Weyl Law for Magnetic Laplacian on manifolds with cusps, and counting function of the embedded eigenvalues of the Laplace operator

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Outline

- The hyperbolic setup
- The constant magnetic Laplacian on the Poincaré half-plane
- Geometrically finite hyperbolic surfaces
  - The spectrum of the magnetic Laplacian
  - The Weyl formula (case of finite area)
- Higher dimensions: manifolds with cusps
  - The spectrum of the magnetic Laplacian
  - The Weyl formula with sharp remainder
  - Estimate on the embedded values of the Laplacian
  - Outline of proofs

Joint work with A. Morame (Nantes)

Framework

- Let $(\mathcal{M}, g)$ be a connected and oriented Riemannian manifold of dimension n.
- For any real one-form A on $\mathcal{M}$, define

$$-\Delta_A \; = \; (i\,d{+}A)^\star(i\,d{+}A)\;,\;((i\;d+A)u = i\;du + uA\;,\;\forall\;u\;\in\;C_0^\infty(\mathcal{M}))\;\;.$$

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- The magnetic field is the two-form dA .
- To dA is associated the linear operator B defined on the tangent space by $dA(X, Y) \;=\; g(B.X, Y)\;;\quad \forall\;X\;,\;Y\;\in T\mathcal{M}\times T\mathcal{M}$.
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- The magnetic intensity $\mathbf{b}$ is given by $\mathbf{b} = \frac{1}{2} tr\left((B^\star B)^{1/2}\right)$.
- If $dim(\mathcal{M}) = 2$, then
  - $dA = \widetilde{\mathbf{b}}\,dv$, with $|\widetilde{\mathbf{b}}| = \mathbf{b}$,
    dv the Riemannian measure on $\mathcal{M}$.
  - The magnetic field is constant $\iff$ $\widetilde{\mathbf{b}}$ is constant.

The Poincaré half-plane

Let $\mathcal{M} = \mathbb{H}$ be the hyperbolic half-plane :

$$\mathbb{H} = \mathbb{R}\times]0,+\infty[\,, \quad g = \frac{dx^2+dy^2}{y^2}\,.$$

- $-\Delta_A = y^2(D_x - A_1)^2 + y^2(D_y - A_2)^2$,
  with $A = A_1(x,y)\,dx + A_2(x,y)\,dy$, and $A_j(x,y) \in C^\infty(\mathbb{H};\mathbb{R})$,

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Properties

- $-\Delta_A$ is essentially self-adjoint on $L^2(\mathbb{H})$.
- We are interested on its spectrum : $\mathrm{sp}(-\Delta_A)$.
  Gauge invariance:

$$\mathrm{sp}(-\Delta_A) = \mathrm{sp}(-\Delta_{A+d\varphi})\,; \quad \forall\,\varphi \in C^\infty(\mathbb{H};\mathbb{R})\,.$$

The spectrum for constant magnetic field

- The spectrum of $-\Delta_{A^{\mathbf{b}}}$ is essential:
  $\mathrm{sp}(-\Delta_{A^{\mathbf{b}}}) = \mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \bigcup S(\mathbf{b})$.
- Its absolutely continuous part is given by
  $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \;=\; [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.

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  $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) = [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.
- If $0 \leq \mathbf{b} \leq 1/2$ the remaining part $S(\mathbf{b})$ is empty,
- if $\mathbf{b} > 1/2$ it is formed by a finite number of eigenvalues of infinite multiplicity given by

$$S(\mathbf{b}) = \{(2j+1)\mathbf{b} - j(j+1) \; ; \; j \in \mathbb{N} \, , \; j < \mathbf{b} - \frac{1}{2}\} \, .$$

Related results : hyperbolic context

- Results on the hyperbolic space
  - Maass Laplacian: Elstrodt(73), Grosche(88), Comtet(87), Ikeda-Matsumoto (99), Kim-Lee(02)
  - Asympt. constant magnetic fields ,Pauli operators : Inahama-Shirai (03)
  - Asymptotic distribution for Schrödinger operators : Inahama-Shirai (04)
  - Asymptotic distribution for magnetic bottles : Morame-Truc (08)
- conformally cusp manifolds :
  Asymptotic distribution for Schrödinger operators :
  Golénia-Moroianu (08)
  Weyl law for magnetic Laplacian : Golénia-Moroianu (08)
- Geometrically finite hyperbolic surfaces
  Asymptotic distribution for magnetic bottles : Morame-Truc (09)

Geometrically finite hyperbolic surface of infinite area

- Definition

  $(\mathcal{M},\ g)$: a smooth connected Riemannian manifold of dimension two $\mathcal{M} = \left(\bigcup_{j=0}^{J_1} M_j \right) \bigcup \left(\bigcup_{k=1}^{J_2} F_k \right)$;

  - M_j , F_k open sets of $\mathcal{M}$, M_0 compact closure,
  - ($j \neq 0$: M_j isometric to $\mathbb{S} \times]a_j^2, +\infty[$, (cuspidal ends)

$$ds_j^2 = y^{-2}(L_j^2 \, d\theta^2 + dy^2)$$

$(a_j$ and L_j are strictly positive constants)

  - F_k isometric to $\mathbb{S} \times]\alpha_k^2, +\infty[$, (funnel ends)

$$ds_k^2 = \tau_k^2 \cosh^2 t \, d\theta^2 + dt^2$$

$(\alpha_k$ and τ_k are strictly positive constants) .

"Constant" magnetic field on $\mathcal{M}$

- $$\mathcal{M} = \left(\bigcup_{j=0}^{J_1} M_j \right) \bigcup \left(\bigcup_{k=1}^{J_2} F_k \right) ;$$
- It is not always possible to have a constant magnetic field on $\mathcal{M}$, but

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- It is not always possible to have a constant magnetic field on $\mathcal{M}$, but
- $\forall (b, \beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2}$, $\exists\, A$, s. t. dA satisfies

$$dA = \widetilde{\mathbf{b}}(z) dm \quad \begin{cases} \widetilde{\mathbf{b}}(z) = b_j \;\forall\; z \in M_j \\ \widetilde{\mathbf{b}}(z) = \beta_k \;\forall\; z \in F_k \end{cases}$$

The essential spectrum of magnetic Laplacians (1)

Theorem 1

- If $J_1 = 0$ and $J_2 > 0$,

$$\mathrm{sp}_{ess}(-\Delta_A) = [\frac{1}{4} + \inf_k \beta_k^2 \, , \, +\infty[\bigcup \left(\bigcup_{k=1}^{J_2} S(\beta_k) \right) \, .$$

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- if $\beta_k > 1/2$ it is formed by a finite number of eigenvalues of infinite multiplicity given by

$$S(\beta_k) = \{(2j+1)\beta_k - j(j+1) \; ; \; j \in \mathbb{N} \, , \, j < \beta_k - \frac{1}{2}\} .$$

The essential spectrum of magnetic Laplacians (2)

If $J_1 > 0$, then

- $\forall j\ ,\ 1 \leq j \leq J_1$ and $\forall\ z \in M_j$
  $\exists$ a unique closed curve $\mathcal{C}_{j,z}$ through z , in $(M_j,\ g)$, not contractible and with zero $g-$curvature.
- The following limit exists and is finite:

$$[\mathbf{A}]_{\mathbf{M_j}} = \lim_{d(z)\to+\infty} \int_{\mathcal{C}_{j,z}} A\ .$$

- Define : $J_1^A = \{j \in \mathbb{N}\ ,\ 1 \leq j \leq J_1\ s.t.\ [\mathbf{A}]_{\mathbf{M_j}} \in 2\pi\mathbb{Z}\ \}$.

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Theorem 2

Assume that $J_1 > 0$, $J_2 > 0$, and $J_1^A \neq \emptyset$. Then

- $\mathrm{sp}_{ess}(-\Delta_A) =$
  $$[\frac{1}{4} + \min\{ \inf_{j \in J_1^A} b_j^2 \,,\, \inf_{1 \leq k \leq J_2} \beta_k^2 \, \} \,,\, +\infty[\bigcup \left(\bigcup_{k=1}^{J_2} S(\beta_k) \right) \,.$$

Weyl law ($\mathcal{M}$ of finite area, A "non exact at ∞")

- Theorem 3

  Consider a geometrically finite hyperbolic surface $(\mathcal{M},\ g)$ of finite area, $(J_2 = 0)$, and assume that $J_1^A\ =\ \emptyset$. Then

  - (i) $\mathrm{sp}_{ess}(-\Delta_A)\ =\ \emptyset$:

    $-\Delta_A$ has purely discrete spectrum , (its resolvent is compact).

  - (ii) $N(\lambda, -\Delta_A)\ =\ \lambda\dfrac{|\mathcal{M}|}{4\pi}\ +\ \mathbf{O}(\sqrt{\lambda}\ln\lambda)$.

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- (Definitions) In the case when $\mathrm{sp}(-\Delta_A)$ is discrete,
  - denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue repeated according to its multiplicity)
  - define $N(\lambda, -\Delta_A)\ =\ \sum_{\lambda_j<\lambda} 1$.
  - Weyl law for the Laplacian on a compact n-dim Riemannian manifold M

$$N(\lambda, -\Delta_0^M)\ =\ \lambda^{n/2}\frac{|M|}{(4\pi)^{n/2}\Gamma(n/2+1)}\ +\ o(\lambda^{n/2})\ .$$

Remarks (1)

- Theorem 3 is a consequence of the following
  Proposition (Weyl law for one cusp)
  - Consider $M = \mathbb{S}\times]\alpha^2, +\infty[$ equipped with the metric $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ for some $\alpha > 0$ and $L > 0$.
  - Denote by $-\Delta_A^M$ the Dirichlet op. on M , ass. to $-\Delta_A$.
  - $$N(\lambda, -\Delta_A^M) = \lambda \frac{|M|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda) .$$
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- Stability under perturbation of the metric
  Theorem 3 still holds if the metric of $\mathcal{M}$ is modified in a compact set.
- One can see the case $J^A = \emptyset$ as an Aharonov-Bohm phenomenon :
  - the magnetic field dA is not sufficient to describe $-\Delta_A$
  - the use of the magnetic potential A is essential :
    one can have magnetic bottle (magnetic Laplacian with compact resolvent) with null intensity.

Remarks (2)

- When $A = 0$, $-\Delta = -\Delta_0$ has embedded eigenvalues in its essential spectrum : $(\mathrm{sp}_{ess}(-\Delta) = [\frac{1}{4}, +\infty[)$.
  - Denote by $N_{ess}(\lambda, -\Delta)$ the number of these eigenvalues in $[\frac{1}{4}, \lambda[$, then
  - $N_{ess}(\lambda, -\Delta) \ \leq \ \lambda \frac{|\mathcal{M}|}{4\pi}$; (Colin de Verdière, Hejhal)

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  - $N_{ess}(\lambda, -\Delta) \; \leq \; \lambda \frac{|\mathcal{M}|}{4\pi}$; (Colin de Verdière, Hejhal)
- Locally symmetric spaces and automorphic forms
  Consider $\mathcal{M} = \Gamma(N) \backslash \mathbb{H}$, with

$$\Gamma(N) = \{\gamma \in SL(2,\mathbb{Z}) : \gamma = Id \bmod N\}$$

then (Müller '07)

$$N_{ess}(\lambda, -\Delta) \; = \; \lambda \frac{|\mathcal{M}|}{4\pi} \; + \; \mathbf{O}(\sqrt{\lambda} \ln \lambda) \; .$$

Remarks (3)

- Definition

  A Maass automorphic form is a smooth function $\mathbb{H} \to \mathbb{C}$ s.t.

  - $f(\gamma z) = f(z) \quad \forall \gamma \in \Gamma(N)$
  - $\exists \lambda$ s.t. $\Delta f = \lambda f$
  - f is slowly increasing

- Exemples : Eisenstein series, cusp forms

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  - f is slowly increasing
- Exemples : Eisenstein series, cusp forms
- Selberg formula (derived from the trace formula)
  For any lattice $\Gamma \in SL(2, \mathbb{R})$:

$$N_\Gamma(\lambda, -\Delta) + M_\Gamma(\lambda, -\Delta) \sim \lambda \frac{|\Gamma \backslash \mathbb{H}|}{4\pi} .$$

  - $M_\Gamma(\lambda, -\Delta)$: winding number of the determinant of the scattering matrix (given by the zeroth Fourier coefficients of the Eisenstein series)

Higher dimensions

$(\mathcal{M},\ g)$: a smooth connected n-dim. Riemannian manifold

$\mathcal{M} \ = \ M_0 \cup \left(\bigcup_{j=1}^{J} M_j \right)$;

- $M_j \cap M_k = \emptyset$,
- M_j open sets of $\mathcal{M}$, M_0 compact closure,
- $\forall j > 1, \exists$ a closed compact $(n-1)$-dim. Riem. manifold $(X_j, \mathbf{h}_j)$ s.t. M_j is isometric to $X_j \times]a_j^2, +\infty[$, $(a_j > 0)$ (M_j $j \neq 0$ cuspidal ends)

$$ds_j^2 \ = \ y^{-2\delta_j}(\mathbf{h}_j \ + \ dy^2 \ ) \ ; \quad (1/n < \delta_j \leq 1)$$

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$$ds_j^2 \ = \ y^{-2\delta_j}(\mathbf{h}_j \ + \ dy^2 \ ) \ ; \quad (1/n < \delta_j \leq 1)$$

$\Longrightarrow \forall j > 1, \exists$ a smooth real one-form $A_j \in T^\star(X_j)$, s.t.

- $dA_j \neq 0$ or
- $dA_j = 0$ and $[A_j]$ is not integer

($\exists$ a smooth closed curve γ in X_j such that $\int_\gamma A_j \ \notin \ 2\pi\mathbb{Z}$.)

- One can find $A \in T^\star(\mathcal{M})$ s.t.
  $\forall\, j,\ 1 \leq j \leq J, \quad A = A_j \quad \text{on } M_j$.
- Define the magnetic Laplacian,
  $-\Delta_A = (i\,d + A)^\star (i\,d + A)$,
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- $\mathcal{M}$ complete metric space
  $\Longrightarrow$ by Hopf-Rinow theorem $\mathcal{M}$ is geodesically complete
  $\Longrightarrow -\Delta_A$ has a unique self-adjoint extension on $L^2(\mathcal{M})$.

The essential spectrum

- Theorem 4 (The case $A = 0$)
  Under the above assumptions on $\mathcal{M}$,
  the essential spectrum of $-\Delta = -\Delta_0$ on $\mathcal{M}$ is given by
  - $\mathrm{sp}_{\mathrm{ess}}(-\Delta) = [0, +\infty[\quad \text{if} \quad 1/n < \delta < 1$
  - $\mathrm{sp}_{\mathrm{ess}}(-\Delta) = [\frac{(n-1)^2}{4}, +\infty[\quad \text{if} \quad \delta = 1 \; .$
  - $\delta = \min_{1 \leq j \leq J} \delta_j \; .$

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  - $\delta = \min_{1 \leq j \leq J} \delta_j\ .$

- Theorem 5 (The case $A \neq 0$)
  Under the above assumptions on $\mathcal{M}$ and A,
  - the magnetic Laplacian $-\Delta_A$ has a compact resolvent.
  - $\mathrm{sp}(-\Delta_A) = \mathrm{sp}_d(-\Delta_A) = \{\lambda_j, j \in \mathbb{N}^\star\}$ with
    - $\lambda_j \leq \lambda_{j+1}$
    - $\lim_{j \to +\infty} \lambda_j = +\infty,$
    - $\lambda_0 > 0.$
  - the sequence of normalized eigenfunctions $(\varphi_j)_{j \in \mathbb{N}^\star}$ is a Hilbert basis of $L^2(\mathcal{M})$.

Weyl formula

- Theorem 6 (The case $A \neq 0$)
  Under the above assumptions on $\mathcal{M}$ and on A,
  we have the Weyl formula with remainder as $\lambda \to +\infty$

$$N(\lambda, -\Delta_A) \; = \; |\mathcal{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} \; + \; \mathbf{O}(r(\lambda)) \; ,$$

with $r(\lambda) \; = \begin{cases} \lambda^{(n-1)/2} \ln(\lambda), & \text{if} \quad 1/(n-1) \leq \delta \\ \lambda^{1/(2\delta)}, & \text{if} \quad 1/n < \delta < 1/(n-1) \end{cases}$

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- $\delta = \min_{1 \leq j \leq J} \delta_j$,
- $|\mathcal{M}|$ is the Riemannian measure of $\mathcal{M}$
- ω_d the euclidian volume of the unit ball of $\mathbb{R}^d$: $\omega_d = \dfrac{\pi^{d/2}}{\Gamma(1 + \frac{d}{2})}$.

- Remarks
  - The asymptotic formula is given without remainder by Golenia -Moroianu (in a larger context),
  - Theorem 3 is a particular case of Theorem 6 (for $n = 2$ and $\delta_j = 1$ for any $1 \leq j \leq J$) .

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  - The asymptotic formula is given without remainder by Golenia -Moroianu (in a larger context),
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- Consequences
  - The Laplace-Beltrami operator $-\Delta = -\Delta_0$ may have embedded eigenvalues in its essential spectrum $\mathrm{sp}_{\mathrm{ess}}(-\Delta)$.
  - Denote by $N_{\mathrm{ess}}(\lambda, -\Delta)$ the number of eigenvalues of $-\Delta$, (counted according to their multiplicity), less then λ.
  - Theorem 6 can be mimicked to get an upperbound for $N_{\mathrm{ess}}(\lambda, -\Delta)$.

Theorem 7

There exists a constant $C_{\mathcal{M}}$ such that, for any $\lambda >> 1$,

$$N_{\text{ess}}(\lambda, -\Delta) \;\leq\; |\mathcal{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} \;+\; C_{\mathcal{M}} r_0(\lambda) \qquad (*),$$

with $r_0(\lambda)$ defined by $r_0(\lambda) \;=\; \begin{cases} \lambda^{\frac{n-1}{2}} \ln(\lambda), & \text{if} \quad 2/n \leq \delta \leq 1 \\ \lambda^{\frac{n-(n\delta-1)}{2}}, & \text{if} \quad 1/n < \delta < 2/n \end{cases}$;

δ defined as in Theorem 6 .

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- Remarks
  - $(*) \Longrightarrow$ any eigenvalue of $-\Delta$ has finite multiplicity.
  - $(*)$ is sharp when $n = 2$. (Müller '07: $\mathcal{M} = \Gamma(N) \backslash \mathbb{H}$, with $\Gamma(N) = \{\gamma \in SL(2, \mathbb{Z}) : \gamma = Id \bmod N\}$)
  - when $n = 2$, a compact perturbation of the metric can destroy all embedded eigenvalues (Colin de Verdière '83).

Proof of Theorems 4-5

- $\mathrm{sp}_{\mathrm{ess}}(-\Delta_A) = \bigcup_{j=1}^{J} \mathrm{sp}_{\mathrm{ess}}(-\Delta_A^{M_j,D})$ (The essential spectrum of an elliptic operator on a manifold is invariant by compact perturbation)
  - $-\Delta_A^{M_j,D}$: the self-adjoint operator on $L^2(M_j)$ associated to $-\Delta_A$ with Dirichlet boundary conditions on the boundary ∂M_j of M_j.

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- $\mathrm{sp}_{\mathrm{ess}}(-\Delta_A) \ = \ \bigcup_{j=1}^{J} \mathrm{sp}_{\mathrm{ess}}(-\Delta_A^{M_j,D})$ (The essential spectrum of an elliptic operator on a manifold is invariant by compact perturbation)
  - $-\Delta_A^{M_j,D}$: the self-adjoint operator on $L^2(M_j)$ associated to $-\Delta_A$ with Dirichlet boundary conditions on the boundary ∂M_j of M_j.

- Consider a cusp $M_j = X_j \times]a_j^2, +\infty[$ equipped with the metric $ds_j^2 \ = \ y^{-2\delta_j}(\mathbf{h}_j \ + \ dy^2 \ ) \ ; \quad (1/n < \delta_j \leq 1) \quad .$
  - Then for any $u \in C^2(M_j)$,
    $-\Delta_A^{M_j,D} u \ = \ -y^{2\delta_j} \Delta_{A_j}^{X_j} u \ - \ y^{n\delta_j} \partial_y (y^{(2-n)\delta_j} \partial_y u) \ ,$
  - $\Delta_{A_j}^{X_j}$ is the magnetic Laplacian on X_j :
  - if for local coordinates $\mathbf{h}_j = \sum_{k,\ell} G_{k\ell} \ dx_k dx_\ell$ and $A_j = \sum_{k=1}^{n-1} a_{j,k} \ dx_k$, then
    $-\Delta_{A_j}^{X_j} \ = \ \frac{1}{\sqrt{det(G)}} \sum_{k,\ell} (i\partial_{x_k} + a_{j,k}) \left(\sqrt{det(G)} G^{k\ell} (i\partial_{x_\ell} + a_{j,\ell}) \right) \ .$

- Perform the change of variables $y = e^t$,
- define the unitary operator $U : L^2(X_j \times]2\ln(a_j), +\infty[) \rightarrow L^2(M_j)$
  $U(f) := y^{(n\delta_j - 1)/2} f$.
- $-U^\star \Delta_A^{M_j,D} U f =$
  $-e^{2\delta_j t} \Delta_{A_j}^{X_j} f + \frac{(n\delta_j - 1)[3 + \delta_j(n-4)]}{4} e^{2t(\delta_j - 1)} f - \partial_t (e^{2t(\delta_j - 1)} \partial_t f)$.

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- $(\mu_\ell(j))_{\ell \in \mathbb{N}}$: the increasing sequence of eigenvalues of $-\Delta_{A_j}^{X_j}$
- $\mathrm{sp}(-\Delta_A^{M_j,D}) \ = \ \mathrm{sp}(\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D) \ ,$
- $L_{j,\ell}^D$: the Dirichlet operator on $L^2(]2\ln(a_j), +\infty[)$ associated to
  $$L_{j,\ell} \ = \ e^{2\delta_j t} \mu_\ell(j) \ + \ \frac{(n\delta_j - 1)}{4} [3 + \delta_j(n-4)] e^{2t(\delta_j - 1)} \ - \ \partial_t (e^{2t(\delta_j - 1)} \partial_t) \ .$$

- If $\mu_\ell(j) = 0$ then $\mathrm{sp}(L^D_{j,\ell}) = \mathrm{sp}_{\mathrm{ess}}(L^D_{j,\ell}) = [\alpha_n, +\infty[$,
  - $\alpha_n = 0$ if $\delta_j < 1$,
  - $\alpha_n = (n-1)^2/4$ if $\delta_j = 1$.
- $A = 0, \Longrightarrow \mu_0(j) = 0 \Longrightarrow \mathrm{sp}_{\mathrm{ess}}(-\Delta_0) = [\alpha_n, +\infty[$.

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- $A = 0, \Longrightarrow \mu_0(j) = 0 \Longrightarrow \mathrm{sp}_{\mathrm{ess}}(-\Delta_0) = [\alpha_n, +\infty[$.

- If $\mu_\ell(j) > 0$ then $\mathrm{sp}(L^D_{j,\ell}) = \mathrm{sp}_d(L^D_{j,\ell}) = \{\mu_{\ell,k}(j);\ k \in \mathbb{N}\}$,
  - $(\mu_{\ell,k}(j))_{k\in\mathbb{N}}$ the increasing sequence of eigenvalues of $L^D_{j,\ell}$, $\lim_{k\to+\infty} \mu_{\ell,k}(j) = +\infty$.
- If A satisfies the previous assumptions, then $0 < \mu_0(j) \leq \mu_\ell(j)$ for all j and ℓ, , $\Longrightarrow$
  - $\mathrm{sp}(-\Delta^{M_j,D}_A) = \{\mu_{\ell,k}(j);\ (\ell,k) \in \mathbb{N}^2\}$.
  - $\lim_{\ell\to+\infty} \mu_{\ell,k}(j) = +\infty, \Longrightarrow$ each $\mu_{\ell,k}(j)$ is an eigenvalue of $-\Delta^{M_j,D}_A$ of finite multiplicity,
    $\Longrightarrow \mathrm{sp}(-\Delta^{M_j,D}_A) = \mathrm{sp}_d(-\Delta^{M_j,D}_A) \Longrightarrow \mathrm{sp}_{\mathrm{ess}}(-\Delta_A) = \emptyset \ \square$

Proof of Theorem 6

$\Longrightarrow$ wanted: Weyl formula for M_j, with $-\Delta_A^{M_j,D}$ instead of $-\Delta_A$.

- $\delta_j = 1 \Longrightarrow$ same change of variables and functions as before
- $1/n < \delta_j < 1$, $\Longrightarrow$ set $y = [(1-\delta_j)t]^{1/(1-\delta_j)}$, and define the unitary operator

  $U : L^2(X_j \times]\alpha_j, +\infty[) \to L^2(M_j)$, ($\alpha_j = \frac{a_j^{2(1-\delta_j)}}{1-\delta_j}$)

  $U(f) = y^{(n-1)\delta_j/2} f$.

  - $\mathrm{sp}(-\Delta_A^{M_j,D}) = \mathrm{sp}(\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D)$,

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  - $\mathrm{sp}(-\Delta_A^{M_j,D}) = \mathrm{sp}(\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D)$,

- $L_{j,\ell}^D$: the Dirichlet operator on $L^2(]\frac{a^{2(1-\delta_j)}}{1-\delta_j}, +\infty[)$ associated to

$$L_{j,\ell} = V_{j,\ell} - \partial_t^2 .$$

$$V_{j,\ell}(t) = \mu_\ell(j)[(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} + \frac{(n-1)\delta_j[(n-3)\delta_j + 2]}{4(1-\delta_j)^2 t^2}$$

- Titchmarsh's method $\Longrightarrow$ there exists $C > 1$ so that for any $\lambda >> 1$ and any $\ell \in K_\lambda = \{l \in \mathbb{N};\ \mu_\ell(j) \in [0, \lambda/a_j^2[\}$,
$$|N(\lambda, L^D_{j,\ell})\ -\ \frac{1}{\pi} w_{j,\ell}(\lambda)|\ \leq\ C \ln(\lambda)$$, with
$$w_{j,\ell}(\mu)\ =\ \int_{\alpha_j}^{+\infty} [\mu - V_{j,\ell}(t)]_+^{1/2}\, dt$$

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$$w_{j,\ell}(\mu) \ = \ \int_{\alpha_j}^{+\infty} [\mu - V_{j,\ell}(t)]_+^{1/2}\, dt$$
- Use
  - the sharp asymptotic Weyl formula of L. Hörmander (SAWFH)
  - + the fact that $N(\lambda, L_{j,\ell}^D) = 0$ when $\ell \ \notin \ K_\lambda$,
  - + the formula $N(\lambda, -\Delta_A^{M_j,D}) \ = \ \sum_{\ell=0}^{+\infty} N(\lambda, L_{j,\ell}^D)$
- to get $|N(\lambda, -\Delta_A^{M_j,D}) \ - \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda)| \ \leq \ C \lambda^{(n-1)/2} \ln(\lambda)$.
- (SAWFH) There exists $C > 0$ so that for any $\mu >> 1$
$$|N(\mu, -\Delta_{A_j}^{\mathbf{X}_j}) \ - \ \frac{\omega_{n-1}}{(2\pi)^{n-1}} |\mathbf{X}_j| \mu^{(n-1)/2}| \ \leq \ C \mu^{(n-2)/2}$$.

Estimate on $\Theta_j(\lambda) = \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda)$

- Write $V_{j,\ell}(t) = \mu_\ell(j) V_j(t) + W_j(t)$
- $\Longrightarrow \Theta_j(\lambda) = \frac{1}{\pi} \int_{\alpha_j}^{T_j(\lambda)} V_j^{1/2}(t) R_j(\frac{\lambda - W_j(t)}{V_j(t)}) dt$. with
- $R_j(\mu) = \sum_{\ell=0}^{+\infty} [\mu - \mu_\ell(j)]_+^{1/2}$.

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- $R_j(\mu) = \sum_{\ell=0}^{+\infty} [\mu - \mu_\ell(j)]_+^{1/2}$.

- Use then
  - (SAWFH) and
  - $R_j(\mu) = \frac{1}{2}\int_0^{+\infty} [\mu - s]_+^{-1/2} N(s, -\Delta_{A_j}^{X_j}) ds,$

  to get that
- There exists a constant $C > 0$ such that, for any $\mu >> 1$,
  $$|R_j(\mu) - \frac{\omega_{n-1}}{2(2\pi)^{n-1}} |X_j| \int_0^{+\infty} [\mu - s]_+^{-1/2} s^{(n-1)/2} ds| \leq C\mu^{(n-1)/2} .$$
- and use the definitions of V_j and W_j to conclude.

Proof of Theorem 7

- Proposition 1 When A satisfies the previous assumptions (A) and if A_j verifies property (P) for any $j \in \{1, \ldots, J\}$, , then as $\lambda \to +\infty$,

$$N(\lambda, -\Delta_{(\lambda^{-\rho} A)}) \; = \; |\mathcal{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} \; + \; \mathbf{O}(r_0(\lambda)) \; ,$$

with $\rho \; = \begin{cases} 1/2, & \text{if} \quad 2/n \leq \delta \leq 1 \\ (n\delta - 1)/2, & \text{if} \quad 1/n < \delta < 2/n \end{cases}$

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- Property (P)
  There exists $\tau_0 = \tau_0(A_j) > 0$ and $C = C(A_j) > 0$ such that, if $e(\tau, j)$ denotes the first eigenvalue of $-\Delta^{X_j}_{\tau A_j}$, then
  $e(\tau, j) \;\geq\; C\tau^2 \;; \quad \forall\, \tau \in]0, \tau_0] \;.$

- Proposition 2 For any $j \in \{1, \ldots, J\}$, there exists a one-form A_j verifying assumptions (A) and property (P).

Proof of Proposition 1

- A satisfies (P) $\Longrightarrow C/\lambda^{2\rho} \leq \mu_0(j) \quad \text{and} \quad C \leq \mu_1(j)$.
  - $(\mu_\ell(j))_{\ell\in\mathbb{N}}$: the increasing sequence of eigenvalues of $-\Delta^{X_j}_{\lambda^{-\rho}A_j}$.
- mimick the proof of Theorem 6.
  - Titchmarsh's method holds for any $\ell \in K_\lambda, \ell \neq 0$
  - it remains to prove that $N(\lambda, L^D_{j,0}) \ = \ \mathbf{O}(r_0(\lambda))$.

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  - it remains to prove that $N(\lambda, L^D_{j,0}) \; = \; \mathbf{O}(r_0(\lambda))$.
- $\delta_j = 1$ $(\rho = 1/2) \Longrightarrow N(\lambda, L^D_{j,0}) \; \leq \; N(\lambda + C, L^{D,\lambda}) \; \leq \; C\lambda^{1/2}\ln(\lambda)$, where $L^{D,\lambda}$ is the Dirichlet operator on $]0, +\infty[$ associated to $\dfrac{C}{\lambda}e^{2t} - \partial_t^2$.
- $0 < \delta_j < 1 \Longrightarrow$ $N(\lambda, L^D_{j,0}) \; \leq \; N((\lambda + C)^{1+2\rho(1-\delta_j)}, L^D) \; \leq \; C\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)}$, where L^D is the Dirichlet operator on $]0, +\infty[$ associated to $\dfrac{1}{C^2}t^{\frac{2\delta_j}{1-\delta_j}} - \partial_t^2$.

End of proof of Theorem 7

- Take a one-form A satisfying the assumptions (A) and (P).
- any eigenfunction u of $-\Delta$ on $\mathcal{M}$ associated to an eigenvalue in $]\inf \mathrm{sp}_{\mathrm{ess}}(-\Delta), +\infty[$, verifies
$$\forall j = 1, \ldots, J, \quad \int_{X_j} u(x_j, y)dx_j = 0 \,, \quad \forall y \in]a_j^2, +\infty[\,,$$
- this implies that
  - $$\|u\|^2_{L^2(X_j)} \leq \frac{1}{\mu_1(j)} \|idu\|^2_{L^2(X_j)} \,, \Longrightarrow$$
  - $$\|idu + \tau uA\|^2_{L^2(\mathcal{M})} \leq (1 + \tau C_A)\|idu\|^2_{L^2(\mathcal{M})} + C_A\|u\|^2_{L^2(\mathcal{M})} \,.$$
- Denote by H_λ the subspace of $L^2(\mathcal{M})$ spanned by eigenfunctions of $-\Delta$ associated to eigenvalues in $]0, +\lambda[$. Do $\tau = 1/\lambda^\rho$,

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- Denote by H_λ the subspace of $L^2(\mathcal{M})$ spanned by eigenfunctions of $-\Delta$ associated to eigenvalues in $]0, +\lambda[$. Do $\tau = 1/\lambda^\rho$,
- $\forall u \in H_\lambda, \quad \|idu + \frac{1}{\lambda^\rho} u A\|^2_{L^2(\mathcal{M})} \; \leq \; (1 + \frac{C_A}{\lambda^\rho})\|du\|^2_{L^2(\mathcal{M})} + C_A \|u\|^2_{L^2(\mathcal{M})}$
$\Longrightarrow$
$$dim(H_\lambda) \; \leq \; N((1 + \frac{C_A}{\lambda^\rho})\lambda + C_A, -\Delta_{(\lambda^{-\rho} A)}) \,.$$
- finally notice that $\lambda^{n/2}/\lambda^\rho = \mathbf{O}(r_0(\lambda))$

Proof of Theorems 1,2, 3(i)

- $$\mathrm{sp}_{ess}(-\Delta_A) = \left(\bigcup_{j=1}^{J_1} \mathrm{sp}_{ess}(-\Delta_A^{M_j}) \right) \bigcup \left(\bigcup_{k=1}^{J_2} \mathrm{sp}_{ess}(-\Delta_A^{F_k}) \right) \ ;$$

Proof of Theorems 1,2, 3(i)

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- Lemma 1

$$\mathrm{sp}_{ess}(-\Delta_A^{F_k}) = [\frac{1}{4} + \beta_k^2 , +\infty[\bigcup \left(\bigcup_{k=1}^{J_2} S(\beta_k)\right) .$$

- Lemma 2

  If $1 \le j \le J_1$ and $j \notin J_1^A$, then

$$\mathrm{sp}_{ess}(-\Delta_A^{M_j}) = \emptyset .$$

  If $j \in J_1^A$, then

$$\mathrm{sp}_{ess}(-\Delta_A^{M_j}) = [\frac{1}{4} + b_j^2 , +\infty[.$$

Proof of Lemma 2 (1)

- M_j isometric to $\mathbb{S}\times]a_j^2,+\infty[$, (cuspidal ends)
  $ds_j^2 \;=\; y^{-2}(\,L_j^2\,d\theta^2\;+\;dy^2\,) \quad (a_j > 0 \text{ and } L_j > 0\,)$
- Use the coordinate $t = \ln y$ instead of y , so
  - $M_j \;=\; \mathbb{S}\times]\alpha_j^2,+\infty[\quad \text{and} \quad ds_j^2 \;=\; L_j^2 e^{-2t} d\theta^2 \;+\; dt^2$;
    $(\alpha_j = e^{a_j})$.

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    $(\alpha_j = e^{a_j})$.

-
  - $-\Delta_A^{M_j} = L_j^{-2} e^{2t}(D_\theta - A_1)^2 + e^t(D_t - A_2)(e^{-t}(D_t - A_2))$,
  - $\widetilde{\mathbf{b}} = b_j = L_j^{-1} e^t(\partial_\theta A_2 - \partial_t A_1)$ and $dm = L_j e^{-t} d\theta dt$.

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  $ds_j^2 \;=\; y^{-2}(\; L_j^2\; d\theta^2 \;+\; dy^2\;) \quad (a_j > 0 \text{ and } L_j > 0\;)$
- Use the coordinate $t = \ln y$ instead of y , so
  - $M_j \;=\; \mathbb{S}\times]\alpha_j^2,+\infty[\quad \text{and} \quad ds_j^2 \;=\; L_j^2 e^{-2t} d\theta^2 \;+\; dt^2$;
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-
  - $-\Delta_A^{M_j} \;=\; L_j^{-2} e^{2t}(D_\theta - A_1)^2 + e^t(D_t - A_2)(e^{-t}(D_t - A_2))$,
  - $\widetilde{\mathbf{b}} = b_j \;=\; L_j^{-1} e^t(\partial_\theta A_2 - \partial_t A_1)$ and $dm = L_j e^{-t} d\theta dt$.
- we have

$$A - \widetilde{A} \;=\; d\varphi \text{ if } \widetilde{A} = (\xi + L_j b_j e^{-t})d\theta\;,\;\; (\text{for some constant } \xi)\;.$$

- $\Longrightarrow$ we can assume that $A = \widetilde{A}$.

Proof of Lemma 2 (2)

- Define $Uf = \sqrt{L_j} e^{-t/2} f$,
- $\Longrightarrow P = -U \Delta_A^{M_j} U^\star = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}$.

Proof of Lemma 2 (2)

- Define $Uf = \sqrt{L_j} e^{-t/2} f$,
- $\Longrightarrow P = -U\Delta_A^{M_j} U^\star = L_j^{-2} e^{2t}(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}$.
- $\Longrightarrow \mathrm{sp}(-\Delta_A^{M_j}) = \mathrm{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \mathrm{sp}(P_\ell)$

$$P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L_j} + b_j \right)^2 ,$$

for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_j^2, +\infty[$.

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- $\Longrightarrow \mathrm{sp}(-\Delta_A^{M_j}) = \mathrm{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \mathrm{sp}(P_\ell)$

  $$P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L_j} + b_j \right)^2 ,$$

  for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_j^2, +\infty[$.
- When $\ell + \xi \neq 0$, the spectrum of P_ℓ is discrete.
- More precisely : $\mathrm{sp}(P_\ell) \;=\; \mathrm{sp}(P^\pm)$ $\qquad P^\pm = D_t^2 + \dfrac{1}{4} + (\pm e^t + b_j)^2$

  for the Dirichlet condition on

  $L^2(I_{j,\ell}; dt)$; $I_{j,\ell} =]\alpha_j^2 + \ln(|\ell + \xi|/L_j), +\infty[$, and $\pm = \dfrac{\ell + \xi}{|\ell + \xi|}$.

Proof of Lemma 2 (3)

- $\Longrightarrow$ if $\ell + \xi \neq 0 \; \forall \ell \in \mathbb{Z}$, $\lim_{|\ell| \to \infty} \inf \mathrm{sp}(P_\ell) \; = \; +\infty$,
- $\Longrightarrow$ the spectrum of $-\Delta_A^{M_j}$ is discrete.

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- $\Longrightarrow$ the spectrum of $-\Delta_A^{M_j}$ is discrete.

- What means this condition ? Recall
  - $A = (\xi + L_j b_j e^{-t}) d\theta$,
  - $[\mathbf{A}]_{\mathbf{M_j}} \ = \ \lim_{d(z) \to +\infty} \int_{\mathcal{C}_{j,z}} A$

    $\Longrightarrow [\mathbf{A}]_{\mathbf{M_j}} = \lim_{t \to +\infty} \int_0^{2\pi} (\xi + L_j b_j e^{-t}) d\theta = 2\pi\xi$, so
- $\ell + \xi \neq 0 \ \forall \ell \in \mathbb{Z} \quad \Longleftrightarrow \quad [\mathbf{A}]_{\mathbf{M_j}} \notin 2\pi\mathbb{Z} \quad \Longleftrightarrow \quad J_1^A \ = \ \emptyset$.

Proof of Lemma 2 (3)

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- $\ell + \xi \neq 0 \; \forall \ell \in \mathbb{Z} \quad \Longleftrightarrow \quad [\mathbf{A}]_{\mathbf{M_j}} \notin 2\pi\mathbb{Z} \quad \Longleftrightarrow \quad J_1^A \; = \; \emptyset$.
- If $\ell + \xi = 0$, the spectrum of P_ℓ is absolutely continuous :

$$\mathrm{sp}(P_{-\xi}) \; = \; \mathrm{sp}_{ess}(P_{-\xi}) \; = \; \mathrm{sp}_{ac}(P_{-\xi}) \; = \; [\frac{1}{4} + b_j^2 \; , \; +\infty[\; ;$$

- $\Longrightarrow$ if $[A]_{M_j} \; \in \; 2\pi\mathbb{Z}$, $\quad \mathrm{sp}_{ess}(-\Delta_A^{M_j}) \; = \; [\frac{1}{4} + b_j^2 \; , \; +\infty[$.

Proof of Proposition (1)

- $M = \mathbb{S}\times]\alpha^2, +\infty[$ a cusp
  $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ the metric on M ($\alpha > 0$ and $L > 0$).
- $A = (\xi + Lbe^{-t})d\theta$, (for some constant ξ) .
- $N(\lambda, -\Delta_A^M) = \sum_{\ell\in\mathbb{Z}} N(\lambda, P_\ell)$

  with $P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell+\xi)}{L} \pm b\right)^2$,

  for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha^2, +\infty[$.

Proof of Proposition (1)

- $M = \mathbb{S}\times]\alpha^2, +\infty[$ a cusp
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- Define $Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell+\xi)^2}{L^2} e^{2t}$, for the D. C. on $L^2(I; dt)$.
- Then $Q_\ell - C\sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C\sqrt{Q_\ell}$.

Proof of Proposition (1)

- $M = \mathbb{S}\times]\alpha^2,+\infty[$ a cusp
  $ds^2 \;=\; L^2 e^{-2t}d\theta^2 \;+\; dt^2$ the metric on M ($\alpha > 0$ and $L > 0$).
- $A = (\xi + Lbe^{-t})d\theta$, (for some constant ξ) .
- $$N(\lambda, -\Delta_A^M) \;=\; \sum_{\ell\in\mathbb{Z}} N(\lambda, P_\ell)$$
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- Then $Q_\ell - C\sqrt{Q_\ell} \;\le\; P_\ell \;\le\; Q_\ell + C\sqrt{Q_\ell}$.
- $\Longrightarrow$ $\exists$ a constant $C(b)$, s. t. for any $\lambda >> 1$,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \;\le\; N(\lambda, P_\ell) \;\le\; N(\lambda + \sqrt{\lambda}C(b), Q_\ell) \;;$$

Proof of Proposition (2)

- Lemma

  $\exists C > 1$ s. t. $\forall \mu >> 1$ and $\forall \ell \in X_\mu$,

  $$w_\ell(\mu) - \pi \;\leq\; \pi N(\mu - \frac{1}{4}, Q_\ell) \;\leq\; w_\ell(\mu) + \frac{1}{12} \ln \mu + C$$

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$$w_\ell(\mu) \ = \ \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dt$$

$$X_\mu = \{\ell \ / \ e^{\alpha^2} \frac{|\ell + \xi|}{L} < \sqrt{\mu - 1/4} - b \ \} \ .$$

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- Proof of Lemma (Titchmarsh 's method)
  - define $V_\ell = \frac{(\ell+\xi)^2}{L^2} e^{2t}$, ϕ_μ^ℓ : a solution of $Q_\ell \phi = (\mu - \frac{1}{4})\phi$.
  - Consider x_ℓ and y_ℓ s. t. $V_\ell(x_\ell) = \mu$ and $V_\ell(y_\ell) = \nu$, $0 < \nu < \mu$ to be determined later.
  - n (resp. m): number of zeros of ϕ_μ^ℓ on $]\alpha^2, x_\ell[$ (resp. on $]\alpha^2, y_\ell[$)
  - recall: $n = N(\mu - \frac{1}{4}, Q_\ell)$.

Proof of Proposition (3)

(Titchmarsh 's lemma)

$$m\pi = \int_{\alpha^2}^{y_\ell} [\mu - V_\ell]^{1/2}\, dt + R_\ell$$

with $\quad R_\ell \leq \frac{1}{4}\ln(\mu - V_\ell(\alpha^2)) - \frac{1}{4}\ln(\mu - V_\ell(y_\ell)) + \pi$,

$$\Longrightarrow |n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2}\, dt| \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2} + R_\ell + (n - m)\pi$$

Proof of Proposition (3)

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Sturm comparison theorem $\Longrightarrow (n - m)\pi \ \leq \ (x_\ell - y_\ell)(\mu - \nu)^{1/2} + \pi$

So $\quad$ since $x_\ell - y_\ell = \ (1/2)\ln(\frac{\mu}{\nu})$

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2}\, dt| \leq \ln(\frac{\mu}{\nu})(\mu - \nu)^{1/2} + \frac{1}{4}\ln\mu - \frac{1}{4}\ln(\mu - \nu) + 2\pi$$

Now take $\nu = \mu - \mu^{2/3}$ to get the Lemma.

Proof of Proposition (4)

- We want to compute $\sum_{\ell\in\mathbb{Z}} w_\ell(\mu)$ with

$$w_\ell(\mu) = \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dt.$$

Proof of Proposition (4)

- We want to compute $\sum_{\ell\in\mathbb{Z}} w_\ell(\mu)$ with

$$w_\ell(\mu) \;=\; \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dt.$$

- $\Longrightarrow$ it is enough to compute $\mathcal{I} = \int_{\alpha^2}^{+\infty} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dxdt$,
- $\mathcal{I}$ is equivalent to $\mu L e^{-\alpha^2} \int_{\mathbb{R}} \left[1 - x^2\right]_+^{1/2} dx$
- Use $|M| = 2\pi L e^{-\alpha^2}$ to conclude.