WEYL LAW FOR LAPLACIANS WITH CONSTANT MAGNETIC FIELD ON NON COMPACT HYPERBOLIC SURFACES OF FINITE AREA

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Outline

- The hyperbolic context
- The Poincaré half-plane
  - The constant magnetic Laplacian
  - Related results
- Geometrically finite hyperbolic surfaces
  - The essential spectrum of constant magnetic Laplacians
  - The Weyl formula in the case of finite area with a non-integer class one-form
  - Remarks
  - Outline of proofs

Joint work with Abderemane Morame, University of Nantes

Framework

- Let (M, g) be a connected and oriented Riemannian manifold of dimension n.
- For any real one-form A on M , define

$$-\Delta_A = (i\, d+A)^\star (i\, d+A)\ , \ ((i\ d + A)u = i\ du + uA\ ,\ \forall\ u\ \in\ C_0^\infty(M))\ .$$

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- The magnetic field is the two-form dA .
- To dA is associated the linear operator B defined on the tangent space by $dA(X, Y) \ = \ g(B.X, Y)\ ; \quad \forall\ X\ ,\ Y\ \in TM \times TM$.
- The magnetic intensity $\mathbf{b}$ is given by $\mathbf{b} \ = \ \frac{1}{2} tr\left((B^\star B)^{1/2}\right)$.

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- The magnetic intensity $\mathbf{b}$ is given by $\mathbf{b} \;=\; \frac{1}{2}tr\left((B^\star B)^{1/2}\right)$.
- If $dim(M)=2$, then
  - $dA \;=\; \widetilde{\mathbf{b}}\;dv$, with $|\widetilde{\mathbf{b}}| \;=\; \mathbf{b}$,
    dv the Riemannian measure on M.
  - The magnetic field is constant $\Longleftrightarrow$ $\widetilde{\mathbf{b}}$ is constant.

The Poincaré half-plane

Let $M = \mathbb{H}$ be the hyperbolic half-plane :

$$\mathbb{H} = \mathbb{R}\times]0,+\infty[\,, \quad g = \frac{dx^2+dy^2}{y^2}\,.$$

- $-\Delta_A = y^2(D_x - A_1)^2 + y^2(D_y - A_2)^2$,
  with $A = A_1(x,y)\,dx + A_2(x,y)\,dy$, and $A_j(x,y) \in C^\infty(\mathbb{H};\mathbb{R})$,

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- $\widetilde{\mathbf{b}} = y^2\,(\partial_x A_2 - \partial_y A_1)$
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Properties

- $-\Delta_A$ is essentially self-adjoint on $L^2(\mathbb{H})$.
- We are interested on its spectrum : $\mathrm{sp}(-\Delta_A)$.
  Gauge invariance:

$$\mathrm{sp}(-\Delta_A) = \mathrm{sp}(-\Delta_{A+d\varphi})\,; \quad \forall\,\varphi \in C^\infty(\mathbb{H};\mathbb{R})\,.$$

The spectrum for constant magnetic field

- The spectrum of $-\Delta_{A^{\mathbf{b}}}$ is essential:
  $\mathrm{sp}(-\Delta_{A^{\mathbf{b}}}) = \mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \bigcup S(\mathbf{b})$.
- Its absolutely continuous part is given by
  $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \; = \; [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.

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  $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) = [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.
- If $0 \leq \mathbf{b} \leq 1/2$ the remaining part $S(\mathbf{b})$ is empty,
- if $\mathbf{b} > 1/2$ it is formed by a finite number of eigenvalues of infinite multiplicity given by

$$S(\mathbf{b}) = \{(2j+1)\mathbf{b} - j(j+1) \; ; \; j \in \mathbb{N} \, , \; j < \mathbf{b} - \frac{1}{2}\} \, .$$

Related results : hyperbolic context

- Results on the hyperbolic space
  - Maass Laplacian: Elstrodt(73), Grosche(88), Comtet(87), Ikeda-Matsumoto (99), Kim-Lee(02)
  - Asympt. constant magnetic fields ,Pauli operators : Inahama-Shirai (03)
  - Asymptotic distribution for Schrödinger operators : Inahama-Shirai (04)
  - Asymptotic distribution for magnetic bottles : Morame-Truc (08)
- conformally cusp manifolds :
  Asymptotic distribution for Schrödinger operators :
  Golénia-Moroianu (08)
- Geometrically finite hyperbolic surfaces
  Asymptotic distribution for magnetic bottles : Morame-Truc (09)

Geometrically finite hyperbolic surface of infinite area

- Definition

  $(\mathcal{M},\ g)$: a smooth connected Riemannian manifold of dimension two $\mathcal{M} \ = \ \left(\bigcup_{j=0}^{J_1} M_j \right) \bigcup \left(\bigcup_{k=1}^{J_2} F_k \right)$;

  - M_j , F_k open sets of $\mathcal{M}$, M_0 compact closure,
  - ($j \neq 0$: M_j isometric to $\mathbb{S}\times]a_j^2, +\infty[$, (cuspidal ends)

$$ds_j^2 \ = \ y^{-2}(\ L_j^2 \ d\theta^2 \ + \ dy^2 \ )$$

  $(a_j$ and L_j are strictly positive constants)

  - F_k isometric to $\mathbb{S}\times]\alpha_k^2, +\infty[$, (funnel ends)

$$ds_k^2 \ = \ \tau_k^2 \cosh^2 t \ d\theta^2 \ + \ dt^2$$

  $(\alpha_k$ and τ_k are strictly positive constants) .

Constant magnetic field on $\mathcal{M}$

- $\mathcal{M} = \left(\bigcup_{j=0}^{J_1} M_j \right) \bigcup \left(\bigcup_{k=1}^{J_2} F_k \right)$;
- It is not always possible to have a constant magnetic field on $\mathcal{M}$, but

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- $\forall (b, \beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2}$, $\exists\, A$, s. t. dA satisfies

$$dA = \widetilde{\mathbf{b}}(z) dm \quad \begin{cases} \widetilde{\mathbf{b}}(z) = b_j \ \forall\, z \in M_j \\ \widetilde{\mathbf{b}}(z) = \beta_k \ \forall\, z \in F_k \end{cases}$$

The essential spectrum of magnetic Laplacians (1)

Theorem 1

- If $J_1 = 0$ and $J_2 > 0$,

$$\mathrm{sp}_{ess}(-\Delta_A) = [\frac{1}{4} + \inf_k \beta_k^2 \,,\, +\infty[\bigcup \left(\bigcup_{k=1}^{J_2} S(\beta_k) \right) .$$

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- If $0 \leq \beta_k \leq 1/2$ $S(\beta_k)$ is empty,
- if $\beta_k > 1/2$ it is formed by a finite number of eigenvalues of infinite multiplicity given by

$$S(\beta_k) = \{(2j+1)\beta_k - j(j+1)\ ;\ j \in \mathbb{N}\ ,\ j < \beta_k - \frac{1}{2}\} .$$

The essential spectrum of magnetic Laplacians (2)

If $J_1 > 0$, then

- $\forall j$, $1 \leq j \leq J_1$ and $\forall z \in M_j$ $\exists$ a unique closed curve $\mathcal{C}_{j,z}$ through z , in (M_j, g) , not contractible and with zero $g-$curvature.
- The following limit exists and is finite:

$$[\mathbf{A}]_{\mathbf{M_j}} = \lim_{d(z)\to+\infty} \int_{\mathcal{C}_{j,z}} A \, .$$

- Define : $J_1^A = \{j \in \mathbb{N} \, , \, 1 \leq j \leq J_1 \; s.t. \; [\mathbf{A}]_{\mathbf{M_j}} \in 2\pi\mathbb{Z} \, \}$.

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Theorem 2

Assume that $J_1 > 0$, $J_2 > 0$, and $J_1^A \neq \emptyset$. Then

- $\mathrm{sp}_{ess}(-\Delta_A) =$
  $$[\frac{1}{4} + \min\{ \inf_{j \in J_1^A} b_j^2 \, , \, \inf_{1 \leq k \leq J_2} \beta_k^2 \} \, , \, +\infty[\bigcup \left(\bigcup_{k=1}^{J_2} S(\beta_k) \right) \, .$$

Weyl law for $\mathcal{M}$ of finite area,with a non-integer class 1-form A

- Theorem 3

  Consider a geometrically finite hyperbolic surface $(\mathcal{M},\ g)$ of finite area, $(J_2 = 0)$, and assume that $J_1^A = \emptyset$. Then

  - (i) $\mathrm{sp}_{ess}(-\Delta_A) = \emptyset$:
    $-\Delta_A$ has purely discrete spectrum , (its resolvent is compact).
  - (ii) $N(\lambda, -\Delta_A) = \lambda \frac{|\mathcal{M}|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda)$.

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- (Definitions) In the case when $\mathrm{sp}(-\Delta_A)$ is discrete,

  - denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue repeated according to its multiplicity)
  - define $N(\lambda, -\Delta_A) = \sum_{\lambda_j < \lambda} 1$.
  - Weyl law for the Laplacian on a compact n-dim Riemannian manifold M

    $$N(\lambda, -\Delta_0^M) = \lambda^{n/2} \frac{|M|}{(4\pi)^{n/2}\Gamma(n/2+1)} + o(\lambda^{n/2}) .$$

Remarks (1)

- Theorem 3 relies on the Proposition (Weyl law for one cusp)
  - Consider $M = \mathbb{S}\times]\alpha^2, +\infty[$ equipped with the metric $ds^2 \ = \ L^2 e^{-2t} d\theta^2 \ + \ dt^2$ for some $\alpha > 0$ and $L > 0$.
  - Denote by $-\Delta_A^M$ the Dirichlet op. on M , ass. to $-\Delta_A$.
  - $$N(\lambda, -\Delta_A^M) \ = \ \lambda \frac{|M|}{4\pi} \ + \ \mathbf{O}(\sqrt{\lambda} \ln \lambda) \ .$$
- Stability under perturbation of the metric

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- Stability under perturbation of the metric

  Theorem 3 still holds if the metric of $\mathcal{M}$ is modified in a compact set.
- One can see the case $J^A = \emptyset$ as an Aharonov-Bohm phenomenon :
  - the magnetic field dA is not sufficient to describe $-\Delta_A$
  - the use of the magnetic potential A is essential :

    one can have magnetic bottle (magnetic Laplacian with compact resolvent) with null intensity.

Remarks (2)

- When $A = 0$, $-\Delta = -\Delta_0$ has embedded eigenvalues in its essential spectrum : $\quad (sp_{ess}(-\Delta) = [\frac{1}{4}, +\infty[)$.
  - Denote by $N_{ess}(\lambda, -\Delta)$ the number of these eigenvalues in $[\frac{1}{4}, \lambda[$, then
  - $N_{ess}(\lambda, -\Delta) \ \leq \ \lambda \frac{|\mathcal{M}|}{4\pi}$; (Colin de Verdière, Hejhal)

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  - $N_{ess}(\lambda, -\Delta) \ \leq \ \lambda \frac{|\mathcal{M}|}{4\pi}$; (Colin de Verdière, Hejhal)
- Locally symmetric spaces and automorphic forms
  Consider $\mathcal{M} = \Gamma(N)\backslash\mathbb{H}$, with

$$\Gamma(N) = \{\gamma \in SL(2, \mathbb{Z}) : \gamma = Id \bmod N\}$$

then (Müller '07)

$$N_{ess}(\lambda, -\Delta) \ = \ \lambda \frac{|\mathcal{M}|}{4\pi} \ + \ \mathbf{O}(\sqrt{\lambda} \ln \lambda) \ .$$

Remarks (3)

- Definition

  A Maass automorphic form is a smooth function $\mathbb{H} \to \mathbb{C}$ s.t.

  - $f(\gamma z) = f(z) \quad \forall \gamma \in \Gamma(N)$
  - $\exists \lambda$ s.t. $\Delta f = \lambda f$
  - f is slowly increasing

- Exemples : Eisenstein series, cusp forms

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- Selberg formula (derived from the trace formula)

  For any lattice $\Gamma \in SL(2, \mathbb{R})$:

$$N_\Gamma(\lambda, -\Delta) + \; M_\Gamma(\lambda, -\Delta) \; \sim \; \lambda \frac{|\Gamma \backslash \mathbb{H}|}{4\pi} \; .$$

  - $M_\Gamma(\lambda, -\Delta)$: winding number of the determinant of the scattering matrix (given by the zeroth Fourier coefficients of the Eisenstein series)

Proof of Theorems 1,2, 3(i)

- $$\mathrm{sp}_{ess}(-\Delta_A) = \left(\bigcup_{j=1}^{J_1} \mathrm{sp}_{ess}(-\Delta_A^{M_j})\right) \bigcup \left(\bigcup_{k=1}^{J_2} \mathrm{sp}_{ess}(-\Delta_A^{F_k})\right) \; ;$$

Proof of Theorems 1,2, 3(i)

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- Lemma 1

$$\mathrm{sp}_{ess}(-\Delta_A^{F_k}) = [\frac{1}{4} + \beta_k^2 , +\infty[\bigcup \left(\bigcup_{k=1}^{J_2} S(\beta_k)\right) .$$

- Lemma 2

  If $1 \leq j \leq J_1$ and $j \notin J_1^A$, then

$$\mathrm{sp}_{ess}(-\Delta_A^{M_j}) = \emptyset .$$

  If $j \in J_1^A$, then

$$\mathrm{sp}_{ess}(-\Delta_A^{M_j}) = [\frac{1}{4} + b_j^2 , +\infty[.$$

Proof of Lemma 2 (1)

- M_j isometric to $\mathbb{S}\times]a_j^2, +\infty[$, (cuspidal ends)
  $ds_j^2 \;=\; y^{-2}(\; L_j^2 \; d\theta^2 \;+\; dy^2 \;) \quad (a_j > 0 \text{ and } L_j > 0\;)$
- Use the coordinate $t = \ln y$ instead of y , so
  - $M_j \;=\; \mathbb{S}\times]\alpha_j^2, +\infty[\quad \text{and} \quad ds_j^2 \;=\; L_j^2 e^{-2t} d\theta^2 \;+\; dt^2$;
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  - $-\Delta_A^{M_j} \;=\; L_j^{-2} e^{2t}(D_\theta - A_1)^2 + e^t(D_t - A_2)(e^{-t}(D_t - A_2))$,
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- - $-\Delta_A^{M_j} = L_j^{-2} e^{2t}(D_\theta - A_1)^2 + e^t(D_t - A_2)(e^{-t}(D_t - A_2))$,
  - $\widetilde{\mathbf{b}} = b_j = L_j^{-1} e^t(\partial_\theta A_2 - \partial_t A_1)$ and $dm = L_j e^{-t} d\theta dt$.
- we have

$$A - \widetilde{A} = d\varphi \text{ if } \widetilde{A} = (\xi + L_j b_j e^{-t})d\theta\ , \ (\text{for some constant } \xi)\ .$$

- $\Longrightarrow$ we can assume that $A = \widetilde{A}$.

Proof of Lemma 2 (2)

- Define $Uf = \sqrt{L_j} e^{-t/2} f$,
- $\Longrightarrow P = -U \Delta_A^{M_j} U^\star = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \dfrac{1}{4}$.

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- $\Longrightarrow \mathrm{sp}(-\Delta_A^{M_j}) = \mathrm{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \mathrm{sp}(P_\ell)$

  $$P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L_j} + b_j \right)^2 ,$$

  for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_j^2, +\infty[$.

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  for the Dirichlet condition on $L^2(I;dt)$; $I =]\alpha_j^2, +\infty[$.
- When $\ell + \xi \neq 0$, the spectrum of P_ℓ is discrete.
- More precisely : $\mathrm{sp}(P_\ell) \;=\; \mathrm{sp}(P^\pm) \qquad P^\pm = D_t^2 + \frac{1}{4} + (\pm e^t + b_j)^2$

  for the Dirichlet condition on

  $L^2(I_{j,\ell};dt)$; $I_{j,\ell} =]\alpha_j^2 + \ln(|\ell+\xi|/L_j), +\infty[$, and $\pm = \dfrac{\ell+\xi}{|\ell+\xi|}$.

Proof of Lemma 2 (3)

- $\Longrightarrow$ if $\ell + \xi \neq 0 \; \forall \ell \in \mathbb{Z}$, $\lim_{|\ell| \to \infty} \inf \mathrm{sp}(P_\ell) \; = \; +\infty$,
- $\Longrightarrow$ the spectrum of $-\Delta_A^{M_j}$ is discrete.

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- $\Longrightarrow$ the spectrum of $-\Delta_A^{M_j}$ is discrete.
- What means this condition ? Recall
  - $A = (\xi + L_j b_j e^{-t}) d\theta$,
  - $[\mathbf{A}]_{\mathbf{M_j}} \ = \ \lim_{d(z) \to +\infty} \int_{\mathcal{C}_{j,z}} A$

    $\Longrightarrow [\mathbf{A}]_{\mathbf{M_j}} = \lim_{t \to +\infty} \int_0^{2\pi} (\xi + L_j b_j e^{-t}) d\theta = 2\pi\xi$, so
- $\ell + \xi \neq 0 \ \forall \ell \in \mathbb{Z} \quad \Longleftrightarrow \quad [\mathbf{A}]_{\mathbf{M_j}} \notin 2\pi\mathbb{Z} \quad \Longleftrightarrow \quad J_1^A \ = \ \emptyset$.

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  $\Longrightarrow [\mathbf{A}]_{\mathbf{M_j}} = \lim_{t \to +\infty} \int_0^{2\pi} (\xi + L_j b_j e^{-t}) d\theta = 2\pi\xi$, so
- $\ell + \xi \neq 0 \ \forall \ell \in \mathbb{Z} \iff [\mathbf{A}]_{\mathbf{M_j}} \notin 2\pi\mathbb{Z} \iff J_1^A = \emptyset$.

- If $\ell + \xi = 0$, the spectrum of P_ℓ is absolutely continuous :

$$\mathrm{sp}(P_{-\xi}) = \mathrm{sp}_{ess}(P_{-\xi}) = \mathrm{sp}_{ac}(P_{-\xi}) = [\frac{1}{4} + b_j^2 \ , \ +\infty[\ ;$$

- $\Longrightarrow$ if $[A]_{M_j} \in 2\pi\mathbb{Z}$, $\mathrm{sp}_{ess}(-\Delta_A^{M_j}) = [\frac{1}{4} + b_j^2 \ , \ +\infty[$.

Proof of Proposition (1)

- $M = \mathbb{S}\times]\alpha^2, +\infty[$ a cusp
  $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ the metric on M ($\alpha > 0$ and $L > 0$).
- $A = (\xi + Lbe^{-t})d\theta$, (for some constant ξ) .
- $N(\lambda, -\Delta_A^M) = \sum_{\ell \in \mathbb{Z}} N(\lambda, P_\ell)$
  with $P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L} \pm b \right)^2$,
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  $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ the metric on M ($\alpha > 0$ and $L > 0$).
- $A = (\xi + Lbe^{-t})d\theta$, (for some constant ξ) .
- $N(\lambda, -\Delta_A^M) = \sum_{\ell \in \mathbb{Z}} N(\lambda, P_\ell)$
  with $P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L} \pm b\right)^2$,
  for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha^2, +\infty[$.
- Define $Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell + \xi)^2}{L^2} e^{2t}$, for the D. C. on $L^2(I; dt)$.
- Then $Q_\ell - C\sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C\sqrt{Q_\ell}$.

Proof of Proposition (1)

- $M = \mathbb{S}\times]\alpha^2, +\infty[$ a cusp
  $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ the metric on M ($\alpha > 0$ and $L > 0$).
- $A = (\xi + Lbe^{-t})d\theta$, (for some constant ξ) .
- $N(\lambda, -\Delta_A^M) = \sum_{\ell\in\mathbb{Z}} N(\lambda, P_\ell)$
  with $P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell+\xi)}{L} \pm b\right)^2$,
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- Define $Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell+\xi)^2}{L^2} e^{2t}$, for the D. C. on $L^2(I; dt)$.
- Then $Q_\ell - C\sqrt{Q_\ell} \le P_\ell \le Q_\ell + C\sqrt{Q_\ell}$.
- $\Longrightarrow$ $\exists$ a constant $C(b)$, s. t. for any $\lambda >> 1$,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \le N(\lambda, P_\ell) \le N(\lambda + \sqrt{\lambda}C(b), Q_\ell) ;$$

Proof of Proposition (2)

- Lemma

  $\exists C > 1$ s. t. $\forall \mu >> 1$ and $\forall \ell \in X_\mu$,

  $$w_\ell(\mu) - \pi \ \leq \ \pi N(\mu - \frac{1}{4}, Q_\ell) \ \leq \ w_\ell(\mu) + \frac{1}{12} \ln \mu + C$$

Proof of Proposition (2)

- Lemma

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  $$w_\ell(\mu) - \pi \ \leq \ \pi N(\mu - \frac{1}{4}, Q_\ell) \ \leq \ w_\ell(\mu) + \frac{1}{12} \ln \mu + C$$

$$w_\ell(\mu) \ = \ \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t} \right]_+^{1/2} dt$$

$$X_\mu = \{ \ell \ / \ e^{\alpha^2} \frac{|\ell + \xi|}{L} < \sqrt{\mu - 1/4} - b \ \} \ .$$

Proof of Proposition (2)

- **Lemma**

  $\exists C > 1$ s. t. $\forall \mu >> 1$ and $\forall \ell \in X_\mu$,

  $$w_\ell(\mu) - \pi \;\leq\; \pi N(\mu - \frac{1}{4}, Q_\ell) \;\leq\; w_\ell(\mu) + \frac{1}{12}\ln\mu + C$$

$$w_\ell(\mu) \;=\; \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dt$$

$$X_\mu = \{\ell \;/\; e^{\alpha^2} \frac{|\ell + \xi|}{L} < \sqrt{\mu - 1/4} - b\;\}\;.$$

- Proof of Lemma (Titchmarsh 's method)
  - define $V_\ell = \frac{(\ell+\xi)^2}{L^2} e^{2t}$, ϕ_μ^ℓ : a solution of $Q_\ell \phi = (\mu - \frac{1}{4})\phi$.
  - Consider x_ℓ and y_ℓ s. t. $V_\ell(x_\ell) = \mu$ and $V_\ell(y_\ell) = \nu$, $0 < \nu < \mu$ to be determined later.
  - n (resp. m): number of zeros of ϕ_μ^ℓ on $]\alpha^2, x_\ell[$ (resp. on $]\alpha^2, y_\ell[$)
  - recall: $n = N(\mu - \frac{1}{4}, Q_\ell)$.

Proof of Proposition (3)

(Titchmarsh 's lemma)

$$m\pi = \int_{\alpha^2}^{y_\ell} [\mu - V_\ell]^{1/2}\, dt + R_\ell$$

with $R_\ell \le \frac{1}{4}\ln(\mu - V_\ell(\alpha^2)) - \frac{1}{4}\ln(\mu - V_\ell(y_\ell)) + \pi$,

$$\Longrightarrow |n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2}\, dt| \le (x_\ell - y_\ell)(\mu - \nu)^{1/2} + R_\ell + (n - m)\pi$$

Proof of Proposition (3)

(Titchmarsh 's lemma)

$m\pi = \int_{\alpha^2}^{y_\ell} [\mu - V_\ell]^{1/2}\, dt + R_\ell$

with $R_\ell \leq \frac{1}{4}\ln(\mu - V_\ell(\alpha^2)) - \frac{1}{4}\ln(\mu - V_\ell(y_\ell)) + \pi$,

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Sturm comparison theorem $\Longrightarrow (n - m)\pi \ \leq \ (x_\ell - y_\ell)(\mu - \nu)^{1/2} + \pi$

So since $x_\ell - y_\ell = \ (1/2)\ln(\frac{\mu}{\nu})$

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2}\, dt| \leq \ln(\frac{\mu}{\nu})(\mu - \nu)^{1/2} + \frac{1}{4}\ln\mu - \frac{1}{4}\ln(\mu - \nu) + 2\pi$$

Now take $\nu = \mu - \mu^{2/3}$ to get the Lemma.

Proof of Proposition (4)

- We want to compute $\sum_{\ell\in\mathbb{Z}} w_\ell(\mu)$ with

$$w_\ell(\mu) \;=\; \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dt.$$

Proof of Proposition (4)

- We want to compute $\sum_{\ell\in\mathbb{Z}} w_\ell(\mu)$ with

$$w_\ell(\mu) \;=\; \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dt.$$

- $\Longrightarrow$ it is enough to compute $\mathcal{I} = \int_{\alpha^2}^{+\infty} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dx dt$,
- $\mathcal{I}$ is equivalent to $\mu L e^{-\alpha^2} \int_{\mathbb{R}} \left[1 - x^2\right]_+^{1/2} dx$
- Use $|M| = 2\pi L e^{-\alpha^2}$ to conclude.