SINGULAR LAGRANGIAN MANIFOLDS and SEMI-CLASSICAL ANALYSIS

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March 10, 2011

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Abstract

Lagrangian submanifolds of symplectic manifolds are very central objects in classical mechanics and microlocal analysis. These manifolds are frequently singular (integrable systems, bifurcations, reduction). There has been a lot of works on singular Lagrangian manifolds initiated by Arnold, Givental and others. The goal of our paper is to extend the classical and semi-classical normal forms of completely integrable systems near non degenerate (Morse-Bott) singularities to more singular systems. It turns out that there is a nicely working way to do that, leading to normal forms and universal unfoldings. We obtain this way natural Ansatz's extending the WKB-Maslov Ansatz. We give more details on the simplest non Morse example, the cusp, which corresponds to a saddle-node bifurcation[1].

Keywords: singular Lagrangian manifolds, integrable Hamiltonian systems, bifurcations, quasi-homogeneous singularities, Bohr-Sommerfeld rules, WKB, semi-classics, normal forms, versal deformations.
Mathematics Subject Classification 2000: 32S05, 32S40, 34E20, 35P20, 53D12, 70H06.

[1] **Acknowlegments:** the papers of Eric Delabaere and Frédéric Pham as well as discussions with them were important sources of inspiration. Emmanuel Ferrand provided me a basic list of references on singular Lagrangian manifolds. I thank also Bernard Malgrange for remarks on preliminary versions, discussions and "Theorem 1" and San Vũ Ngọc for carefully reading a preliminary version. Pertinent remarks of the referees were helpful in order to make the paper correct and legible.

Introduction

In the papers [12], [13], [14], [11], [37] and [16], we studied semi-classical completely integrable Hamiltonian systems whose singularities are of Morse-Bott type using normal forms of Birkhoff type. In the nice paper [33] which was an important source of inspiration for us, Frédéric Pham showed the universality of solutions of semi-classical Schrödinger equations with polynomial potentials. Our goal is to extend this analysis allowing (more general) canonical transformations in order to study for example

- the saddle-node bifurcation
- the Birkhoff normal form in case of $k : 1$ resonances with $k \geq 3$ in the spirit of [16]
- the bifurcation of periodic orbits of a Hamiltonian system where the Poincaré map of a periodic orbit admits an eigenvalue which is a cubic root of 1
- the adiabatic limit or the Born-Oppenheimer approximation with crossings of more than 2 eigenvalues.

This way, we propose a general setting inspired by "Thom's catastrophe theory" (see [3]) and present a sketchy study of the *saddle-node bifurcation* (the cusp) $\xi^2 + x^3 = 0$.

A more algebraic (co-homological approach) is presented in [40].

The subject is really the study of the *singularities of Lagrangian manifolds*, of their *deformations* (or *bifurcations*) and of the associated *semi-classical Ansatz's*. Building up classical and semi-classical normal forms leads to study model problems depending on a finite number of parameters among whose the simplest were already described in the litterature: cubic oscillators (see [10], [9], [20]), quartic oscillators (see [19], and polynomial potentials (see [42])). A noticeable fact is that we can use the same methods for the classical and the semi-classical bifurcations and in particular the codimension of the singularities are the same. Of course, the study of the classical Hamiltonian dynamic in a 2D phase space is trivial, but this is no more the case for the semi-classical case which we reduce to special functions.

The reader should take care of the fact that caustic singularities is a different problem for which Lagrangian manifolds usually are smooth. We strongly use canonical transformations which eliminate the problem of caustics.

The main idea is to forget the equations of the manifolds and to focus on the *ideal of functions* which vanish on it. The same idea turned out to be very important in algebraic geometry. On the quantum side, we do the same change of point of view: we consider *left ideals of pseudo-differential operators.* We can do that because any solution of $\hat{P}u = 0$ satisfies also $\hat{B}\hat{P}u = 0$ for any operator $\hat{B}$. It appears that usual singularities, at least for 1 degree of freedom, do admit normal forms and their deformations have a universal model depending of a finite number of parameters. The solutions of this model are the ad'hoc special functions: the smooth case corresponds that way to the BKW-Maslov Ansatz, the Morse-Bott case corresponds to Lagrangian intersections (hyperbolic case) or to coherent states (elliptic case)... An important part of the *programme* is the study of these special functions.

In the case of the cusp $\xi^2 + x^3 = 0$, it is enough to study the differential equation (cubic Schrödinger equation):

$$-u'' + (x^3 + Ax + B)u = 0 \ .$$

We give the general definitions for any dimension and we restrict after section 2 to the case of a 2 dimensional phase space.

The main non trivial result is Theorem 6 which is an holomorphic versal deformation result for quasi-homogeneous isolated singularities of curves.

The semi-classical results follow then from the techniques already developped in [14].

Contents

1 Singular Lagrangian manifolds

1.1 Definitions

There are several possible definitions of germs of singular Lagrangian manifolds. We will work in the real analytic context. We will denote by $(Z^{2d}, \omega; z_0)$ a germ of non-singular real analytic symplectic manifold of dimension $2d$ which, by Darboux theorem, can be identified with $(T^\star R^d, \sum d\xi_i \wedge dx_i; 0)$. $\mathcal{E}$ will denote the algebra of germs of real valued analytic functions (or smooth functions).

Definition 1 *1. A (germ of)* singular Lagrangian manifold *L in Z^{2d} is a germ of real analytic variety (ie complex variety invariant by complex conjugation) of dimension d which is Lagrangian near all smooth points. We will denote by $\mathcal{L}$ or $\mathcal{L}_L$ the ideal of $\mathcal{E}$ of functions vanishing on L. If $F_j,\ j = 1, \cdots, n$ is a system of generators of $\mathcal{L}$ we will denote $\mathcal{L} =< F_1, \cdots, F_n >$. This ideal is* involutive *meaning that $\{\mathcal{L}, \mathcal{L}\} \subset \mathcal{L}$.*

2. A (germ of)singular Lagrangian manifold L is a complete intersection *if the ideal $\mathcal{L}_L$ is generated by d functions.*

3. A (germ of) singular Lagrangian manifold L is a singular leaf of a Lagrangian foliation *if $\mathcal{L}_L$ admits a set of generators $F_j,\ j = 1, \cdots, d$ such that $\{F_j, F_k\} = 0$ for all j, k.*

In the first case we will speak about a (germ of) singular Lagrangian manifold*, in the second of a* (germ of) singular Lagrangian manifold which is a complete intersection *and in the third of a* (germ of) singular leaf of a completely integrable system.

We can ask the

Question 1 *Are cases 2 and 3 really distinct: does every singular Lagrangian manifold which is a complete intersection admits Poisson commuting generators $F_j,\ j = 1, \cdots, d$?*

1.2 Examples

Example 1.1 *Let $d = 1$ and $f : Z^2 \to \mathbb{R}$ an analytic map. If 0 is a critical value of f, the curve $\{f = 0\}$ is a Lagrangian singular manifold with respect to all possible definitions. If f is a Morse function the level sets $L_E = f^{-1}(E)$ are smooth except for a discrete set of energies.*

Example 1.2 *Let us start with an anharmonic oscillator with only one resonance, like $H = |z_1|^2 + |z_2|^2 + \sum_{j=3}^d \omega_j |z_j|^2 + O(|z|^3)$ where $z_j = x_j + i\xi_j$ and $(1, \omega_3, \cdots, \omega_d)$ are linearly independent over the rationals. We get an integrable system using the truncated Birkhoff normal form. The Hamiltonians are $F_1 = |z_1|^2 + |z_2|^2,\ F_2 = |z_3|^2,\ F_{d-1} = |z_d|^2, F_d = K$ where $K = O(|z|^3)$ is a polynomial. Reducing by the action of T^{d-1} given by the $d-1$ first hamiltonians we get a complex projective line depending of $d-1$ parameters; K can be seen as a function on this projective line depending of $d-1$ parameters and hence the Lagrangian foliation admits generically all singularities of codimension $\leq d$ of functions of 2 variables (this example was described to me by Marc Joyeux, see section 4.2).*

Example 1.3 *The "normal bundle" L of the cusp $9x^2 - 8y^3 = 0$, namely the closure of the normal bundle to the non singular part of it, is a singular Lagrangian manifold parametrized by*

$$m(u, v) = (u^3/3, u^2/2; v, -uv)\ .$$

The ideal $\mathcal{L}$ of functions vanishing on L is minimally generated by:

$$F_1 = 9x^2 - 8y^3,\ F_2 = 3x\xi + 2y\eta,\ F_3 = \eta^2 - 2y\xi^2, F_4 = 3x\eta + 4y^2\xi\ ,$$

(as computed by Marcelo Morales) hence it is not a complete intersection.

Example 1.4 *The (open) swallowtail S (see [1]) can be defined as the subset S of the set Z of polynomials*

$$P = x^5 + ax^3 + bx^2 + cx + d$$

admitting a zero of order at least 3. We can write $P = (x-u)^3(x^2+3ux+v)$ which give a parametrization of S. There exists a natural symplectic structure on Z for which S is Lagrangian. This manifold is obtained in a generic way in the following problem: if $X \subset \mathbb{R}^3$ is a surface and V a vector field on X whose integral curves are geodesics, the set of affine lines generated by the vectors $V(m),\ m \in X$ is a (singular) Lagrangian manifold in the symplectic manifold of affine lines in $\mathbb{R}^3$. It can be shown that S is not a complete intersection[2].

1.3 Phase functions

The WKB-Maslov Ansatz allows to associate to any smooth Lagrangian submanifold of $T^\star\mathbb{R}^n$ a family of h-dependent functions $u_h(x)$ (h is a small positive parameter) given by oscillatory integrals of the form

$$u_h(x) = \int_{\mathbb{R}^N} e^{i\varphi(x,\theta)/h} a(x,\theta) d\theta$$

whose microsupport is the reduced Lagrangian manifold

$$L_\varphi = \{(x, \partial_x\varphi) \mid \partial_\theta\varphi = 0\}\ .$$

Locally every Lagrangian submanifold of $T^\star\mathbb{R}^n$ can be defined by the previous formula using a so called **non degenerate phase function** φ (see [22] p.31). This construction is a special case of *symplectic reduction*. In order to do the same thing for a singular Lagrangian manifold, one could try to use degenerate phase functions.

Definition 2 *Let $L \subset T^\star\mathbb{R}^n$ a germ of a singular Lagrangian manifold whose singular part is denoted by L_0 and the smooth n dimensional stratum by L_1. A germ of smooth function $\varphi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ is a phase function for L if the map $j_\varphi : C_\varphi \to L$, where $C_\varphi = \{(x,\theta)|\partial_\theta\varphi = 0\}$ and $j_\varphi(x,\theta) = (x, \partial_x\varphi)$, is an homeomorphism and φ restricted to the open set of C_φ where the $\partial_{\theta_j}\varphi,\ j = 1, \cdots, N$'s are independent is a non degenerate phase function for L_1.*

Lemma 1 *If L admits a phase function, the Maslov index of any loop included in L_1 vanishes.*

Proof.–

> It is a consequence of the fact that the Maslov index can be defined (see [26] p. 154-163 and also the appendix by Arnold to the book [29]) in Čech cohomology by the cocycle $\mathrm{ind}(\partial_{\theta,\theta}\varphi_i) - \mathrm{ind}(\partial_{\theta,\theta}\varphi_j)$ ("ind (q)" is the Morse index of the quadratic form q) which in our case gives a trivial cocycle because there is only one open set in the covering.

□

[2] I thank very much Marcelo Morales for the computations of these examples

Question 2 Give a characteristic property of singular germs of Lagrangian manifolds which admits a degenerate phase function.

With respect to this question, we propose the following example:

Example 1.5 *Let* $\varphi(x,\theta) = \theta(x^2 - \theta^2/3)$*. We get* $L_\varphi = \{\xi^2 - 4x^4 = 0\}$*, so that* φ *is a degenerate phase function for* L_φ*.*

Example 1.6 *We have the following (see also [38]) :*

Proposition 1 *The germ at* 0 *of the normal bundle of the cusp (example 1.3) does not admit a degenerate phase function.*

Proof.–

The Maslov index of any closed curve inside the smooth part of the germ would be zero by Lemma 1. Let us consider the curve

$$\gamma(\theta) = m(\cos\theta, \sin\theta)$$

The Lagrangian vector space tangent to L at the point $\gamma(\theta)$ is generated by the vectors $(0,0;1,-\cos\theta)$, $(\cos^2\theta, \cos\theta; 0, -\sin\theta)$ and by reduction with respect to $\xi = 0$, we get the curve $\theta \to [(\cos\theta; -\sin\theta)]$ inside the Lagrangian Grassmanian of $T^\star\mathbb{R}$ whose Maslov index is ± 2.

□

2 Infinitesimal deformations

We propose below a very naïve approach, restricting ouself to phase spaces of dimension 2: a more precise and algebraic approach in any dimension can be found in [40].

We restrict ourselves in what follows to the case $d = 1$ **(except in sections 3 and 4).**

In this case, every curve is Lagrangian and is a complete intersection. Moreover canonical transformation are just orientation and area preserving diffeomorphisms.

Definition 3 *We will say that the germs of Lagrangian manifolds* $(< F_0 >, \omega_0)$ *and* $(< F >, \omega)$ *are* equivalent *if there exists a germ of diffeomorphism* χ *such that* $F \circ \chi = EF_0$ $(E(0) \neq 0)$ *and* $\chi^\star(\omega) = \omega_0$*. By Darboux theorem, we will often restrict ourself to* $\omega = \omega_0$*.*

One could have taken a stronger form of equivalence by asking $F \circ \chi = \psi \circ F_0$ (as in the paper "Le lemme de Morse isochore" [15]) where ψ is a germ of diffeomorphism of $\mathbb{R}$ fixing 0. One would be lead to the same space of infinitesimal deformations, but it would be inappropriate for our semi-classical business because it forces to use functional calculus for operators which are in general non self-adjoint.

We want to define the *codimension of the set of equivalent germs of Lagrangian manifolds* inside the set of all germs, so we need first to define infinitesimal deformations of germs:

Definition 4 *Given a singular germ of curve* $\mathcal{L}$ *in* $T^\star\mathbb{R}$ *given by* $F_0 = 0$*, the* space of infinitesimal deformations *(as a Lagrangian manifold) of* $\mathcal{L} =< F_0 >$ *is the space of all germs of functions* $\mathcal{E}$*.*

A general deformation of (F_0, ω_0) is given by (F_t, ω_t). Using Darboux, we can reduce to deformations $(F_0 + tK + O(t^2), \omega_0)$. K is an arbitrary germ of real valued function.

Definition 5 *A deformation* $\mathcal{L}_t =< F_t >$ *is* trivial *if there exists a smooth family* χ_t *of canonical transformations and a smooth family of functions* $E_t \in \mathcal{E}$*, such that:*

$$F_t \circ \chi_t = E_t F_0 \ .$$

This implies that there exists germs of functions X *and* Y *such that the infinitesimal deformation* $K = \frac{dF_t}{dt}_{|t=0}$ *satisfies:*

$$K = \{X, F_0\} + Y F_0 \ .$$

We can now give the definition of the codimension of a germ of Lagrangian curve:

Definition 6 *The* codimension $\mu = \mu\,(< F_0 >, \omega_0)$ *of the Lagrangian curve* $\mathcal{L} =< F_0 >$ *is defined by*

$$\mu = \dim\big(\mathcal{E} / \left(\{\mathcal{E}, F_0\} + \mathcal{E}.F_0\right)\big) \ , \tag{1}$$

where $\{.|.\}$ *is the Poisson bracket,*

If μ *is finite, any basis* $K_\alpha \in D_{\mathcal{L}}$*,* $\alpha = 1, \cdots, \mu$*, of a supplementary space of* $\{\mathcal{E}, F_0\} + \mathcal{E}.F_0$ *in* $D_{\mathcal{L}}$ *will define the* (uni)versal deformation of $\mathcal{L}$ *as follows:*

$$< F_0 + \sum_{\alpha=1}^{\mu} a_\alpha K_\alpha > \ .$$

More precisely, we ask that equation (1) is true with $\mathcal{E}(U_j)$ *for a basis* U_j *of neighbourhoods of* O *(with the same functions* K_α*).*

Question 3 *Give a natural extension of the definition 1.1 to the case of systems of operators, i.e. matrix valued germs of functions (see [8]).*

3 Examples

1. **The smooth case:** the differentials dF_j are linearly independent in some neighbourhood of the origine. Then $\mathcal{L}$ is a germ of smooth Lagrangian manifold. This $\mathcal{L}$ is of codimension 0. Moreover Darboux theorem implies that up to canonical transformation $\mathcal{L} =< \xi_1, \cdots, \xi_d >$.

2. **The Morse** $(d = 1)$ **case:** let $F_\varepsilon = F_0 + O(\varepsilon)$ where F_0 is a non degenerate quadratic form on $T^\star\mathbb{R}$. By the *lemme de Morse isochore* (see [15]), there exists χ_ε a germ of canonical transformations smoothly depending of ε and a smooth function Φ_ε such that

   $$F_\varepsilon \circ \chi_\varepsilon = \Phi_\varepsilon \circ F_0$$

   and $\Phi_0'(0) \neq 0$. Hence Φ_ε admits a non degenerate zero $t(\varepsilon)$ and we have

   $$F_\varepsilon \circ \chi_\varepsilon(x, \xi) = E_\varepsilon(x, \xi)(F_0(x, \xi) - t(\varepsilon))$$

   from which it is clear that $< F_0 - t >$ is a versal deformation of $< F_0 >$.

3. **The Eliasson case** ([23] or the non degenerate case of [36], définition 2.1.). It is an extension of the previous case to several quadratic forms. Let

   $$q_1, \cdots q_d$$

be d independent commuting quadratic forms on $T^\star\mathbb{R}^d$ where $(q_1, \cdots, q_d)$ is of type (m_e, m_h, m_f) and $d = m_e + m_h + 2m_f$ where m_e is the number of elliptic forms, m_h the number of hyperbolic one's and m_f the number of focus-focus one's. We have $\mu = d$. This value is minimal for rank 0 singular point of an integrable system.

4. **Cusp** (A_2) : $F_0 = \xi^2 + x^3$ $(d = 1)$ and $\mu = 2$:

$$K_1 = 1,\ K_2 = x\ .$$

We will see that up to canonical transformation any F which admits a non degenerate cusp is equivalent to the standard example $\xi^2 + x^3$.

5. **Quartic oscillator** (A_3^+) $F_0 = \xi^2 + x^4$ $(d = 1)$ and $\mu = 3$: $K_1 = 1,\ K_2 = x,\ K_3 = x^2$.

6. **Quartic anti-oscillator** (A_3^-) $F_0 = \xi^2 - x^4$ or $F_0 = x(x - \xi^2)$ $(d = 1)$ and $\mu = 3$.

7. **Triple crossing** (D_4^-) $F_0 = x\xi(x - \xi)$ $(d = 1)$ and $\mu = 4$:

$$K_1 = 1,\ K_2 = x,\ K_3 = \xi,\ K_4 = x\xi\ .$$

8. **Hyperbolic umbilic** (D_4^+) $F_0 = x(x^2 + \xi^2)$ $(d = 1)$ and $\mu = 4$.

Question 4 *Describe all singular Lagrangian manifolds of small codimension.*

4 Integrable systems

4.1 Singularities of integrable systems

Definition 7 *Let $(Z;\omega)$ be a germ of symplectic manifold of dimension $2d$, a* germ of completely integrable system *is given by d germs of real functions $F_j,\ j = 1, \cdots, d$ such that the Poisson bracket $\{F_i, F_j\}$ all vanish and the differentials of the F_j's are linearly independent almost everywhere.*

The map $z \to (F_j(z))$ of Z into $\mathbb{R}^n$ is called the momentum map.

A singular point *z_0 is a point where the* rank *$r(z_0)$ of the $dF_j(z_0)$'s $(1 \le j \le n)$ is $< d$.*

The separatrix *is the image by the momentum map (F_j) of the set of points z where $r(z) < n$.*

4.2 Singularities of integrable systems and deformations of Lagrangian manifolds

Let $< F(x,\xi;t) = 0 >$, $((x,\xi) \in T^\star\mathbb{R},\ t \in \mathbb{R}^N)$ be a deformation of the germ of (Lagrangian) curve $< F(x,\xi,0) = 0 >$ and assume

$$(\star)\ \frac{\partial F}{\partial t}(0,0;0) \neq 0$$

We can associate to it a germ of completely integrable system in $T^\star(\mathbb{R}^N_{(t',x)})$ in the following way: we choose coordinates $t = (t', t_N)$ such that $\partial_{t_N} F \neq 0$. Then we can rewrite $F(x,\xi,t) = E(x,\xi,t)(H_N(x,\xi;t') - t_N)$. We take the commuting Hamiltonians $t_1, \cdots, t_{N-1}, H_N$ on $T^\star(\mathbb{R}^N_{(t',x)})$ which define an integrable germ.

We can go back to the deformation in the following way: we start with the integrable germ with a singularity of rank $N-1$ and choose $t_1, \cdots, t_{N-1}$ commuting integrals whose differential at the singular point are independent. We reduce the systems and we get for each $a \in \mathbb{R}^{N-1},\ b \in \mathbb{R}$ a 2-dimensional curve $< H_N(x, \xi, a) = b >$ which give the previous deformation.

Proposition 2 *The previous correspondence is an isomorphism between germs of integrable systems of rank $N-1$ (modulo canonical diffeomorphisms) and N parameters deformations of curves (modulo canonical diffeomorphisms) satisfying* $(\star)$.

Moreover, we get that way a correspondence between *universal deformations* (deformations containing the versal deformation) and *stable germs of integrable systems.* A germ of integrable systems will be said to be stable if any small perturbation of the germ of integrable system is equivalent to the unperturbed system by a germ of canonical transformation.

We see, using the previous correspondence, that universal deformations of a germ of curve of codimension $\mu \leq N$ correspond to stable singularities of integrable systems with N degrees of freedom. The image of the set of equivalent singularities by the momentum map is then of codimension $N - \mu$.

4.3 Generic singularities of integrable systems with 2 degrees of freedom

From the previous sections, we get the following list of locally stable singularities of integrable systems with 2 degrees of freedom (see [24] for pictures of these separatrices for classical systems).

1. **Rank 1:**

   E (elliptic) $(x_1^2 + \xi_1^2,\ \xi_2)$ $(\mu = 1)$

   H (hyperbolic) $(x_1\xi_1,\ \xi_2)$ $(\mu = 1)$

   C (cusp) $(x_1^3 + \xi_1^2 + x_1\xi_2, \xi_2)$ $(\mu = 2)$

   This list is the list of codimension ≤ 2 germs of plane curves which are obtained by reduction using the procedure described in section 4.2.

2. **Rank 0:**

   EE (elliptic-elliptic), **EH** (elliptic-hyperbolic), **HH** (hyperbolic-hyperbolic), **L** (loxodromic) $(\mu = 2)$.

From the semi-global picture (see [43]), we get other stable singularities which correspond to codimension 2 singularities in the $\mathbb{Z}_m$-equivariant cases:

1. $m = 2$: $x^4 \pm \xi^2$.
2. $m = 3$: $\Re(z^3)$.
3. $m = 4$: $(x^2 + \xi^2)^2 + a\Re(z^4)$ with $a \in \mathbb{R} \setminus \{0, \pm 1\}$.
4. $m > 4$: $(x^2 + \xi^2)^2 + a\Re(z^m)$ with $a \in \mathbb{R}$.

Question 5 *Are there other stable singularities? Find the corresponding list for* $d = 3, 4, \cdots$.

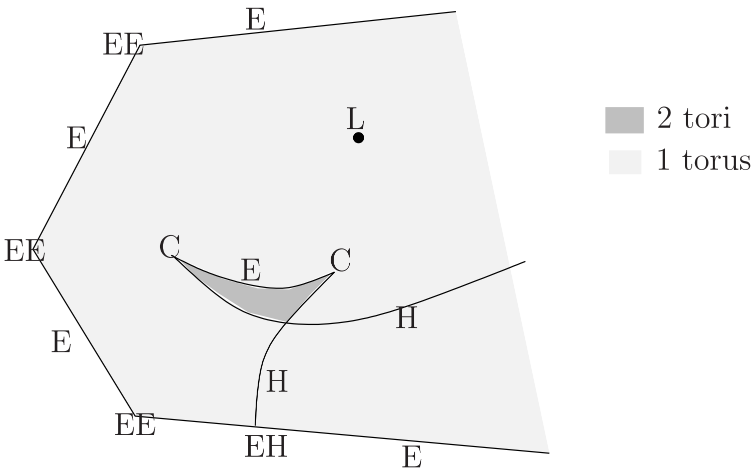

Figure 1: Typical bifurcation diagram for a 2 degrees of freedom system

5 The symplectic codimension of curves with isolated singularities

5.1 Vanishing cohomology: a short review

All results below are described in [30]. Let $F_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function and assume that the origin is an *isolated* critical point of multiplicity μ' or equivalently, if $\mathcal{M}$ is the (maximal) ideal of germs vanishing at 0, there exists k such that $\mathcal{M}^k \subset J(F_0)$ where $J(F_0)$, the jacobian ideal of F_0, is the ideal generated by the partial derivatives of F_0 and $\dim_{\mathbb{C}} \mathcal{E}/J(F_0) = \mu'$. Then, if $\varepsilon > 0$ is small enough and $r = r(\varepsilon) > 0$ is small enough, the map $(x, y) \to t = F_0(x, y)$ is a smooth fibration of $B(0, \varepsilon) \cap \{0 < |F_0(x, y)| < r\}$ on $D^\star = \{0 < t < r\}$ whose fiber X_t is a Riemann surface. The vanishing homology is the family of homologies of X_t which is a vector bundle on $D^\star$. It has been proved by Milnor (see [31]) that X_t has the homotopy type of a bouquet of μ' circles, so that the vanishing homology $H_1^{van}(X_t)$ is a vector space of dimension μ'. It is generated by μ' cycles $\gamma_j(t)$ which can be choosen locally constant w.r. to t. Globally when t goes around the origin we get a monodromy which preserves the lattice generated by the geometric cycles.

In order to make computations, it is usefull to introduce the vanishing cohomology H^1_{van} as follows: we put

$$H^1_{van} = \Omega^1/(\Omega^0 dF_0 + d\Omega^0)$$

where Ω^j is the space of germs of holomorphic differential forms of degree j. One can prove that H^1_{van} is a free module of rank μ' over A where $A = \mathbb{C}\{F_0\}$ is the ring of convergent series in F_0. We have the following

Proposition 3 *If (ω_j) is a basis of H^1_{van} as a module over A, the restrictions of the ω_j's to X_t for $t \in D^\star$ is a basis of the cohomology of X_t and the determinant of the matrix $(\int_{\gamma_i(t)} \omega_j)$ does not vanish.*

The problem is now to find such a basis (ω_j). For that purpose, we introduce the map $\omega \to \omega \wedge dF_0$, it can be shown that this map induces an isomorphism of A-modules from H^1_{van} to $dF_0 \wedge \Omega^1/dF_0 \wedge d\Omega^0$. The space $dF_0 \wedge \Omega^1$ is of finite co-dimension μ' over $\mathbb{C}$ inside Ω^2 and hence, if we make the tensor product with the field M of meromorphic functions of F_0 which is the fraction field of A, we get that

$dF_0\wedge$ induces an isomorphism of M-vector spaces (of dimension μ') from $H^1_{van}\otimes M$ to $(\Omega^2/dF_0\wedge d\Omega^0)\otimes M$. It has been proved by Sebastiani (see [30] p. 416) that the A-module $\Omega^2/dF_0\wedge d\Omega^0$ is free of rank μ'. Hence $\Omega^2/dF_0\wedge d\Omega^0+F_0\Omega^2$ is a $\mathbb{C}$-vector space of dimension μ'. If $[\alpha_j],\ j=1,\cdots,\mu'$ is a basis of the μ'-dimensional $\mathbb{C}$-vector space $\Omega^2/dF_0\wedge d\Omega^0+F_0\Omega^2$, we see that a family of 1-forms ω_j such that $\omega_j\wedge dF_0=\alpha_j$ is a basis of $H^1_{van}\otimes M$.

5.2 Applications

As a first application of the previous discussion, we get the following result observed by Bernard Malgrange [3] (see also [33] for the hyperelliptic case, and [40]):

Theorem 1 *If $F_0:(T^\star\mathbb{R}=\mathbb{R}^2,0)\to(\mathbb{R},0)$ is a germ of analytic function and admits an isolated singularity at 0 whose multiplicity is μ', $<F_0>$ is of codimension μ': we have always $\mu=\mu'$.*

Proof.–

It is clear, by using the isomorphism from $\Omega^0=\mathcal{E}$ into Ω^2 given by $\varphi\to\varphi dx\wedge d\xi$, that we get an isomorphism between $\mathcal{E}/(\{\mathcal{E},F_0\}+F_0\mathcal{E})$ and $\Omega^2/(dF_0\wedge d\Omega^0+F_0\Omega^2)$ which is a μ'-dimensional vector space over $\mathbb{C}$ after the previous subsection.

□

A simple proof of Theorem 1 in the quasi-homogeneous case will be given in section 8.

Theorem 1 admits a very nice geometrical interpretation which we can derive from the paper [33]. If χ is a germ of canonical transformation near the origin, actions integrals over small cycles are preserved. Hence any (uni)versal deformation should be able to reproduce the variations of the action integrals over the vanishing cycles. This is strongly consistent with the fact that μ is also the number of vanishing cycles as shown in [31]. This is exactly the way things work in the quasi-homogeneous case as shown in section 9; we will show there how to get the versal deformation theorem for quasi-homogeneous singularities.

If the singularity is not quasi-homogeneous, $\mathcal{E}F_0+\{\mathcal{E},F_0\}$ is no more the Jacobian ideal; indeed Saito proved in [35] that: $(F\in \mathrm{J}(F))$ *implies (F quasi-homogeneous).* In other words, there are deformations which are trivial as singularities of functions, but not for the symplectic version. There is always a choice of a versal deformation which is valid for both problems: a pair of vector subspaces of the same codimension always admit a common supplementary subspace.

For example of a non quasi-homogeneous singularity, we can take the singularity called Z_{11} ($\mu=11$) in [3], which is given by $F_a=x^3\xi+\xi^5+ax\xi^4$. Different values of a give non-equivalent singularities of functions, but equivalent ideals.

If $F_0=0$ is a germ of singular curve, we can associate to it a de Rham complex as in [25]:

$$0\to\mathcal{E}\to\Omega^1/K\to 0$$

where the non trivial arrow is d and K is the set of 1-form which vanish on the tangent vectors to the smooth stratum of $F_0=0$:

$$K=\{\alpha\in\Omega^1\mid \exists\beta\in\Omega^2,\ \alpha\wedge dF_0=F_0\beta\}\ .$$

Then we define

$$H^1_{\text{de Rham}}(<F_0>)=\Omega^1/(K+d\Omega^0)\ .$$

[3] Oral communication

There is a subspace of the space of infinitesimal deformations which we can identify with $H^1_{de\ Rham}(< F_0 >)$. If $\alpha \in \Omega^1$ is a germ of 1-form, it gives a deformation of (F_0, ω_0) defined by $(F_0, \omega_0 + \varepsilon d\alpha)$. It is easy to check that the cohomology of $[\alpha]$ vanishes if and only if the deformation is trivial.

Definition 8 *The Tyurina number τ of F_0 is defined by:*

$$\tau = \dim \mathcal{E} / \left(\mathcal{E} F_0 + \mathrm{J}(F_0) \right) \ .$$

τ is the dimension of the versal deformation of the ideal generated by F_0.

It follows from the previously quoted result by Saito that $\tau = \mu$ if and only if F_0 is quasi-homogeneous.

We can summarize the situation as follows (see also [40]):

Theorem 2 *The following sequence of $\mathbb{C}$-vector spaces is exact:*

$$0 \to \frac{\Omega^1}{d\Omega^0 + K} \to \frac{\Omega^0}{F_0\Omega^0 + \{F_0, \Omega^0\}} \to \frac{\Omega^0}{F_0\Omega^0 + J(F_0)} \to 0 \ ,$$

where the first non trivial arrow is induced by $\alpha \to dF_0 \wedge \alpha / dx \wedge d\xi$ and the second is the canonical surjection. In particular, we have:

$$\mu = \tau + b_1 \ .$$

This result is certainly not new, but we were unable to locate it in the litterature.

The meaning of the previous exact sequence is: *(deformations with $< F_0 >$ fixed)$\to$ (deformations of $(< F_0 >, \omega_0)$) $\to$ (deformations of $< F_0 >$).* The exactness is easily checked from the definitions.

6 Normal forms

Theorem 3 *Let $(< F_0 >, \omega_0)$ be a singular germ of curve and ω another germ of symplectic form.*

If $\omega/\omega_0 > 0$ and $\omega = \omega_0 + d\alpha$ where the cohomology class of α vanishes,

$$(< F_0 >, \omega) \sim (< F_0 >, \omega_0) \ .$$

Conversely, if $(< F_0 >, \omega_t) \sim (< F_0 >, \omega_0)$ and $\omega_t - \omega_0 = d\alpha_t$, the cohomology class of α_t vanishes.

In particular, if F_0 is quasi-homogeneous and $\omega/\omega_0 > 0$, we have $(< F_0 >, \omega) \sim (< F_0 >, \omega_0)$.

Proof.–

We consider the path of symplectic forms $\omega_t = \omega_0 + t(\omega - \omega_0)$ $(0 \leq t \leq 1)$. We need to find a diffeomorphism ψ such that ψ preserves the curves $F_0 = 0$ and $\psi^\star(\omega) = \omega_0$. We can use the homotopy (Moser) trick, following [25] Theorem 1: we try to find a family ψ_t of germs of diffeomorphisms associated to the time dependent vector field X_t by $X_t(\psi_t(x)) = \frac{d}{dt}\psi_t(x)$, such that

1. $\psi_t^\star(\omega_t) = \omega_0$
2. The curve $F_0 = 0$ is invariant by ψ_t

The condition **1** is satisfied iff $d(\iota(X_t)\omega_t) + \omega - \omega_0 = 0$, which, if $d\alpha = \omega - \omega_0$, is implied by $\iota(X_t)\omega_t = -(\alpha - df)$ for any function f. The condition **2** is then satisfied as soon as $\alpha - df$ vanishes on the tangent vectors to the smooth part of $F_0 = 0$. This last condition can be fulfilled iff $[\alpha] = 0$.

□

Question 6 *Does* $(< F_0 >, \omega) \sim (< F_0 >, \omega_0)$, *with* $\omega/\omega_0 > 0$, *imply* $\omega - \omega_0 = d\alpha$ *with* $[\alpha] = 0$*?*

7 Versal deformations: the formal case

Theorem 4 *Let* $\mathcal{L}_0 =< F_0 >$ $(d = 1)$ *be a germ of a singular Lagrangian manifold of codimension* μ *in the sense of definition 6 and let us denote by* $F_0 + \sum_{\alpha=1}^{\mu} a_\alpha K_\alpha$ *a versal deformation of* F_0. *Let* $\mathcal{L}_\varepsilon =< F_\varepsilon >$ *with* $F_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k F_k + O(\varepsilon^\infty)$ *be a smooth deformation of* $\mathcal{L}_0 =< F_0 >$.

Then there exists a smooth family of canonical transformations χ_ε, *a smooth invertible function* $E_\varepsilon(x, \xi)$ *and smooth functions* $a_\alpha(\varepsilon) = O(\varepsilon)$ *such that*

$$F_\varepsilon \circ \chi_\varepsilon = E_\varepsilon\big(F_0 + \sum_{\alpha=1}^{\mu} a_\alpha(\varepsilon) K_\alpha\big) + O(\varepsilon^\infty)) \ .$$

Question 7 *We may conjecture on the basis of the Morse case and of the proof that the formal series* $a_\alpha(\varepsilon)$ *are uniquely defined.*

Proof.–

We assume that

$$F_\varepsilon = F_0 + \sum_{\alpha=1}^{\mu} a_\alpha(\varepsilon) K_\alpha + \varepsilon^n R_n + O(\varepsilon^{n+1})$$

We need to find $\chi_\varepsilon = Id + O(\varepsilon^n) = \exp(\varepsilon^n Z) + O(\varepsilon^{n+1})$ where Z is the Hamiltonian vector field of X such that

$$F_\varepsilon \circ \chi_\varepsilon = (1 + \varepsilon^n E)(F_0 + \sum_{\alpha=1}^{\mu} a_\alpha(\varepsilon) K_\alpha + \varepsilon^n (\sum b_\alpha K_\alpha)) + O(\varepsilon^{n+1}) \ .$$

By identification of terms in ε^n, we get the following equation:

$$\{F_0, X\} - EF_0 = -R_n + \sum b_\alpha K_\alpha \ ,$$

which can be solved in a fixed open set by the hypothesis of finite codimension.

□

We assume that $< F_0 >$ is of codimension μ. Let $< F_\varepsilon >$ be a smooth deformation of $< F_0 >$. A basic question is the following one:

do there exists a smooth canonical deformation of the identity χ_ε, a smooth deformation E_ε of the function 1 and smooth functions $a_\alpha(\varepsilon) = O(\varepsilon)$ such that

$$F_\varepsilon \circ \chi_\varepsilon = E_\varepsilon(F_0 + \sum_{\alpha} a_\alpha(\varepsilon) K_\alpha) \ ? \qquad (2)$$

The transformations χ_ε move then the deformation $< F_\varepsilon >$ of $< F_0 >$ into the universal one $< F_0 + \sum a_\alpha(\varepsilon) K_\alpha >$. The condition of finite codimension allows to solve the linearized problem, so it is natural to ask the following:

Question 8 *Does there exists in this context an implicit function theorem “à la Mather”?*

The answer is yes for all simple singularities of curves in the holomorphic case (see section 9.4).

8 The quasi-homogeneous case

8.1 Definitions

We give the:

Definition 9 *$F = F(x,\xi)$ is $(a,b,N)-$quasi-homogeneous, where a,b and N are positive integers with a and b coprime, if F is a polynomial satisfying the identity:*

$$F(t^a x, t^b \xi) = t^N F(x,\xi) \ .$$

We denote by $\mathcal{E}_{a,b}^N$ this space of polynomials.

Any monomial $x^p\xi^q$ is in $\mathcal{E}_{a,b}^{pa+qb}$. The algebra $\mathbb{R}[[x,\xi]]$ of formal series with the usual products is gradued by

$$\mathbb{R}[[x,\xi]] = \oplus_{N=0}^{\infty} \mathcal{E}_{a,b}^N \ .$$

Concerning Poisson brackets, we have:

$$\{\mathcal{E}_{a,b}^l, \mathcal{E}_{a,b}^m\} \subset \mathcal{E}_{a,b}^{l+m-(a+b)} \ .$$

If $< F >$ is quasi-homogeneous ($F \in \mathcal{E}_{a,b}^N$) of finite codimension, we can choose quasi-homogeneous K_α and for any k:

$$\mathcal{E}_{a,b}^{N+k} = \{\mathcal{E}_{a,b}^{k+a+b}, F\} + \mathcal{E}_{a,b}^k F + \sum_{K_\alpha \in \mathcal{E}_{a,b}^{N+k}} \mathbb{R} K_\alpha,$$

where the last sum is finite.

8.2 Using Euler identity

Theorem 5 *If F is a quasi-homogenous isolated singularity of Milnor number μ, i.e.*

$$\dim \big(\mathcal{E}/\mathrm{J}(F)\big) = \mu \ ,$$

then $< F >$ is of codimension μ. More precisely

$$\mathrm{J}(F) = \mathcal{E}F + \{\mathcal{E}, F\}$$

Proof.–

Let us denote $A = \partial F/\partial x, \ B = \partial F/\partial \xi$, we have by Euler identity:

$$a' x A + b' \xi B = F \ ,$$

with $a' = a/N$ and $b' = b/N$. We want to solve:

$$\{X, F\} + Y F = \lambda A + \nu B$$

where λ, ν are given and $X, Y \in \mathcal{E}$ are unknown functions. We get, by replacing $\{X, F\}$ by $A\frac{\partial X}{\partial \xi} - B\frac{\partial X}{\partial x}$:

$$A(\frac{\partial X}{\partial \xi} + a'xY) + B(-\frac{\partial X}{\partial x} + b'\xi Y) = \lambda A + \nu B$$

and it is now enough to solve

$$\frac{\partial X}{\partial x} = -\nu + b'\xi Y, \ \frac{\partial X}{\partial \xi} = \lambda - a'xY \ .$$

The integrability condition is:

$$(a' + b')Y + b'\xi\frac{\partial Y}{\partial \xi} + a'x\frac{\partial Y}{\partial x} = \frac{\partial \lambda}{\partial x} + \frac{\partial \nu}{\partial \xi} \ ,$$

which admits an unique solution Y: we solve first inside formal series, then inside flat functions. We can take for the U_j's a basis of neighbourhoods star-shaped with respect to quasi-homogeneous dilatations.

□

9 Versal deformations for quasi-homogeneous singularities

9.1 A remarkable identity

We have the following:

Proposition 4 *For any quasi-homogeneous singularity* F_0 *with* $F_0(t^a x, t^b \xi) = t^N F_0(x, \xi)$ *we have the following identity:*

$$\sum_{\alpha=1}^{\mu} N_\alpha = \mu\,(N - (a + b)) \ ,$$

where (K_α) *is a family of monomials defining a versal deformation and* $K_\alpha(t^a x, t^b \xi) = t^{N_\alpha} K_\alpha(x, \xi)$.

Proof.–

Following [3], we introduce the Poincaré polynomial $P(t)$ of the singularity as follows:

$$P(t) = \sum_{\alpha=1}^{\mu} t^{N_\alpha} \ .$$

The following result is proved in the previous reference pages 166-168:

$$P(t) = \frac{t^{N-a} - 1}{t^a - 1} . \frac{t^{N-b} - 1}{t^b - 1} \ .$$

It is clear that $P(1) = \mu$ and hence

$$\mu = \frac{N - a}{a} . \frac{N - b}{b} \ .$$

We see also that $\sum N_\alpha = P'(1)$. By computing the derivative at $t = 1$, we get the result.

□

9.2 Non vanishing of the Jacobian determinant of action integrals

Lemma 2 *Let $F_a(x,\xi)$ $((x,\xi)\in\mathbb{C}^2,\ a\in\mathbb{C}^\mu)$ be a versal deformation of a* quasi-homogeous singularity *and γ_j a locally constant basis of the vanishing homology. Then the Jacobian determinant $J(a)$ of $a\rightarrow(\int_{\gamma_j(a)}\xi dx)$ which is well defined outside the discriminant set (the set of a's for which the curve $F_a=0$ is singular) extends to $\mathbb{C}^\mu$ as a* non vanishing *holomorphic function. If we take the versal deformation generated by monomials, J is constant.*

As a corollary we get that there exists a canonical measure on the versal deformation (because the vanishing homology has a canonical Lebesgue measure). It would be nice to have a geometric definition of that measure.
Proof.–

- We first check that:

$$\frac{\partial}{\partial a_\alpha}\int_{\gamma(a)}\xi dx=\int_{\gamma(a)}K_\alpha dt$$

  where dt is the time for the dynamics induced by the Hamiltonian F_a on the surface $F_a=0$.
- We then prove using Picard-Lefschetz formula that J is univalent: the Poincaré group of the complement of the discriminant is generated by small loops around the stratum corresponding to 1 vanishing cycle say γ_1. Following such a loop will add to the lines of the Jacobian determinant a linear combination of the first one.
- J is bounded near the codimension 1 stratum of the discriminant. Hence J is holomorphic near the codimension 1 strata and, by Hartogs theorem, everywhere.
- J is quasi-homogeneous of degree 0:

$$J(t^{N-N_1}a_1,\cdots,t^{N-N_\mu}a_\mu)=J(a_1,\cdots,a_\mu)$$

  Proof: $F_a(x,\xi)$ gets multiplied by t^N if $x\mapsto t^a x$, $\xi\mapsto t^b\xi$, $a_\alpha\mapsto t^{N-N_\alpha}a_\alpha$. This implies that $F=0$ is invariant under the latter transformation. Under this transformation the integral of ξdx over $\gamma(a)$, where the latter curve is determined by the condition that it is contained in $F=0$, get multiplied by t^{a+b}. It follows that the derivative with respect to a_α gets multiplied by $t^{a+b-N+N_\alpha}$ and therefore J gets multiplied by t^k, with $k=\mu(a+b-N)+\sum_\alpha N_\alpha$, if $a_\alpha\mapsto t^{N-N_\alpha}a_\alpha$. The Proposition 4 implies $k=0$.
- If we choose the versal deformation so that $K_1=1$, $J(a_1,0,\cdots,0)$ is non vanishing for $a_1\neq0$: it is a corollary of the discussion of the subsection 5.1 and of the explicit computation of the Jacobian. Because we have

$$J(t^N,0,\cdots,0)=J(1,0,\cdots,0)\ ,$$

  we see that $J(0)\neq0$.

□

9.3 Lifting isotopies

Let us give $F_a(x,\xi) = F_0(x,\xi) + \sum a_\alpha K_\alpha(x,\xi)$ a (mini-)versal deformation of F_0 where F_0 admits an isolated singular point at the origin. *We do not assume in this section that F_0 is quasi-homogeneous.* Let us denote by

- $z = (x,\xi) \in \mathbb{C}^2$
- $F(z,a) = F_a(z)$
- $\pi : \mathbb{C}^{2+\mu} \to \mathbb{C}^\mu$ the canonical projection
- $Z = \{(z,a) \mid F_a(z) = 0\}$
- $X_a = \{z \mid F_a(z) = 0\}$
- δ the *critical set*

$$\delta = \{(z,a) \mid z \text{ is a singular point of } X_a\}$$

- $\Delta = \pi(\delta)$ the *discriminant set*
- Δ_1 the set of a's for which X_a admits an unique singular point which is non degenerated (a double point). Δ_1 is a submanifold of codimension 1 in $\mathbb{C}^\mu$, whose closure is Δ.

Lemma 3 *The critical set δ is smooth.*

This fact is well known and much more general. Here is a simple proof in our case:
Proof.–

If we can take $K_1 = 1,\ K_2 = x,\ K_3 = \xi$, δ is a graph $(a_1,a_2,a_3) = G(a_4,\cdots,a_\mu,x,\xi)$. Otherwise F_0 is an A_k singularity $(\xi^2 \pm x^{k+1})$ (there exists at least one derivative of F_0 of order 2 nonvanishing at 0) and the result can easily be checked.

□

We have the following:

Lemma 4 *Let $X = \sum A_\alpha(a)\frac{\partial}{\partial a_\alpha}$ be an (germ of) holomorphic vector field in $\mathbb{C}^\mu$ which is tangent to Δ_1. There exists an holomorphic (germ of) vector field $\tilde{X}$ on $\mathbb{C}^{2+\mu}$ tangent to Z which satisfies $\pi_\star(\tilde{X}_{|Z}) = X$.*

Proof.–

The vector field $\tilde{X}$ should satisfy

$$\tilde{X} = \sum A_\alpha(a)\frac{\partial}{\partial a_\alpha} + U(z,a)\frac{\partial}{\partial x} + V(z,a)\frac{\partial}{\partial \xi}$$

and $\tilde{X}F_a$ vanishes on Z. In other words, $XF = \sum A_\alpha K_\alpha$ belongs to the ideal $\mathcal{J}$ generated by F and its partial derivatives w.r. to x and ξ. $\mathcal{J}$ is the ideal of definition of the smooth manifold δ. Hence, it is enough to prove that XF vanishes on δ. Let us fix $a_0 \in \Delta_1$ and let z_0 be the singular point of X_{a_0}. We have $\pi'(z_0,a_0)(T_{(z_0,a_0)}Z) = T_{a_0}\Delta_1$: this is because the map $\sigma : a \to z$ from Δ_1 to δ where z is the Morse singular point of X_a is a section of π over Δ_1. Let $W_0 \in T_{(z_0,a_0)}Z$ be such that $\pi'(z_0,a_0)(W) = X(a_0)$. We have $W_0(F) = X(a_0)(F)$, because the derivatives of F w.r. to z vanish at that point. We deduce $X(a_0)F = 0$, because W_0 is tangent to Z. It follows that XF, vanishing on δ, the closure of $\sigma(\Delta_1)$, belongs to $\mathcal{J}$.

□

We will need the following:

Corollary 1 *If $a \to \varphi_t(a)$ is a smooth isotopy which preserves Δ_1, it can be lifted to a smooth isotopy $\Phi_t(a,z)$ on $\mathbb{C}^{2+\mu}$ which preserves Z.*

In other words we have $\Phi_t(z,a) = (\psi_t^a(z), \varphi_t(a))$ where ψ_t^a is a (germ of) diffeomorphism of $\mathbb{C}^2$ which maps X_a onto $X_{\varphi_t(a)}$.

Proof.–

The proof is just by integrating the lift $\tilde{X}_t$, builded using Lemma 4, of the time dependent vector field $X_t(\varphi_t(a)) = \frac{d}{dt}\varphi_t(a)$.

□

9.4 Versal deformation theorem in the holomorphic case

We will prove the versal deformation theorem for all quasi-homogeneous singularities. Using the strategy of Pham in [33], we can prove the following:

Theorem 6 *Let $< F_0 >$ be a* quasi-homogeneous singularity *with $F_a = F_0 + \sum a_\alpha K_\alpha$ (K_α monomials) as a versal deformation. Let $< F_t >$ be any analytic deformation of $< F_0 >$. There exists an analytic family of germs of canonical diffeomorphisms χ_t such that $< F_t \circ \chi_t > = < F_{a(t)} >$, i.e.*

$$F_t \circ \chi_t = E_t \left(F_0 + \sum a_\alpha(t) K_\alpha \right)$$

where the functions $a_\alpha(t)$, E_t are analytic.

Remark: the previous result could be extended, with the same proof, to every *isolated singularity* if we were able to prove the non-vanishing of an appropriate Jacobian determinant.
Proof.–

We will give the proof for A_2 (the cusp), it is then trivial to see how to extend the proof to the general case.

Using Moser's method, the idea is to fit the action integrals. The details run as follows:

- We can assume, using the versal deformation theorem (see [3]), that our deformation is embedded into the deformation ($F_a = F_0 + a_1 x + a_2, \omega_t$) where $\omega_t = \omega_0 + O(t)$. We choose λ_t such that $d\lambda_t = \omega_t - \omega_0$ and we assume that $\lambda_t = O(|t|)$.
- Let $\Delta = \{4a_1^3 + 27a_2^2 = 0\}$ be the discriminant set. We want to define a smooth family of holomorphic diffeomorphisms (an isotopy) $a \to \varphi_t(a) = a'$ such that $\varphi_0 = Id$ and for all cycles γ_j of $X_a = \{F_a = 0\}$ we have

$$\int_{\gamma_j(a)} \xi dx = \int_{\gamma_j(a')} (\xi dx + \lambda_t) \tag{3}$$

  This implicit equation can be uniquely solved for t small enough outside Δ because the Jacobian determinant of $a \to (\int_{\gamma_j(a)} \xi dx)_{j=1,2}$ is a nonzero constant (see Lemma 2).

- Near the stratum of the regular part Δ_1 of the discriminant set where the vanishing cycle is γ_1, the integrals $\int_{\gamma_1}$ and $\int_{\gamma_2} \pm \int_{\gamma_1} \log \int_{\gamma_1}$ are univalent and holomorphic, thanks to the Picard-Lefschetz formula, and the Jacobian determinant is the same: so we can also solve equation (3).
- Now we have solved equation (3) outside a set of codimension 2, we can solve it everywhere using the fact that holomorphic functions have no singularities of codimension ≥ 2 (Hartog's theorem).
- Using Corollary 1, we get a diffeomorphism Φ_t which lifts φ_t. We have then
$$\int_{\gamma_j(a)} \xi dx = \int_{\gamma_j(a)} (\psi_t^a)^\star(\xi dx + \lambda_t) \ .$$
We put $\omega_0 = d\xi \wedge dx$ and $\omega_t' = (\psi_t^a)^\star(\omega_t)$. The deformations $(F_{a'}, \omega_t)$ and (F_a, ω_t') are clearly equivalent. The difference of the 2 symplectic forms ω_t' and ω_0 is $d\beta_t^a$ where the integrals of β_t^a over all vanishing cycles of all X_a's vanish.
- It remains now to find $f_{a,t}(x,\xi)$ whose differential on X_a is β_t^a. We will build $f_{a,t}$ so that $f_{(a_1,b),t} = g_{a_1,t}$ is independent of b. The differential of $g_{a_1,t}$ restricted to $X_{a_1,b}$ where b varies is given by the restriction of $\beta_t^{a_1,b}$ to $X_{a_1,b}$. We get $g_{a_1,t}$ by integrating from a point $m_{a_1,b} \in Z_{a_1,b} \cap \{\|z\| = 1\}$ which can be choosen an analytic function of (a_1, b) of the differential forms $\beta_t^{a_1,b}$. The smoothness of g outside Δ is clear. Moreover g is holomorphic outside Δ and bounded near Δ_1, hence holomorphic everywhere.
- We can then apply Moser's method.

□

10 Semi-classics

In this section, we will *quantize* everything in order to get semi-classical objects.

10.1 Semi-classical normal forms

Theorem 7 *Let $< H_0 >$ be of finite codimension μ with a (classical) real versal deformation generated by K_α, $\alpha = 1, \cdots, \mu$. Let $\widehat{H}$ be a pseudo-differential operator on $\mathbb{R}$ whose principal symbol is H_0. There exists then some elliptic pseudo-differential operators $\hat{U}$ and $\hat{V}$ and formal series $a_\alpha(h) = O(h)$ such that we have microlocally near* 0

$$\hat{U}\widehat{H}\hat{V} = \widehat{H_0} + \sum_\alpha a_\alpha(h)\widehat{K_\alpha} + O(h^\infty)$$

where $\widehat{Q}$ is the Weyl quantization of Q. If $\hat{H}$ is self-adjoint, we can choose $\hat{U}$ and $\hat{V}$ so that the a_α's are real valued.

The proof by induction on the powers of h is similar to that of section 7.

10.2 Mixed case

We consider now a smooth family $\widehat{H_\varepsilon}$ of semi-classical Hamiltonians and denote by H_0 the principal symbol of $\widehat{H_0}$. We assume that $< H_0 >$ is of finite codimension μ. The following result is an extension of Theorems 4 ($h = 0$) and 7 ($\varepsilon = 0$).

Theorem 8 *There exist elliptic pseudo-differential operators* $\widehat{U_\varepsilon}$ *and* $\widehat{V_\varepsilon}$ *and formal series* $a_\alpha(\varepsilon, h) = O(|h| + |\varepsilon|)$ *such that*

$$\widehat{U_\varepsilon}\widehat{H_\varepsilon}\widehat{V_\varepsilon} = \widehat{H_0} + \sum_\alpha a_\alpha(\varepsilon, h)\widehat{K_\alpha} + O(\varepsilon^\infty + h^\infty) \ .$$

The proof is by induction on the powers of h and for each power of h by induction on the powers of ε.

10.3 The case of quasi-homogeneous singularities

In the holomorphic quasi-homogeneous case, using the tools of section 9.4, we get a much better result:

Definition 10 *We will say that* $\hat{H}_E = \mathrm{Op}_W\left(\sum h^j H_j(E; x, \xi)\right)$ *is an* analytic family of pseudo-differential operators near 0*, if, for all indices* j*,* $H_j(E, x, \xi)$ *extends to an holomorphic function in some complex neighbourhood* Ω *of* 0 *independent of* j*.*

Theorem 9 *If* $\hat{H}_E$ *is an analytic family of pseudo-differential operators of order* 0 *such that* $H_0(0; x, \xi)$ *is a quasi-homogeneous singularity, there exists, for* E *small enough, an analytic family of unitary Fourier integral operators* U_E*, an analytic family of elliptic pseudo-diffential operators* F_E *and symbols (analytic w.r.t. to* E*)* $a_\alpha(E, h)$ *such that we have, microlocally near* 0*:*

$$U_E^\star \hat{H}_E U_E = F_E \circ \left(\hat{H}_0 + \sum a_\alpha(E, h)\hat{K}_\alpha) + O(h^\infty)\right) \ .$$

Proof.–

Proceeding by induction on the powers of h, we get the following equation to solve where $X(E; x, \xi)$, $Y(E; x, \xi)$, $c_\alpha(E)$ are the unknown functions:

$$\{H_0 + \sum a_\alpha(E)K_\alpha, X\} + Y(H_0 + \sum a_\alpha(E)K_\alpha) = R(E; x, \xi) - \sum c_\alpha(E)K_\alpha(x, \xi)$$

This equation express that on the Riemann surface $H_0 + \sum a_\alpha(E)K_\alpha = 0$, $R(E; x, \xi) - \sum c_\alpha(E)K_\alpha(x, \xi)$ is the derivative with respect to the time of the function X. We need first to choose $c_\alpha(E)$ so that the integrals

$$\int_{\gamma_j(E)} (R(E; x, \xi) - \sum c_\alpha(E)K_\alpha(x, \xi))dt$$

all vanish. This is possible outside the discriminant set because of the non vanishing of the determinant $\int_{\gamma_j(E)} K_\alpha dt$ (see Lemma 2). The solution is bounded near the discriminant, hence can be extended to an holomorphic function. The proof is then finished using the same arguments as in the proof of Theorem 6.

□

11 Singular Bohr-Sommerfeld rules: the general scheme

From the local model and the WKB solutions, we define the scattering matrices and singular holonomies. We show how one can take the principal part of the regular holonomies in order to get the singular holonomies. We can then derive the Bohr-Sommerfeld rules using the same combinatorial recipe as in [14] (maximal trees ...).

11.1 The context

We will assume that $\widehat{H}_E$ is a pseudo-differential operator of order 0 on the real line and denote by H_E his principal symbol. H is supposed to be real valued and we assume that the energy surface $Z = H_0^{-1}(0)$ admits only finite codimension singularities $z_j,\ j = 1, \cdots, N$ with normal forms

$$\widehat{U_j}\widehat{H}\widehat{V_j} = \widehat{H_j} + \sum_{\alpha=1}^{\mu_j} a_{j,\alpha}(E,h)\widehat{K_{j,\alpha}} + O(h^\infty)\ , \tag{4}$$

with $a_{j,\alpha}(E,h)$ symbols in h and $\widehat{K_{j,\alpha}}$ are Weyl quantizations of the real versal deformation.

11.2 Local models and scattering matrices

In this section we want to describe the solutions of the local model which is mapped on our problem near the singular point z_j.

We will omit the index j in this section.

We fix a neighbourhood Ω of 0 in the (y, η) symplectic plane. We denote by $H_a = H_0 + \sum a_\alpha K_\alpha$ the versal deformation of the model and $\widehat{H_a}$ his (Weyl)-quantized version. We will denote $\gamma_l,\ l = 1, \cdots, L = 2L'$ $(L \geq 2)$ the real branches of the germ $Z_a = H_a^{-1}(0)$. We chose to orient the γ_l's according to the dynamics of H_a. There are now L' ingoing and L' outgoing branches. We choose open sets $\Omega_l \subset \Omega$ with empty mutual intersections and such that $\Omega_l \cap \gamma_l$ is a nonempty connected arc. We assume that a is small enough so that $\Omega_l \cap Z_a$ with $Z_a = H_a^{-1}(0)$ is also a nonempty connected arc.

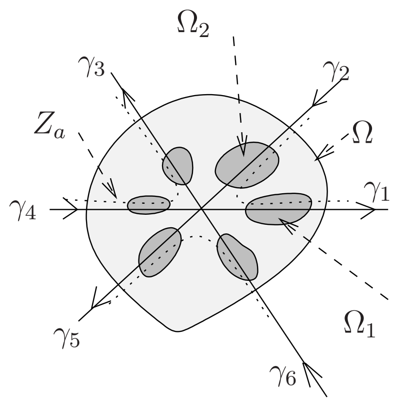

Figure 2: the model problem: $L = 6$

We are looking for the following equation

$$(\widehat{H_0} + \sum a_\alpha \widehat{K_\alpha})u = O(h^\infty)\ , \tag{5}$$

where u is a microfunction in Ω. It is in general not difficult to prove that the space of microfunctions solutions of equation (5) in Ω is a free module of rank $L' = L/2$ over the moderate growth functions of h. We choose microlocal solutions $u_l(a)$ of equation (5) inside Ω_l smoothly dependent of a of the form (in case of no caustics):

$$u_l(a;x) \sim \Big(\sum_{k=0}^{\infty} c_{k,l}(a,x)h^k\Big)e^{iS_l(a;x)/h} \tag{6}$$

with $c_{k,l}$ and S_l smoothly depending on a. Any solution u of equation (5) in Ω restricts to $x_l u_l(a)$ in Ω_l. Given $(x_l) = (x_{in}, x_{out})$ we can express the condition that $x_l u_l(a)$ are the restrictions to Ω_l of some solution u of equation (5) by a matrix

$$x_{out} = \mathcal{S}(a,h) x_{in} \ , \tag{7}$$

where $\mathcal{S}(a,h))$ is called the *scattering matrix*.

11.2.1 Unitarity

We assume that the operator $\hat{H}_a$ is formally self-adjoint. Let us choose Π a pseudo-differential operator of order 0 compactly supported in Ω and equal to Id near the origin. More precisely, we assume that

$$Z_a \cap \{\Pi(Id - \Pi) \neq 0\} \subset \cup_l (\Omega_l \cap Z_a)$$

We define the following inner products on microfunctions in Ω:

$$J_a(u,v) = \frac{i}{h} < [\Pi, \hat{H}_a] u, v >$$

It is clear that

1. If $u, v \in \ker(\hat{H}_a)$, $J(u,v) = O(h^\infty)$
2. If $u_{|\Omega_l} = x_l u_l$ and $v_{|\Omega_l} = y_l u_l$, we have: $J(u,v) = \sum_l x_l \bar{y}_l J(u_l, u_l)$
3. If the principal symbol of u_l is $|dt|^{\frac{1}{2}}$, we have $J(u_l, u_l) = \pm 1 + O(h)$ where we have a $+$ sign if the arc γ_l is ingoing and a $-$ sign if it is outgoing.

From that we deduce that $\mathcal{S}(a,h)$ is unitary (with maybe some domain).

11.3 Singular holonomies

Let γ_0 be a cycle of Z_0, we want to define the singular holonomy (of $\widehat{H_E}$) along γ and compute it. For simplicity we will assume that there exists only one singular point z_1 in γ_0 at which we have a normal form given by equation (4). We can therefore omit the index j. We first cover the cycle γ_0 by open sets $U_1, \cdots U_n$ such that we can find WKB solutions v_j of $\widehat{H_0} v = O(h^\infty)$ inside U_j, points $\zeta_j = (a_j, b_j) \in U_j \cap U_{j+1}$ and such that the Ω_j's covering the singular point z_0 ($j = 1, n$) are the image by the canonical transformation χ of some open sets Ω_l, $l = 1,2$ introduced in the previous section. We choose $v_1 = \widehat{V_1} u_1$ and $v_n = \widehat{V_1}(u_2)$. We define then then the singular holonomy $\mathrm{HolS}(\widehat{H_0}, \gamma_0)$ by

$$\mathrm{HolS}(\widehat{H_0}, \gamma_0) = \prod_{j=1}^{n-1} \frac{v_j(a_j)}{v_{j+1}(a_j)} \tag{8}$$

It is clear from the theory of WKB-Maslov Ansatz that $\mathrm{HolS}(\widehat{H_0}, \gamma_0) = e^{i(\sum_{k=-1}^{\infty} B_k h^k)}$ so that we go to some *Log* scale and put

$$\mathrm{LHolS} = -i \log \mathrm{HolS} \sim \sum_{k=-1}^{\infty} B_k h^k \ .$$

It is easily checked that singular holonomies are independent of all choices (including χ and the associated FIO's) except for the choosen WKB solutions u_l of the model problem. As we will see singular holonomies and scattering matrices are enough to derive Bohr-Sommerfeld rules.

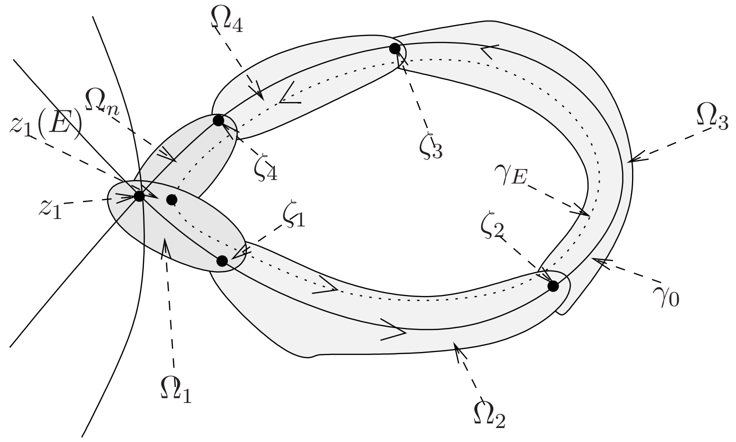

Figure 3: singular holonomy

11.4 Regularization

We will now choose a deformation $\widehat{H_E}$, $E \geq 0$, of $\widehat{H}$ such that $H_E^{-1}(0) = Z_E$ is smooth and a cycle γ_E of Z_E such that $\gamma_E \to \gamma_0$ as $E \to 0^+$. The goal is to derive $\mathrm{LHolS}(\widehat{H}, \gamma_0)$ as a regularization of the usual holonomy (the log of) $\mathrm{LHol}(\widehat{H_E}, \gamma_E) \sim \sum_{k=-1}^{\infty} A_k(E)h^k$. In general the A_k's are divergent as $E \to 0^+$ but we can substract the divergent part using the scattering matrix. More precisely, assume $E > 0$, we have then $v_n = s_{1,n}(E,h)v_1$ where $s_{1,n}$ is the corresponding entry of the local scattering matrix. We deduce:

$$\mathrm{LHol}(E,h) = \mathrm{LHolS}(E,h) + \frac{1}{i}\log s_{1,n}(E,h) \ .$$

For fixed $E > 0$, we have then

$$A_k(E) = B_k(E) + \sigma_{1,n}^k(E) \ ,$$

where

$$\frac{1}{i}\log s_{1,n}(E,h) \sim \sum_{k=-1}^{\infty} \sigma_{1,n}(E)h^k \ .$$

We get that way:

$$B_k(0) = \lim_{E\to 0^+}(A_k(E) - \sigma_{1,n}^k(E)) \ .$$

11.5 Singular Bohr-Sommerfeld rules

Once the singular holonomies are defined, the Bohr-Somerfeld rules follow the same combinatorial picture as in [14].

12 The cusp

The saddle-node bifurcation occurs generically for a 1-dimensional system depending on some extra parameter: it is the generic way to change the number of critical points for a Morse function.

Definition 11 *We will say that the planar curve $L =< H >$ admits at z_0 a* non degenerate cusp *if z_0 is a degenerate (non Morse) critical point of H such that $H''(z_0)$ is of rank* 1 *and the polynomial of degree 3 in the Taylor expansion does not vanish on the kernel of $H''(z_0)$.*

12.1 Classics

Theorem 10 *Let H be an Hamiltonian such that $< H >$ admits at z_0 a non degenerate cusp, there exists a canonical transformation χ and a smooth function E non vanishing at z_0 such that $H \circ \chi = EH_0$ with $H_0 = \xi^2 + x^3$:*

$$< H \circ \chi >= H_0 \ .$$

By Theorem 3, it is enough to know that H and $\xi^2 + x^3$ are equivalent germs. This result can be proved easily as follows: apply first Morse lemma, we get $\xi^2 + f(x)$ where the third derivative of f does not vanish. See [3] chapter 2.

12.2 Semi-classics

Let $\widehat{H_t} u = 0$ be an analytic family of semi-classical equations such that the principal symbol H_0 of $\widehat{H_0}$ vanishes at z_0 with a non degenerate cusp. Using Theorems 6, 8 and 10, we get the following pseudo-differential equation as a microlocal normal form:

$$-h^2 u'' + (x^3 + a(t,h)x + b(t,h))u = O(h^\infty)$$

where $a \sim \sum_{j=0}^{\infty} a_j(t)h^j$ and $b \sim \sum_{j=0}^{\infty} b_j(t)h^j$ are formal series in h.

12.3 Computation of the first coefficients $a_{1,0}$ et $b_{1,0}$

Let us start with F_0 having a cusp at 0. By a rotation, we can assume that

$$F_0 = A\xi^2 + Bx^3 + O(7)$$

where $f = O(N)$ means $F(t^3\xi, t^2x) = O(t^N)$. By a canonical diagonal linear transformation, we get

$$F_0 = (A^3B^2)^{1/5}(\xi^2 + x^3 + \alpha x^2\xi + \beta x\xi^2 + \gamma x^4 + O(9)) \ ,$$

and then, removing the constant prefactor:

$$F_0 = \xi^2 + (x + \frac{\alpha}{3}\xi)^3 + (\beta - \frac{\alpha^2}{3})x\xi^2 + \gamma x^4 + O(9)) \ ,$$

and putting $\xi_1 = \xi, \ x_1 = x + \frac{\alpha}{3}\xi$:

$$F_0 = \xi_1^2 + x_1^3 + (\beta - \frac{\alpha^2}{3})x_1\xi_1^2 + \gamma x_1^4 + O(9)) \ .$$

We want to find χ so that:

$$F_0 \circ \chi = (1 + ex_1)(\xi_1^2 + x_1^3) + O(9)$$

We compute easily

$$x\xi^2 = \frac{3}{7}x(\xi^2 + x^3) + \{S, \xi^2 + x^3\}, x^4 = \frac{4}{7}x(\xi^2 + x^3) + \{S', \xi^2 + x^3\}$$

and we get that way

$$e = \frac{1}{7}(3\beta + 4\gamma - \alpha^2)$$

We have now:

$$(F_0 + tK) \circ \chi = (1 + ex_1)\left(\xi_1^2 + x_1^3 + t\frac{K \circ \chi}{1 + ex_1}\right)) + O(9)$$

and we get by projecting the deformation onto the versal deformation

$$(F_0 + tK) \circ \chi_t = E_t(x_1, \xi_1)\left(\xi_1^2 + x_1^3 + t(a_{1,0}x_1 + b_{1,0})\right) + O(t^2)\ .$$

We put $k_0 = K(0),\ k_1 = \partial_{x_1} K(0)$ and we get

$$a_{1,0} = k_1 - ek_0,\ b_{1,0} = k_0\ .$$

The same formulae holds for $a_{0,1}$ and $b_{0,1}$ be replacing K by the subprincipal symbol of $\hat{H}_0$.

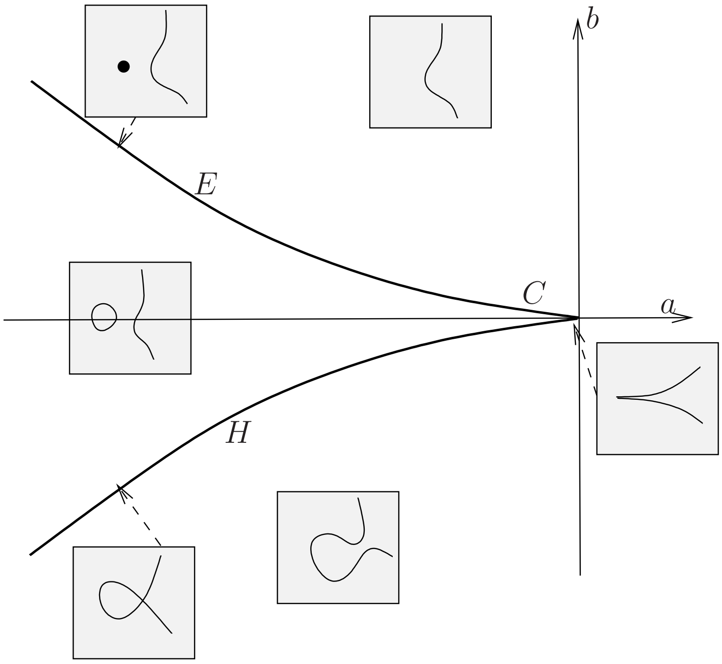

Figure 4: bifurcation diagram of the cusp

12.4 The model problem

Let $\hat{P}v(y) = -v''(y) + (y^3 + Ay + B)v(y)$ with $A, B \in \mathbb{R}$. We may define the reflexion coefficient $R(A, B)$ in the following way: the equation $\hat{P}v = 0$ admits 2 exact solutions $v_\pm(y)$, smoothly depending on A and B, which admits WKB expansions at infinity ($v_-(y) = \overline{v_+(y)}$) and, as $y \to -\infty$:

$$v_+(y) = |y|^{-\frac{3}{4}} e^{i(\frac{2}{5}|y|^{5/2} + A|y|^{1/2})}\left(1 + \sum_{\alpha=1}^{\infty} a_\alpha(A, B)|y|^{-\alpha/2}\right) + O(\ |y|^{-\infty});$$

existence of solutions with a given asymptotic expansion is a classical fact. They are clearly unique (for a general approach concerning asymptotic solutions, see [6], [34], [39]). There exists an unique function $R(A, B)$ (of modulus 1), called the *reflexion coefficient* or *scattering matrix*, such that $v = v_- + R(A, B)v_+ \in L^2([0, +\infty[, dy)$. This function $R(A, B)$ is the *special function* of our problem. It can be related with Stokes multipliers.

Question 9 *Describe as much as possible the function $R(A, B)$.*

12.5 The semi-classical bifurcation

12.5.1 The scattering matrix

We choose exact solutions $u_{\pm,a,b}$ of equation

$$-h^2u'' + (x^3 + ax + b)u = O \tag{9}$$

(with a and b real valued) which admit the following WKB expansions:

$$u_{\pm,a,b}(x,h) = e^{iS_{a,b}(x)/h}(\sum_{j=0}^{\infty} a_j(x;a,b)h^j) + O(h^\infty)$$

normalized by $S_{a,b}(-1) = 0$ and $\sum_{j=0}^{\infty} a_j(-1;a,b)h^j = 1$. We obtain that way the *semi-classical scattering matrix* $\sigma(a,b;h)$, well defined modulo $O(h^\infty)$ by asking that $u_{+,a,b} + \sigma(a,b;h)u_{-,a,b}$ extends to an admissible function.

12.5.2 Renormalization

Let us start with the semi-classical model problem given by equation (9) and assume that a and b can be h dependent. We will denote by

$$\|a,b\| = (|a|^3 + b^2)^{5/12}$$

And we will measure the distance to the bifurcation using τ defined by:

$$\|a,b\| = h\tau = \eta$$

We can now use the renormalisation $x = \eta^{2/5}y$ which gives:

$$-\tau^{-2}v''_{y^2} + (y^3 + Ay + B)v = 0$$

with $a = A\eta^{4/5},\ b = B\eta^{6/5}$. Now A and B are of order 1. We have 3 domains:

1. The domain where τ is bounded (w.r. to h) where the bifurcation really takes place and there is no further asymptotics.
2. The Log domain where $1 << \tau = O(|\log h|)$ where we can use the semi-classical asymptotics w.r. to τ including tunneling effect which is not $O(h^\infty)$.
3. The domain where $\tau >> |\log h|$ where we can apply usual formulae without looking at the bifurcation problem: the semi-classical spectrum splits into 2 parts; one associated to the real vanishing circle, the other to the bigg closed cycle.

12.5.3 The bifurcation domain

In this domain ($\|a,b\| = O(h)$), τ is bounded.

If we use $a = Ah^{4/5},\ b = Bh^{6/5}$, the renormalized equation is

$$-v'' + (x^3 + Ax + B)v = O \tag{10}$$

where A and B are bounded.

In this domain, we have the following relationship between R and σ:

$$\sigma(a,b;h) = R(A,B)e^{-\frac{i}{h}(\frac{4}{5}+2Ah^{4/5})}\left(1 + \sum_{\alpha=1}^{\infty}\gamma_\alpha(A,B))h^{\alpha/5}\right) + O(h^\infty)\ , \tag{11}$$

with $A = ah^{-4/5},\ B = bh^{-6/5}$ and the γ_α's can be computed from the a_α's .

12.5.4 The Log domain

In this domain we can compute the τ semi-classical solution using tunneling effect (see [21], [17]).

12.6 Bohr-Sommerfeld rules

From the previous sections, we can compute the singular holonomy using the asymptotic behaviour of $\sigma(a, b; h)$ for (a, b) non zero and $h \to 0$. We can then derive the Bohr-Sommerfeld rules from $R(A, B)$ using equation (11).

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