MAGNETIC BOTTLES ON POINCARE HALF-PLANE

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Outline

- The hyperbolic context
- The constant magnetic Laplacian
- Magnetic bottles : compacity of the resolvent, spectral asymptotics
- Related results
- Proof outline
- Developpements : dimension 3, hyperbolic surfaces of infinite area

Joint work with Abderemane Morame, University of Nantes

Framework

- Let (M, g) be a connected and oriented Riemannian manifold of dimension n.
- For any real one-form A on M , define

$$-\Delta_A = (i\,d+A)^\star(i\,d+A)\;,\;((i\,d+A)u = i\,du + uA\;,\;\forall\,u\in C_0^\infty(M$$

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- The magnetic field is the two-form dA .
- To dA is associated the linear operator B defined on the tangent space by

$$dA(X,Y) \;=\; g(B.X,Y)\;; \quad \forall\;X\;,\;Y\;\in TM\times TM\;.$$

- The magnetic intensity $\mathbf{b}$ is given by

$$\mathbf{b} \;=\; \frac{1}{2}tr\left((B^{\star}B)^{1/2}\right)\;.$$

The Poincaré half-plane

- If $dim(M) = 2$, then $dA \; = \; \widetilde{\mathbf{b}} \, dv$, with $|\widetilde{\mathbf{b}}| \; = \; \mathbf{b}$, dv the Riemannian measure on M.
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- Let $M = \mathbb{H}$ be the hyperbolic half-plane :
$$\mathbb{H} = \mathbb{R}\times]0,+\infty[\,, \quad g = \frac{dx^2 + dy^2}{y^2}\,.$$
    - $-\Delta_A = y^2(D_x - A_1)^2 + y^2(D_y - A_2)^2$, with $A = A_1(x,y)\,dx + A_2(x,y)\,dy$, and $A_j(x,y) \in C^\infty(\mathbb{H};\mathbb{R})$,
    - $\widetilde{\mathbf{b}} = y^2\,(\partial_x A_2 - \partial_y A_1)$
    - $\mathbf{b} = |\widetilde{\mathbf{b}}|$,
    - $dv = y^{-2} dx dy$.

Properties

- $-\Delta_A$ is essentially self-adjoint on $L^2(\mathbb{H})$.
- We are interested on its spectrum : $\mathrm{sp}(-\Delta_A)$.
  We will use that it is gauge invariant:

$$\mathrm{sp}(-\Delta_A) = \mathrm{sp}(-\Delta_{A+d\varphi}) \; ; \quad \forall \, \varphi \in C^\infty(\mathbb{H};\mathbb{R}) \; .$$

The spectrum for constant magnetic field

- The spectrum of $-\Delta_{A^{\mathbf{b}}}$ is essential: $\mathrm{sp}(-\Delta_{A^{\mathbf{b}}}) = \mathrm{sp}_{es}(-\Delta_{A^{\mathbf{b}}})$.
- Its absolutely continuous part is given by
  $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \;=\; [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.

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  $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \; = \; [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.
- The remaining part of its spectrum is empty if $0 \leq \mathbf{b} \leq 1/2$.
- Otherwise it is formed by a finite number of eigenvalues of infinite multiplicity given by

$$\mathrm{sp}_p(-\Delta_{A^{\mathbf{b}}}) \; = \; \{(2j+1)\mathbf{b} - j(j+1) \; ; \; j \; \in \; \mathbb{N} \, , \; j < \mathbf{b} - \frac{1}{2}\} \, ,$$

$$(\text{if} \quad \frac{1}{2} < \mathbf{b} \, .)$$

Magnetic bottle : compact resolvent

$$-\Delta_A \;=\; y^2(D_x - A_1)^2 \;+\; y^2(D_y - A_2)^2$$

Magnetic bottle-type assumptions (MB)

- $\mathbf{b}(x,y) \;\to\; +\infty \quad \text{as} \quad d(x,y) \;\to\; +\infty$, $d(x,y)$: the hyperbolic distance of (x,y) to $(0,1)$.
- $\exists\, C_0 > 0$ such that, for any vector field X on $\mathbb{H}$,

$$|X\widetilde{\mathbf{b}}| \leq C_0(|\widetilde{\mathbf{b}}| + 1)\sqrt{g(X,X)} \;;$$

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Theorem

Under the assumptions (MB) $P(A) = -\Delta_A$ has a compact resolvent.

Magnetic bottle: spectral asymptotics

For any real $\lambda \leq \inf \mathrm{sp}_{es}(P)$, we denote by $N(\lambda; P)$ the number of eigenvalues of P , which are in $]-\infty, \lambda[$.

Theorem

Under the assumptions (MB) and for any $\delta \in]\frac{1}{3}, \frac{2}{5}[$, there exists a constant $C > 0$ such that

$$\frac{1}{2\pi}\int_{\mathbb{H}}(1-\frac{C}{a_\delta(m)})\mathbf{b}(m)\sum_{k=0}^{+\infty}[\lambda(1-C\lambda^{-3\delta+1})-\frac{1}{4}-(2k+1)\mathbf{b}(m)]_+^0\, dv$$

$$\leq\; N(\lambda,-\Delta_A)\;\leq$$

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$$[\rho]_+^0 \;=\; 1\;,\ \text{if } \rho>0 \;\text{ and }\; 0\;\text{ otherwise}\;,$$

$$a_\delta(m)\;:=(\mathbf{b}(m)+1)^{(2-5\delta)/2}.$$

Corollary

- If moreover $\omega(\lambda) := \int_{\mathbb{H}} [\lambda - \mathbf{b}(m)]_+^0 \, dv$ satisfies (*) :

$$\exists\, C_1 > 0 \text{ s.t. } \forall\ \lambda > C_1\ ,\ \forall\ \tau\ \in\ ]0,1[\ ,$$

$$\omega\ ((1+\tau)\ \lambda) - \omega(\lambda) \leq C_1\ \tau\ \omega(\lambda)\ ,$$

then

$$N(\lambda; -\Delta_A)\ \sim\ \frac{1}{2\pi} \int_{\mathbb{H}} \mathbf{b}(m) \sum_{k \in \mathbb{N}} [\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 \ dv\ .$$

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- (*) is satisfied when $\omega(\lambda)\ \sim\ \alpha\lambda^k \ln^j \lambda$ with $k > 0$, or $k = 0$ and $j > 0$.

Example

- $\mathbf{b}(x,y) \;=\; \left(\frac{x}{y}\right)^{2j} \;+\; p_1(y) \;+\; p_2(1/y)$,
  with $j \in \mathbb{N}^\star$, $p_1(s)$, $p_2(s)$ polynomial functions of order ≥ 1 for large s .

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- $\omega(\lambda) \; \sim \; \alpha\lambda^{\frac{1}{2j}} \ln\lambda$ when $\lambda \to +\infty$,

- $N(\lambda; -\Delta_A) \; \sim \; \dfrac{C}{2\pi}\lambda^{1+1/2j} \ln\lambda$ when $\lambda \to +\infty$.

Related results :Euclidean magnetic bottles

- If (M,g) is the euclidean space $\mathbb{R}^d$,
  $-\Delta_A = \sum_{j=1}^{d}(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j)^2$. $\forall x \in \mathbb{R}^d$, there exists $(e_j(x))$ s.t.
  $B(x) = \sum_{j=1}^{r(x)} b_j(x) dx_j \wedge dy_j,\ \ b_1(x) ... \geq b_r(x) > 0$.
  The magnetic intensity is the norm of $B(x) := (b_j(x))_j$.

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- (Y. Colin de Verdière)
  Under (EMB) conditions $-\Delta_A$ has compact resolvent and

$$N_B^{as}[\lambda(1-o(1))] \leq N(\lambda, -\Delta_A) \leq N_B^{as}[\lambda(1+o(1))] \quad (\lambda \to +\infty) \ .$$

  - $N_B^{as}(\lambda) = \int_{\mathbb{R}^d} \nu_{B(x)}(\lambda) dx$
  - $\nu_{B(x)}(\lambda) = C_{k,r} \sum^* \left(\lambda - \sum_{i=1}^{r} ((2n_i+1) b_i(x)\right)_+^{k/2} \prod_{i=1}^{r} b_i(x)$
  - $\sum^* = \sum_{(n_1, \dots n_r) \in \mathbb{N}^r}$, $\quad d = 2r + k, \quad C_{k,r} = \frac{\gamma_k}{(2\pi)^{k+r}}$
  - $\gamma_k =$ volume of the unit ball in R^k .

Related results : EMB(2)

- (EMB) conditions
  - (B_1) $\lim_{\|x\|\to\infty} \|B(x)\| = \infty$
  - (B_2) $\|x - x'\| \leq 1$, $\|B(x)\| \leq C\|B(x')\|$
  - (B_3) $M(x) = o(\|B(x)\|^{\frac{3}{2}})$ when $\|x\| \to \infty)$
    ($M(x) = \max_{|\beta|=2} \left(\sup_{\|x-x'\|\leq 1} \|D^\beta A(x')\|\right)$.)

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    ($M(x) = \max_{|\beta|=2} \left(\sup_{\|x-x'\|\leq 1} \|D^\beta A(x')\|\right)$.)
- Remark :
  - Take $d = 2$ in Colin de Verdière's result. Then
    $N_B^{as}(\lambda) = \frac{1}{2\pi} \int_{R^2} \mathbf{b}(x) \sum_{k\in\mathbb{N}} \left[\lambda - (2k+1)\mathbf{b}(x)\right]_+^0 dx$
    ($b_1(x) = \|B(x)\| = \mathbf{b}(x)$)
  - to be compared with
    $N(\lambda; -\Delta_A) \sim \frac{1}{2\pi} \int_{\mathbb{H}} \mathbf{b}(m) \sum_{k\in\mathbb{N}} [\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 \, dv$.
    $[\rho]_+^0 = 1$ if $\rho > 0$, $[\rho]_+^0 = 0$ if $\rho \leq 0$.

Related results : hyperbolic context

- Results on the hyperbolic space
  - Maass Laplacian: Elstrodt(73), Grosche(88), Comtet(87), Ikeda-Matsumoto (99), Kim-Lee(02)
  - Asympt. constant magnetic fields ,Pauli operators : Inahama-Shirai (03)
  - Asymptotic distribution for Schrödinger operators : Inahama-Shirai (04)
- conformally cusp manifolds :
  Asymptotic distribution for Schrödinger operators :
  Golénia-Moroianu (08)

Proof outline

- Minimax technique on quadratic forms

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- Main ideas
  - Get a problem in $\mathbb{R}^2$,
  - find a good gauge,
  - localize in a "good" rectangle $\Longrightarrow$ replace the initial problem by a problem with a constant magnetic field,
  - write spectral asymptotics in the constant case for a rectangle
  - use a partition of unity.

Change of variables

- diffeomorphism: $\phi \ : \ \mathbb{R}^2 \ \to \ \mathbb{H} \ , \quad (x, y) = \phi(x, t) := (x, e^t)$
- unitary operator: $\widehat{U} \ : \ L^2(\mathbb{H}; dv) \ \to \ L^2(\mathbb{R}^2; dxdt)$
  $(\widehat{U}u)(x, t) := e^{-t/2}u(x, e^t)$ (for any $u \in L^2(\mathbb{H})$.)

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- quadratic form related to $P(A) = -\Delta_A$:

$$q(u) := \int_{\mathbb{H}} \left[|y(D_x - A_1)u|^2 + |y(D_y - A_2)u|^2 \right] \frac{dxdy}{y^2} \, , (u \in L^2(\mathbb{H}))$$

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$$= \int_{\mathbb{R}^2} \left[|e^t(D_x - \tilde{A}_1)w|^2 + |(D_t - e^t\tilde{A}_2)w|^2 + 1/4|w|^2 \right] dxdt \, .$$

with

$$\tilde{A}_i(x,t) := A_i(x,e^t) \quad , \ i = \ 1,2 \ .$$

Gauge

- Work with a gauge such that $A_2 = 0$.
  Since $\widetilde{\mathbf{b}} \; = \; y^2 \left(\partial_x A_2 - \partial_y A_1 \right) \; ,$ take

$$A_1(x,y) = -\int_1^y \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds$$

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- The associated quadratic form is

$$\widehat{q}^{\tilde{A}}(w) = \int_{\mathbb{R}^2} \left[|e^t (D_x - \tilde{A}_1) w|^2 + |D_t w|^2 + 1/4 |w|^2 \right] dx dt \ ,$$

with $\tilde{A}(x,t) = (\tilde{A}_1, 0)$.

Partition of $\mathbb{R}^2$ of suitable rectangles

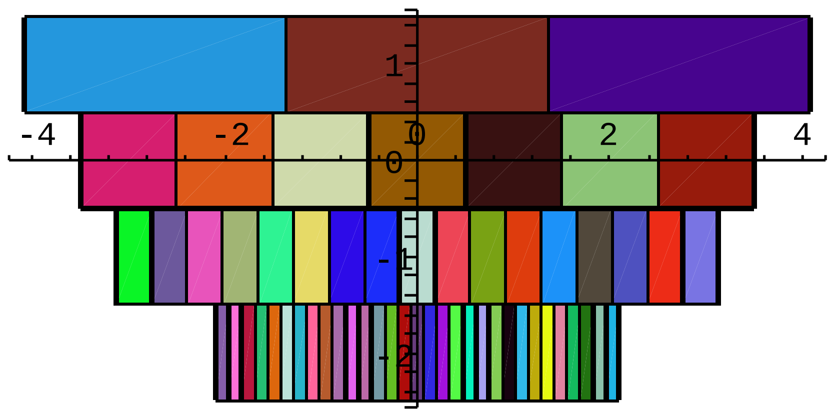
For any $\alpha \in \mathbb{Z}^2$, we denote the rectangle

$$K(\alpha) = \left] -\frac{e^{\alpha_2}}{2} + e^{\alpha_2}\alpha_1 \, , \; e^{\alpha_2}\alpha_1 + \frac{e^{\alpha_2}}{2} \right[\times \left] -\frac{1}{2} + \alpha_2 \, , \; \alpha_2 + \frac{1}{2} \right[.$$

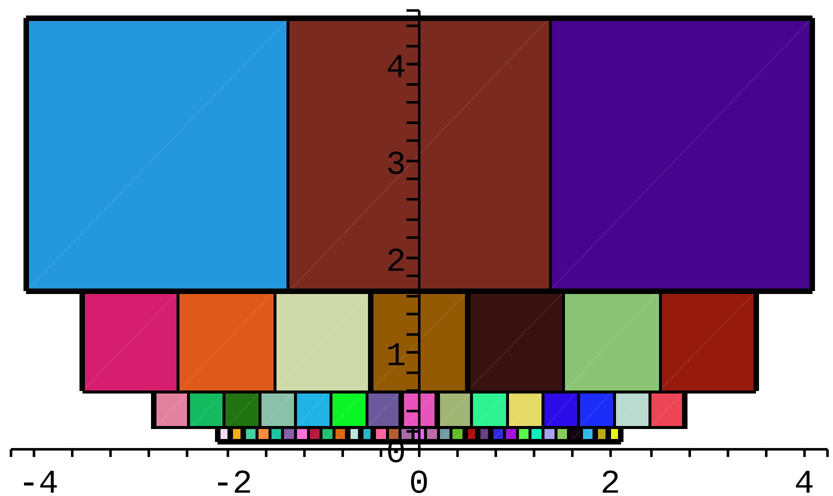
So $\mathbb{R}^2 = \cup_\alpha \overline{K}(\alpha)$

and $K(\alpha) \cap K(\beta) = \emptyset$ if $\alpha \neq \beta$.

Covering of $\mathbb{R}^2$ with the rectangles $\overline{K}(\alpha)$



Covering of $\mathbb{H}$ with the cubes $\phi(\overline{K}(\alpha))$



$$\phi \ : \quad \mathbb{R}^2 \ \to \ \mathbb{H} \ , \qquad \phi(x,t) \ = \ (x, e^t) \ \ .$$

A partition adapted to $\mathbf{b}$

Let a_0 and δ_0 such that

$$1 \ < \ a_0 \quad \text{and} \quad \delta_0 \ \in \,]\frac{1}{3}, \frac{2}{5}[\ .$$

Each $K(\alpha)$ can be parted, (if necessary), into $M(\alpha)$ rectangles:

$$\overline{K}(\alpha) \ = \ \cup_{j=1}^{M(\alpha)} \overline{K_{\alpha,j}} \ ,$$

$$K_{\alpha,j} \ = \,]-\frac{\epsilon_{\alpha,j} e^{t_{\alpha,j}}}{2} + x_{\alpha,j} \ , \ x_{\alpha,j} + \frac{\epsilon_{\alpha,j} e^{t_{\alpha,j}}}{2}[\times]-\frac{\epsilon_{\alpha,j}}{2} + t_{\alpha,j} \ , \ t_{\alpha,j} + \frac{\epsilon_{\alpha,j}}{2}[\ ,$$

with

$$\frac{1}{a_0 (1 + \mathbf{b}^{\delta_0}(x_{\alpha,j}, e^{t_{\alpha,j}}) \ )} \ \leq \ \epsilon_{\alpha,j} \ \leq \ \frac{a_0}{(1 + \mathbf{b}^{\delta_0}(x_{\alpha,j}, e^{t_{\alpha,j}}) \ )} \ , \qquad (1)$$

and such that $K_{\alpha,k} \cap K_{\alpha,j} \ = \ \emptyset$ if $k \neq j$.

Localization

- Let $X_0 = (x_0, t_0) \in \mathbb{R}^2$ such that $\mathbf{b}(z_0) > 1$; $(z_0 = (x_0, e^{t_0})\ )$; $|X_0|$ can be very large.
- Let $a_0 > 1$ and $\varepsilon_0 \in]0, 1[$ be given.
- Take $a \in]1/a_0, a_0[$ and define $K := X_0 + K_0$, with $K_0 =] - \varepsilon_0 a \frac{e^{t_0}}{2},\ \varepsilon_0 a \frac{e^{t_0}}{2} [\times] - \frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{2} [$.

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Compare :

$$\widehat{q}_K^{\tilde{A}}(w) = \int_K \left[|e^t (D_x - \tilde{A}_1) w|^2 + |D_t w|^2 + 1/4 |w|^2 \right] dx dt$$

with the "frozen" form ($w \in W_0^1(K)$) :

$$\widehat{q}_K^{A^0,0}(w) = \int_K \left[|e^{t_0} (D_x - A_1^0) w|^2 + |D_t w|^2 + 1/4 |w|^2 \right] dx dt \ .$$

First step

- $A^0(x,t) = (A_1^0, 0)\ , \quad A_1^0 := -(t-t_0)\, e^{-t_0}\, \widetilde{\mathbf{b}}(x_0, e^{t_0})\ .$
- $\tilde{A}_1(x,t)\ := -\int_{e^{t_0}}^{e^t} \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds$ (gauge invariance .)
- Key Lemma

$$|\tilde{A}_1(x,t) - A_1^0(x,t)|\ \leq\ C\, \varepsilon_0^2\, e^{-t_0}\, \mathbf{b}(x_0, e^{t_0})\ .$$

($\forall (x,t) \in K$, $C > 0$ depends only on a_0.)

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($\forall (x,t) \in K$, $\;C > 0$ depends only on a_0.)

- This comes from the assumption : For any vector field X on $\mathbb{H}$, $|X\widetilde{\mathbf{b}}| \leq C_0(|\widetilde{\mathbf{b}}| + 1)\sqrt{g(X,X)}$.

Second step

- Want to compare
  - $\widehat{q}_K^{\tilde{A},0}(w) := \int_K \left[|e^{t_0}(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 + 1/4|w|^2\right] dxdt$,
  - $\widehat{q}_K^{A^0,0}(w) = \int_K \left[|e^{t_0}(D_x - A_1^0)w|^2 + |D_t w|^2 + 1/4|w|^2\right] dxdt$.
- Write

$$e^{t_0}(D_x - \tilde{A}_1)w = e^{t_0}(D_x - A_1^0)w - e^{t_0}(\tilde{A}_1 - A_1^0)w \ ,$$

- and use the first step to get :

For any $\tau \in]0,1[$;

$$(1-\tau^2)\ \widehat{q}_K^{A^0,0}(w) + (1 - \frac{1}{\tau^2})\ C\ \varepsilon_0^4\ \mathbf{b}^2(z_0)\|w\|^2 \leq \widehat{q}_K^{\tilde{A},0}(w)$$

$$\leq\ (1+\tau^2)\widehat{q}_K^{A^0,0}(w) + (1 + \frac{1}{\tau^2})\ C\ \varepsilon_0^4\ \mathbf{b}^2(z_0)\|w\|^2 \ .$$

Third step

It remains to compare

- $\widehat{q}_K^{\tilde{A}}(w) \ = \int_K \left[|e^t (D_x - \tilde{A}_1) w|^2 + |D_t w|^2 + 1/4 |w|^2 \right] dx dt$, and
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Proposition

$$(1 - \varepsilon_0 C)\, \widehat{q}_K^{\tilde{A},0}(w) \leq \widehat{q}_K^{\tilde{A}}(w) \leq (1 + \varepsilon_0 C)\widehat{q}_K^{\tilde{A},0}(w) \ .$$

Spectral asymptotics for a rectangle : upper bound

- **Proposition** For any real λ ,

$$N(\lambda, P_K^0(A^0)) \; \leq \; \frac{|K|\mathbf{b}(x_0, e^{t_0})}{2\pi e^{t_0}} \sum_{k=0}^{+\infty} [\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(x_0, e^{t_0})]_+^0 \; .$$

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$$N(\lambda, P_K^0(A^0)) \leq \frac{|K|\mathbf{b}(x_0, e^{t_0})}{2\pi e^{t_0}} \sum_{k=0}^{+\infty} [\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(x_0, e^{t_0})]_+^0 \; .$$

- **(Colin de Verdière):**
  - $\mathbf{b} > 0$ constant.
  - $\Omega_R =]-\frac{R_1}{2}, \frac{R_1}{2}[\times]-\frac{R_2}{2}, \frac{R_2}{2}[$, $R = (R_1, R_2) \in (\mathbb{R}_+^\star)^2$,
  - $H_{D,\mathbf{b}}^{\Omega_R}$: Dir. pb. associated to
    $H_0 = (D_x - \mathbf{b}\frac{y}{2})^2 + (D_y + \mathbf{b}\frac{x}{2})^2$
  - $$N(\lambda, H_{D,\mathbf{b}}^{\Omega_R}) \leq \frac{\mathbf{b}|\Omega_R|}{2\pi} \sum_{k=0}^{+\infty} [\lambda - (2k+1)\mathbf{b}]_+^0 \; .$$

Spectral asymptotics for a rectangle : lower bound

Spectral asymptotics for a rectangle : lower bound

The same kind of result applies.

Developpements

- Role of curvature :
  - Asymptotics in the 2-dimensional euclidian space
    $N_B^{as}(\lambda) = \frac{1}{2\pi} \int_{R^2} \mathbf{b}(x) \sum_{k \in \mathbb{N}} \left[\lambda - (2k+1)\mathbf{b}(x)\right]_+^0 dx$
  - Asymptotics in the 2-dimensional hyperbolic space
    $N(\lambda; -\Delta_A) \sim \frac{1}{2\pi} \int_{\mathbb{H}} \mathbf{b}(m) \sum_{k \in \mathbb{N}} [\lambda - \mathbf{\frac{1}{4}} - (2k+1)\mathbf{b}(m)]_+^0 \, dv$.
- 3-dimensional case
  - The magnetic field can be identified with a vector field.
  - Hypothesis on the direction should be done.(cf condition (B3) in the euclidean case)
- Application to geometrically finite hyperbolic surface of infinite area

Geometrically finite hyperbolic surface of infinite area

- Definition

  $(\mathcal{M},\ g)$: a smooth connected Riemannian manifold of dimension two $\mathcal{M} \ = \ \left(\bigcup_{j=0}^{J_1} M_j \right) \bigcup \left(\bigcup_{k=1}^{J_2} F_k \right)$;

  - M_j , F_k open sets of $\mathcal{M}$, M_0 compact closure,
  - ($j \neq 0$: M_j isometric to $\mathbb{S}\times]a_j^2, +\infty[$, (cuspidal ends)

$$ds_j^2 \ = \ y^{-2}(\ L_j^2\ d\theta^2 \ + \ dy^2\ )$$

$(a_j$ and L_j are strictly positive constants$)$

  - F_k isometric to $\mathbb{S}\times]\alpha_k^2, +\infty[$, (funnel ends)

$$ds_k^2 \ = \ \tau_k^2 \cosh^2 t\ d\theta^2 \ + \ dt^2$$

$(\alpha_k$ and τ_k are strictly positive constants$)$.

The result

- For z_0 fixed in M_0 define $d \ : \ \mathcal{M} \ \to \ \mathbb{R}_+ \ ; \quad d(z) \ = \ d_g(z, z_0) \ ;$ $d_g(\ . \ , \ . \ )$: distance w. r. to the metric g.
- (HSMB)
  - $\lim_{d(z)\to\infty} \ \mathbf{b}(z) \ = \ +\infty \ .$
  - $$|X\widetilde{\mathbf{b}}(z)| \ \leq \ C_1(\mathbf{b}(z) \ + \ 1)e^{d(z)}|X|_g \ ;$$
    $$\forall \ X \ \in \ T_z\mathcal{M} \ \forall \ z \ \in \ M_j \ , \ \text{and} \ \forall \ j = 1, \ldots J_1 \ .(C_1 \ > \ 0)$$
  - $$|X\widetilde{\mathbf{b}}(z)| \ \leq \ C_2(\mathbf{b}(z) \ + \ 1)|X|_g \ ;$$
    $$\forall \ z \ \in \ F_k \ , \ \forall \ X \ \in \ T_z\mathcal{M} \ \text{ and} \ \forall \ k = 1, \ldots J_2 \ .(C_2 \ > \ 0)$$
- Under (HSMB) assumptions the same asymptotics still hold .