $\mu_{j_m} = \lambda_m + O(m^{-\infty})$.*

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[1] α is given by $\alpha = (m_1 + \frac{1}{2})\theta + p\pi$ where $m_1 \in \mathbb{N}$ is a "transverse" quantum number, the linearized Poincaré map is a rotation of angle θ ($0 < \theta < 2\pi$) and $p = 0$ or 1 is a "Maslov index" of γ

3 Modes and quasi-modes following Arnol'd

Arnol'd [1] has observed that, given a quasi-mode $(u_m)_{m\in\mathbb{N}}$, there do not always exists a sequence $(\varphi_{j_m})_{m\in\mathbb{N}}$ of exact modes close to the quasi-mode $(u_m)_{m\in\mathbb{N}}$. His example is given by a planar domain with a symmetry of order 3.

A simpler example is given by a symmetric double well: let us give $V : \mathbb{R} \rightarrow [0, +\infty[$ a smooth even function with

- $\lim_{x\rightarrow\infty} V(x) = +\infty$
- $V^{-1}(0) = \{-a, a\}$ with $a > 0$
- $V(0) = b > 0$

. If $\hat{H} = -\hbar^2 d_x^2 + V(x)$ is the semi-classical Schrödinger operator, there exists quasi-modes located in the well $V := \{x \mid x > 0 \text{ and } V(x) < b\}$. The exact eigenfunctions are even or odd and hence are not localized in a single well.

The previous examples are in some sense *non generic*. They involve some symmetry of the operator.

4 The main result

Theorem 4.1 *Let us give a closed Riemannian surface (X, g_0) and a smooth non-zero function $f \geq 0$. Let us define the metric $g_t := \exp(-tf)g_0$. Let us assume that there exists some intervals $I_m = [\lambda_m - l_m, \lambda_m + l_m]$, $(m \in \mathbb{N})$,* **independent of** *t, so that, for any $t \in [0, 1]$, there exists at least one eigenvalue of Δ_t inside I_m. Assume that $\lambda_m \rightarrow +\infty$ and $\sum_{m=1}^{\infty} l_m < \infty$. Choose a sequence $q_m \rightarrow 0$ so that $\sum_{m=1}^{\infty} l_m/q_m < \infty$.*

Then, **for almost all** *$t \in [0, 1]$, for any sequence of exact modes $\varphi_m(t)$ of eigenvalues $\mu_m(t)$ with $\mu_m(t) \in I_m$, we have $\int_X f|\varphi_m(t)|^2 dx_t = o(q_m)$.*

In particular, if $\Gamma = \text{support}(f)$, $\varphi_m(t) \rightarrow 0$ in $L^2_{\text{loc}}(X \setminus \Gamma)$.

Remark 4.1 *In applications, the interval I_m is provided from quasi-modes located in the support of f: if u_m is a quasi-mode for each values of t with*

$$\|(\Delta_t - \lambda_m)u_m\|_{L^2(X,dx_0)} \leq C_m \|u_m\|_{L^2(X,dx_0)}$$

with λ_m independent of t, we can take $l_m = cC_m$ with c large enough, depending only on bounds of f.

Remark 4.2 *The quasi-mode is only used in order to find a sequence of intervalls I_m which contains at least one eigenvalue of Δ_t and is independent of t.*

Remark 4.3 *It works with (u_m) the quasi-modes of Theorem 2.1 with $\Gamma = \gamma$ an elliptic generic closed geodesic and f flat on γ, because the functions u_m satisfies an estimates*

$$u_m(x) = O\left(e^{-cd^2(x,\gamma)/\sqrt{\lambda_m}}\right) .$$

We can then take $q_m = O(m^{-\infty})$.

We are unfortunately unable to prove that the modes φ_m are close to linear combinations of the quasi-modes given in Theorem 2.1 in the interval I_m.

The precise statement is

Corollary 4.1 *With the notations of Section 2, there exists a sequence $0 < l_m = 0(m^{-\infty})$ so that, for any $t \in [0, 1]$, $\text{Spectrum}(\Delta_t) \cap [\lambda_m - l_m, \lambda_m + l_m] \neq \emptyset$ and a* *subset $Z \subset [0,1]$ of measure 1, so that, for any sequence $\mu_{j_m}(t) \in [\lambda_m - l_m, \lambda_m + l_m]$ and for any $t \in Z$,*

$$\int_X f|\varphi_{j_m}(t)|^2 dx_0 = 0(m^{-\infty}) \ .$$

Moreover, for any compact $K \subset X$ with $K \cap \gamma = \emptyset$ and for any $k \in \mathbb{N}$, we have

$$\|\varphi_{j_m}(t)\|_{C^k(K)} = 0(m^{-\infty}) \ .$$

Proof.–

The first part is a direct application of Theorem 4.1.

The second part comes from the Sobolev embeddings and the equations $\Delta_t^N \varphi_{j_m}(t) = \mu_{j_m}(t)^N \varphi_{j_m}(t)$ with $\mu_{j_m}(t) = 0(m^2)$.

□

Remark 4.4 *If we have only $l_m \to 0$, on can apply the previous result by taking first a sub-sequence m_k so that $\sum l_{m_k} < \infty$ and choosing then $q_{m_k} \to 0$. This does not work with $l_m = O(1)$ as in the paper [5].*

5 Variation of the eigenvalues

With $g_t = e^{-tf} g_0$, we define $dx_t = e^{-tf} dx_0$ the Riemannian area of g_t and $\Delta_t = e^{tf} \Delta_0$ the Laplace operator. Let us denote by

$$\mu_1(t) = 0 < \mu_2(t) \leq \cdots \leq \mu_j(t) \leq \cdots$$

the eigenvalues of Δ_t and by $(\varphi_j(t))_{j \in \mathbb{N}}$ an associated orthonormal eigenbasis.

Lemma 5.1
- *$\mu_j(t)$ is a continuous increasing function of t*
- *$\mu_j(t)$ is piecewise analytic and, at any regular point, the t-derivative of $\mu_j(t)$ is given by:*

$$\dot{\mu_j}(t) = \mu_j(t) \int_X f \varphi_j(t)^2 dx_t \ . \quad (1)$$

Proof.–

- The Rayleigh quotient $R_t(\varphi)$ is given by

$$R_t(\varphi) = \int_X \|d\varphi\|_{g_0}^2 dx_0 / \int_X e^{-tf} \varphi^2 dx_0$$

  which is an increasing function of t. Applying the min-max characterization of the eigenvalues, we get their monotonicity.
- Because Δ_t is an analytic function of t, we know that $\mu_j(t)$ is continuous and piecewise smooth as well as $\varphi_j(t)$. We can then compute formally the derivative of the eigenfunction's equation

$$e^{tf} \Delta_0 \varphi_j(t) = \mu_j(t) \varphi_j(t) \ ,$$

  and get

$$f \Delta_t \varphi_j(t) + \Delta_t \dot{\varphi}_j(t) = \dot{\mu}_j(t) \varphi_j(t) + \mu_j(t) \dot{\varphi}_j(t) \ ,$$

  and taking the t-scalar product with $\varphi_j(t)$, we get Equation (1).

□

6 The proof

The proof is an adaptation of the argument of [5]. Let us denote by $I_m = [\lambda_m - l_m, \lambda_m + l_m]$. From the Weyl law and the monotonicity of the μ_j's, we deduce the

Lemma 6.1 *For any $t \in [0,1]$, $\#\{j \mid \mu_j(t) \in I_m\} = O(\lambda_m)$ uniformly in t.*

In fact,

$$\#\{j \mid \mu_j(t) \in I_m\} \leq \#\{j \mid \mu_j(t) \leq \lambda_m + l_m\} \leq \#\{j \mid \mu_j(0) \leq \lambda_m + l_m\} \ !$$

We will also need the elementary

Lemma 6.2 *Let $F : [0,1] \to \mathbb{R}$ be an increasing, continuous and piecewise C^1 function. Let us give a Borel set $Y \subset [0,1]$ and I a compact interval of $\mathbb{R}$ so so that $F'(t) \geq m > 0$ for almost all $t \in K = F^{-1}(I) \cap Y$. Then the Lebesgue measure $|K|$ of K satisfies $|K| \leq |I|/m$.*

Let us denote by

$$Z := \left\{ t \in [0,1] \mid \lim_{m\to\infty} q_m^{-1} \left(\sup_{\mu_j(t) \in I_m} \int_X f|\varphi_j(t)|^2 dx_t \right) = 0 \right\} ,$$

(Z is well defined because there exists at least one $\mu_j(t) \in I_m$ for each m) and $Y = [0,1] \setminus Z$. Let us denote also, for $\varepsilon > 0$, by

$$Y_\varepsilon^m := \left\{ t \mid \exists j \text{ with } \mu_j(t) \in I_m, \int_X f|\varphi_j(t)|^2 dx_t \geq \varepsilon q_m \right\} .$$

Using Lemma 6.1, the monotonicity of $t \to \mu_j(t)$ and the lower bound $\dot{\mu}_j(t) \geq \mu_j(t)\varepsilon q_m$ in Lemma 6.2, we have

$$|Y_\varepsilon^m| \leq C\lambda_m \frac{|I_m|}{\varepsilon q_m(\lambda_m - Cl_m)} = O\left(\frac{l_m}{\varepsilon q_m}\right) .$$

Let us give a sequence $\varepsilon_m \to 0$, then, for any m_0, $Y \subset \cup_{m \geq m_0} Y_{\varepsilon_m}^m$. But this implies that $|Y|$ is arbitrarily close to 0 by choosing ε_m so that $\sum_m l_m/\varepsilon_m q_m < \infty$. This proves that $|Z| = 1$ and the Theorem.

7 Null sets in Banach spaces

It is not clear what is a set of measure 0 in a infinite dimensional Banach space because there is no "Lebesgue measure" on it. There are several definitions of sets of measure 0 in a *separable Banach space B*. For Borel sets, it is shown in [4] that the notions of *cube* null sets and *Gaussian* null sets coincide.

Definition 7.1

- *A* cube measure *in B is defined as the distribution of a random variable $\sum_{i\in\mathbb{N}} t_i e_i$ where $\mathbf{t} = (t_i) \in [0,1]^{\mathbb{N}}$ with the Lebesgue measure and the sequence $(e_i)_{i\in\mathbb{N}}$ span a dense subspace of B with $\sum_{i\in\mathbb{N}} \|e_i\| < \infty$.*
- *A* cube null set *is a Borel subset of B which is of measure 0 for every cube measure.*
- *A* Gaussian measure *on B is a Borel measure whose image by any continuous linear form on B is a (non-degenerate) Gaussian measure on R (i.e. of the form $dm = A.\exp(-(x-a)^2/b)dx$ with $A > 0$).*

- *A* Gaussian null set *is a Borel set which is of measure* 0 *for every Gaussian measure.*

It is proved in [4] that cube null sets and Gaussian null sets coincide in every separable Banach space.

We have the:

Lemma 7.1 *Let B be a separable Banach space and $C \subset B$ be a non empty open cone. Let us give a Borel set $Z \subset B$ so that, for any $x \in B,\ y \in C$,*

$$|\{t \mid x + ty \in Z\}| = 0 \ .$$

Then Z is a cube null and Gaussian null set.

Proof.–

Let us show that Z is of measure 0 for every cube measure given from sequence $(e_i)_{i \in \mathbb{N}}$. There exists $k \in \mathbb{N}$ and $(t_1, \cdots, t_k) \in [0,1]^k$ so that $e = \sum_{i=1}^k t_i e_i \in C$. Let us rewrite the Lebesgue measure on $[0,1]^k$ as

$$\int_{[0,1]^k} f(t)dt = \int_X d\mu(d) \int_{d \cap [0,1]^k} f(t)ds \qquad (2)$$

where X is the set of lines parallel to e cutting $[0,1]^k$ and ds is the Lebesgue measure on the line d. Let us denote $t = (t', t'') \in [0,1]^k \times [0,1]^{\mathbb{N}}$ and denote by

$$Z_{t''} := \{t' \mid x + \sum_{i=1}^k t'_i e_i + \sum_{j>k} t''_j e_j \in Z\} \ .$$

Equation (2) shows that $Z_{t''}$ is of measure 0. We can then use Fubini Theorem on $[0,1]^k \times [0,1]^{\mathbb{N} \setminus \{1, \cdots, k\}}$ in order to finish the proof.

□

8 From Theorem 4.1 to almost all metrics

We will apply the previous result to the following situation where (X, g_0) is our smooth closed surface and γ a closed geodesic; let us choose N large (and even) and define B as follows:

$$B = \{f \in C^N(X, \mathbb{R}) \mid \forall \alpha \text{ with } |\alpha| \le N,\ D^\alpha f \text{ vanishes on } \gamma\}$$

and C the open cone of functions of B which satisfy

$$\exists c > 0 \text{ such that } f(x) \ge c d(x, \gamma)^N$$

with d the distance associated to g_0.

Then Theorem 4.1 can be reformulated with *almost all metrics $e^f g_0$ with $f \in B$* instead of *almost all $t \in [0,1]$* . Of course, we can only take l_m of the order of $m^{-N'}$, where N' depends on N.

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