SPECTRAL ASYMPTOTICS FOR MAGNETIC LAPLACIANS IN HYPERBOLIC GEOMETRY

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Outline

- The hyperbolic context
- The Poincaré Half-plane
  - The constant magnetic Laplacian
  - Magnetic bottles : compacity of the resolvent
  - Magnetic bottles : spectral asymptotics
  - Sketch of the proof
  - Related results
- Geometrically finite hyperbolic surfaces
  - Magnetic bottles : spectral asymptotics
  - The spectrum of constant magnetic Laplacians
  - The Weyl formula in the case of finite area with a non-integer class one-form
  - Outline of proofs

Framework

- Let (M, g) be a connected and oriented Riemannian manifold of dimension n.
- For any real one-form A on M , define

$$-\Delta_A = (i\,d+A)^\star(i\,d+A)\ ,\ ((i\,d+A)u = i\,du + uA\ ,\ \forall\, u \in C_0^\infty(M$$

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- The magnetic field is the two-form dA .
- To dA is associated the linear operator B defined on the tangent space by

$$dA(X,Y) \;=\; g(B.X,Y)\;;\quad \forall\;X\;,\;Y\;\in TM\times TM\;.$$

- The magnetic intensity $\mathbf{b}$ is given by

$$\mathbf{b} \;=\; \frac{1}{2}tr\left((B^\star B)^{1/2}\right)\;.$$

The Poincaré half-plane

- If $dim(M) = 2$, then $dA \;=\; \widetilde{\mathbf{b}}\, dv$, with $|\widetilde{\mathbf{b}}| \;=\; \mathbf{b}$, dv the Riemannian measure on M.
- The magnetic field is constant iff $\widetilde{\mathbf{b}}$ is constant.

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- The magnetic field is constant iff $\widetilde{\mathbf{b}}$ is constant.
- Let $M = \mathbb{H}$ be the hyperbolic half-plane :
  $$\mathbb{H} = \mathbb{R}\times]0,+\infty[\,, \quad g = \frac{dx^2 + dy^2}{y^2}\,.$$
  - $-\Delta_A = y^2(D_x - A_1)^2 + y^2(D_y - A_2)^2$, with $A = A_1(x,y)\,dx + A_2(x,y)\,dy$, and $A_j(x,y) \in C^\infty(\mathbb{H};\mathbb{R})$,
  - $\widetilde{\mathbf{b}} = y^2\,(\partial_x A_2 - \partial_y A_1)$
  - $\mathbf{b} = |\widetilde{\mathbf{b}}|$,
  - $dv = y^{-2} dx dy$.

Properties

- $-\Delta_A$ is essentially self-adjoint on $L^2(\mathbb{H})$.
- We are interested on its spectrum : $\mathrm{sp}(-\Delta_A)$.
  We will use that it is gauge invariant:

$$\mathrm{sp}(-\Delta_A) = \mathrm{sp}(-\Delta_{A+d\varphi}) \; ; \quad \forall \, \varphi \in C^\infty(\mathbb{H};\mathbb{R}) \; .$$

The spectrum for constant magnetic field

- The spectrum of $-\Delta_{A^{\mathbf{b}}}$ is essential: $\mathrm{sp}(-\Delta_{A^{\mathbf{b}}}) = \mathrm{sp}_{es}(-\Delta_{A^{\mathbf{b}}})$.
- Its absolutely continuous part is given by $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \; = \; [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.

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- Its absolutely continuous part is given by
  $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \;=\; [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.
- The remaining part of its spectrum is empty if $0 \leq \mathbf{b} \leq 1/2$.
- Otherwise it is formed by a finite number of eigenvalues of infinite multiplicity given by

$$\mathrm{sp}_p(-\Delta_{A^{\mathbf{b}}}) \;=\; \{(2j+1)\mathbf{b} - j(j+1)\;;\; j \;\in\; \mathbb{N}\;,\; j < \mathbf{b} - \frac{1}{2}\}\;,$$

$$(\text{if} \quad \frac{1}{2} < \mathbf{b}\;.)$$

Magnetic bottle : compact resolvent

$$-\Delta_A \;=\; y^2(D_x - A_1)^2 \;+\; y^2(D_y - A_2)^2$$

Magnetic bottle-type assumptions (MB)

- $\mathbf{b}(x,y) \;\to\; +\infty \quad \text{as} \quad d(x,y) \;\to\; +\infty$, $d(x,y)$: the hyperbolic distance of (x,y) to $(0,1)$.
- $\exists\; C_0 > 0$ such that, for any vector field X on $\mathbb{H}$,

$$|X\widetilde{\mathbf{b}}| \leq C_0(|\widetilde{\mathbf{b}}| + 1)\sqrt{g(X,X)}\;;$$

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Theorem

Under the assumptions (MB) $P(A) = -\Delta_A$ has a compact resolvent.

Magnetic bottle: spectral asymptotics

For any real $\lambda \leq \inf \mathrm{sp}_{es}(P)$, we denote by $N(\lambda; P)$ the number of eigenvalues of P , which are in $]-\infty, \lambda[$.

Theorem

Under the assumptions (MB) and for any $\delta \in]\frac{1}{3}, \frac{2}{5}[$, there exists a constant $C > 0$ such that

$$\frac{1}{2\pi}\int_{\mathbb{H}} (1 - \frac{C}{a_\delta(m)})\mathbf{b}(m)\sum_{k=0}^{+\infty}[\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 \, dv$$

$$\leq N(\lambda, -\Delta_A) \leq$$

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$$[\rho]_+^0 = 1 \text{ , if } \rho > 0 \text{ and } 0 \text{ otherwise ,}$$

$$a_\delta(m) := (\mathbf{b}(m) + 1)^{(2-5\delta)/2}.$$

Corollary

- If moreover $\omega(\lambda) := \int_{\mathbb{H}} [\lambda - \mathbf{b}(m)]^0_+ dv$ satisfies (*) :

$$\exists\, C_1 > 0 \text{ s.t. } \forall\ \lambda > C_1\ ,\ \forall\, \tau\ \in\,]0,1[\ ,$$

$$\omega\ ((1+\tau)\ \lambda) - \omega(\lambda) \leq C_1\ \tau\, \omega(\lambda)\ ,$$

then

$$N(\lambda; -\Delta_A)\ \sim\ \frac{1}{2\pi} \int_{\mathbb{H}} \mathbf{b}(m) \sum_{k\in\mathbb{N}} [\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(m)]^0_+\ dv\ .$$

Corollary

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- (*) is satisfied when $\omega(\lambda)\ \sim\ \alpha \lambda^k \ln^j \lambda$ with $k > 0$, or $k = 0$ and $j > 0$.

Example

- $\mathbf{b}(x,y) = a_0^2 \left(\frac{x}{y}\right)^{2m_0} + a_1^2 y^{m_1} + a_2^2 / y^{m_2}$,
  with $a_j > 0$ and $m_j \in \mathbb{N}^\star$, $j = 0, 1, 2$

- $\omega(\lambda) \sim_{+\infty} \alpha \lambda^{\frac{1}{2m_0}} \ln \lambda$

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  with $a_j > 0$ and $m_j \in \mathbb{N}^\star$, $j = 0, 1, 2$
- $\omega(\lambda) \sim_{+\infty} \alpha \lambda^{\frac{1}{2m_0}} \ln \lambda$
- $N(\lambda; -\Delta_A) \sim_{+\infty} \lambda^{1+1/(2m_0)} \ln(\lambda) \alpha(m_0, m_1, m_2)$
- The Weyl formula for a compact hyperbolic hypersurface :
  $N(\lambda) \sim_{+\infty} \frac{\lambda}{4\pi} |M|$.

Proof

- Method: Minimax principle on quadratic forms

Proof

- Method: Minimax principle on quadratic forms
- the main steps:
  - Change variables $\Longrightarrow$ work in $\mathbb{R}^2$,
  - Choose an appropriate gauge,
  - Localize in a "good" rectangle $\Longrightarrow$ replace the initial problem by a problem with a constant magn. field,
  - Write the asymptotics on a rectangle for a constant magn. field,
  - Perform a partition of unity.

Change of variables

- diffeomorphism : $\phi : \mathbb{R}^2 \to \mathbb{H}$, $(x, y) = \phi(x, t) := (x, e^t)$
- unitary operator $\widehat{U} : L^2(\mathbb{H}; dv) \to L^2(\mathbb{R}^2; dxdt)$
  $w(x, t) = (\widehat{U}u)(x, t) := e^{-t/2}\, u(x, e^t) \qquad u \in L^2(\mathbb{H})$.

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- quadratic form associated to $P(A) = -\Delta_A$:

$$q(u) = \int_{\mathbb{H}} \left[|y(D_x - A_1)u|^2 + |y(D_y - A_2)u|^2 \right] \frac{dxdy}{y^2}\, , \quad u \in L^2(\mathbb{H})$$

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$$= \int_{\mathbb{R}^2} \left[|e^t(D_x - \tilde{A}_1)w|^2 + |(D_t - e^t\tilde{A}_2)w|^2 + 1/4|w|^2 \right] dxdt$$

$$w \in L^2(\mathbb{R}^2), \quad \tilde{A}_i(x,t) = A_i(x, e^t) \quad , \ i = \ 1, 2 \ .$$

Gauge, localization

- Choose a magnetic potential s.t. $A_2 = 0$.
  Since $\widetilde{\mathbf{b}} = y^2 \left(\partial_x A_2 - \partial_y A_1\right)$, one can take
  $$A_1(x,y) = -\int_1^y \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds \quad \Longrightarrow \tilde{A}_1(x,t) := -\int_1^{e^t} \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds$$
  - The quadratic form associated is
    $$\widehat{q}(w) = \int_{\mathbb{R}^2} \left[|e^t(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 + 1/4|w|^2 \right] dxdt .$$

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- Assumption (BM) allows to control the magnetic field by a constant one on the cubes of $\mathbb{H}$
  $$\Omega(x_0, y_0, a, \varepsilon_0) := \{(x,y) \;/\; |x - x_0| \leq a\varepsilon_0\, y_0,\; |y - y_0| \leq \varepsilon_0 y_0\} \;:$$

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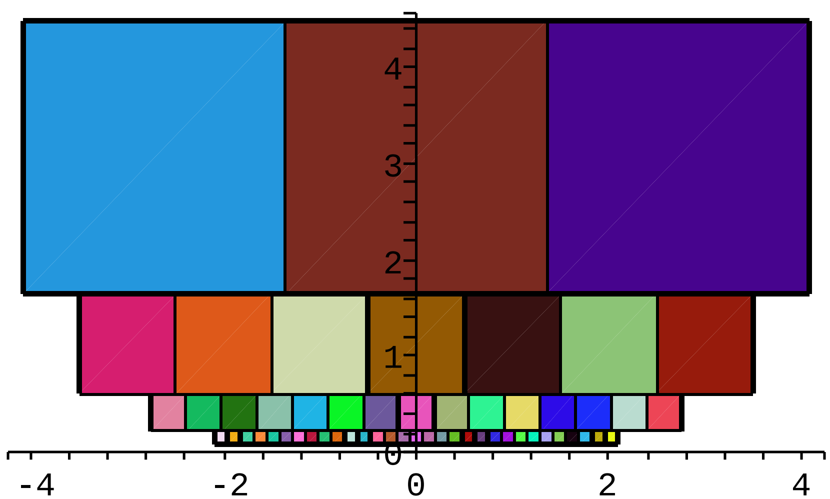
$\exists C_1 > 0$ s.t. , if $\mathbf{b}(x_0, y_0) > 1$,

$$\frac{1}{C_1} \mathbf{b}(x_0,y_0) \le \mathbf{b}(x,y) \le C_1 \mathbf{b}(x_0,y_0) \quad \forall \, (x,y) \in \Omega \, (x_0, y_0, a, \varepsilon_0).$$

Covering of $\mathbb{H}$ with cubes

$$\Omega(x_0, y_0, a, \varepsilon_0) := \{(x,y) \;/\; |x - x_0| \le a\varepsilon_0\, y_0,\; |y - y_0| \le \varepsilon_0 y_0\}$$



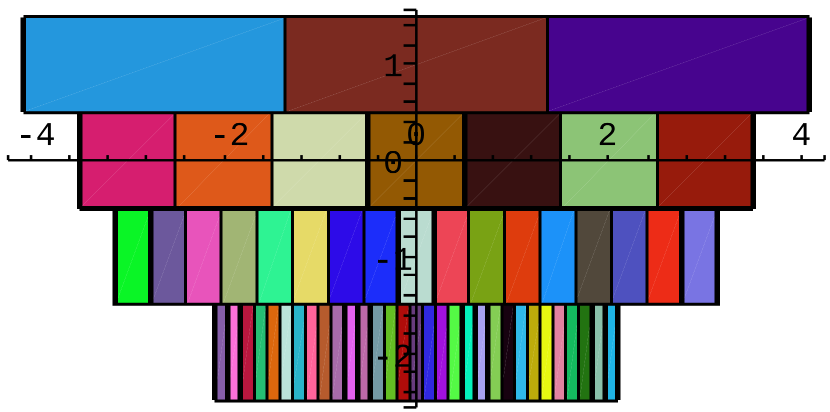
From $\mathbb{H}$ to $\mathbb{R}^2$

- $\Omega(x_0, y_0, a, \varepsilon_0) := \{(x,y) \ / \ |x - x_0| \leq a\varepsilon_0 \ y_0, \ |y - y_0| \leq \varepsilon_0 y_0\}$
- $\phi \ : \ \mathbb{R}^2 \ \rightarrow \ \mathbb{H} \ , \qquad \phi(x,t) \ = \ (x, e^t) \ .$
- For any $\alpha \in \mathbb{Z}^2$, denote by $K(\alpha)$ the rectangle

$$K(\alpha) \ = \left] -\frac{e^{\alpha_2}}{2} + e^{\alpha_2}\alpha_1 \ , \ e^{\alpha_2}\alpha_1 + \frac{e^{\alpha_2}}{2} \right[\times \left] -\frac{1}{2} + \alpha_2 \ , \ \alpha_2 + \frac{1}{2} \right[\quad .$$

- One gets
  - $\mathbb{R}^2 \ = \ \cup_\alpha \overline{K}(\alpha)$ et $K(\alpha) \cap K(\beta) \ = \ \emptyset$ si $\alpha \neq \beta$.
  - the cubes $\phi(\overline{K}(\alpha))$ are the $\Omega(x_0, y_0, a, \varepsilon_0)$

Covering of $\mathbb{R}^2$ with the rectangles $\overline{K}(\alpha)$



$$\phi \ : \quad \mathbb{R}^2 \ \rightarrow \ \mathbb{H} \ , \qquad \phi(x,t) \ = \ (x, e^t) \ \ .$$

Related result : Euclidean magnetic bottles

- If (M,g) is the euclidean space $\mathbb{R}^d$,
  $-\Delta_A = \sum_{j=1}^{d}(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j)^2$. $\forall x \in \mathbb{R}^d$, there exists $(e_j(x))$ s.t.
  $B(x) = \sum_{j=1}^{r(x)} b_j(x) dx_j \wedge dy_j, \quad b_1(x) ... \geq b_r(x) > 0$.
  The magnetic intensity is the norm of $B(x) := (b_j(x))_j$.

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  The magnetic intensity is the norm of $B(x) := (b_j(x))_j$.

- (Y. Colin de Verdière)
  Under (EMB) conditions $-\Delta_A$ has compact resolvent and

$$N_B^{as}[\lambda(1-o(1))] \leq N(\lambda, -\Delta_A) \leq N_B^{as}[\lambda(1+o(1))] \quad (\lambda \to +\infty) \; .$$

  - $N_B^{as}(\lambda) = \int_{\mathbb{R}^d} \nu_{B(x)}(\lambda) dx$
  - $\nu_{B(x)}(\lambda) = C_{k,r} \; \sum^{*} \; (\lambda - \sum_{i=1}^{r} ((2n_i+1) b_i(x))_+^{k/2} \prod_{i=1}^{r} b_i(x)$
  - $\sum^{*} = \sum_{(n_1,\ldots n_r) \in \mathbb{N}^r} \; , \quad d = 2r + k, \quad C_{k,r} = \frac{\gamma_k}{(2\pi)^{k+r}}$
  - $\gamma_k =$ volume of the unit ball in R^k .

- (EMB) conditions
  - (B_1) $\lim_{\|x\|\to\infty}\|B(x)\| = \infty$
  - (B_2) $\|x - x'\| \leq 1$, $\|B(x)\| \leq C\|B(x')\|$
  - (B_3) $M(x) = o(\|B(x)\|^{\frac{3}{2}})$ when $\|x\| \to \infty)$
    ($M(x) = \max_{|\beta|=2}\left(\sup_{\|x-x'\|\leq 1}\|D^\beta A(x')\|\right)$.)

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    ($M(x) = \max_{|\beta|=2} \left(\sup_{\|x-x'\|\leq 1} \|D^\beta A(x')\|\right)$.)
- Remark :
  - Take $d = 2$ in gray Colin de Verdière's result. Then
    $N_B^{as}(\lambda) = \frac{1}{2\pi} \int_{R^2} \mathbf{b}(x) \sum_{k\in\mathbb{N}} \left[\lambda - (2k+1)\mathbf{b}(x)\right]_+^0 dx$
    ($b_1(x) = \|B(x)\| = \mathbf{b}(x)$)
  - to be compared with
    $N(\lambda; -\Delta_A) \sim \frac{1}{2\pi} \int_{\mathbb{H}} \mathbf{b}(m) \sum_{k\in\mathbb{N}} [\lambda - \mathbf{\frac{1}{4}} - (2k+1)\mathbf{b}(m)]_+^0\, dv$.
    $[\rho]_+^0 = 1$ if $\rho > 0$, $[\rho]_+^0 = 0$ if $\rho \leq 0$.

Related results : hyperbolic context

- Results on the hyperbolic space
  - Maass Laplacian: Elstrodt(73), Grosche(88), Comtet(87), Ikeda-Matsumoto (99), Kim-Lee(02)
  - Asympt. constant magnetic fields ,Pauli operators : Inahama-Shirai (03)
  - Asymptotic distribution for Schrödinger operators : Inahama-Shirai (04)
- conformally cusp manifolds :
  Asymptotic distribution for Schrödinger operators :
  Golénia-Moroianu (08)

Geometrically finite hyperbolic surface of infinite area

- Definition

  $(\mathcal{M},\ g)$: a smooth connected Riemannian manifold of dimension two $\mathcal{M} = \left(\bigcup_{j=0}^{J_1} M_j \right) \bigcup \left(\bigcup_{k=1}^{J_2} F_k \right)$;

  - M_j , F_k open sets of $\mathcal{M}$, M_0 compact closure,
  - ($j \neq 0$: M_j isometric to $\mathbb{S}\times]a_j^2, +\infty[$, (cuspidal ends)

$$ds_j^2 = y^{-2}(L_j^2 \, d\theta^2 + dy^2)$$

(a_j and L_j are strictly positive constants)

  - F_k isometric to $\mathbb{S}\times]\alpha_k^2, +\infty[$, (funnel ends)

$$ds_k^2 = \tau_k^2 \cosh^2 t \, d\theta^2 + dt^2$$

(α_k and τ_k are strictly positive constants) .

Magnetic bottle assumptions

- For z_0 fixed in M_0 define $d : \mathcal{M} \to \mathbb{R}_+$; $d(z) = d_g(z, z_0)$; $d_g(\,.\, , \,.\,)$: distance w. r. to the metric g.
- (HSMB)
  - $\lim_{d(z)\to\infty} \mathbf{b}(z) = +\infty$.
  - $$|X\widetilde{\mathbf{b}}(z)| \leq C_1(\mathbf{b}(z) + 1)e^{d(z)}|X|_g ;$$
    $$\forall\, X \in T_z\mathcal{M} \;\forall\, z \in M_j , \text{ and } \forall\, j = 1, \ldots J_1 .(C_1 > 0)$$
  - $$|X\widetilde{\mathbf{b}}(z)| \leq C_2(\mathbf{b}(z) + 1)|X|_g ;$$
    $$\forall\, z \in F_k , \forall\, X \in T_z\mathcal{M} \text{ and } \forall\, k = 1, \ldots J_2 .(C_2 > 0)$$
- Under (HSMB) assumptions the same asymptotics still hold .

Magnetic bottles : Spectral Asymptotics

- Theorem

  Under(HSMB) assumptions, $-\Delta_A$ has a compact resolvent and for any $\delta \in]\frac{1}{3}, \frac{2}{5}[$, there exists a constant $C > 0$ s.t.

  - $$\frac{1}{2\pi}\int_{\mathcal{M}} (1 - \frac{C}{a_\delta(m)})\mathcal{N}(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m))\, dm$$
    $$\leq N(\lambda, -\Delta_A) \leq$$
    $$\frac{1}{2\pi}\int_{\mathcal{M}} (1 + \frac{C}{(a_\delta(m))})\mathcal{N}(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, \mathbf{b}(m))\, dm$$

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  - $\mathcal{N}(\mu, \mathbf{b}(m)) = \mathbf{b}(m)\sum_{k=0}^{+\infty}[\mu - (2k+1)\mathbf{b}(m)]_+^0$ if $\mathbf{b}(m) > 0$, and $\mathcal{N}(\mu, \mathbf{b}(m)) = \mu/2$ if $\mathbf{b}(m) = 0$.

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- The Theorem still holds if we replace $\int_{\mathcal{M}}$ by $\sum_{k=1}^{J_2} \int_{F_k}$.

Example

- on M_j , isometric to $\mathbb{S}\times]a_j^2,+\infty[$, (cuspidal ends)

$$ds_j^2 \;=\; y^{-2}(\; L_j^2\; d\theta^2 \;+\; dy^2\;)$$

$$\mathbf{b}(\theta,y) \;=\; y^{m_1}\;.$$

- on F_k , isometric to $\mathbb{S}\times]\alpha_k^2,+\infty[$, (funnel ends)

$$ds_k^2 \;=\; \tau_k^2\cosh^2 t\; d\theta^2 \;+\; dt^2$$

$$\mathbf{b}(\theta,t) \;=\; (\;1/\cosh(t)\;)^{m_2}\;.$$

Example

- on M_j , isometric to $\mathbb{S}\times]a_j^2,+\infty[$, (cuspidal ends)

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- Theorem

$N(\lambda;-\Delta_A) \;\sim\; \alpha(m_2)\lambda^{1+1/m_2}\;,$

$\alpha>0$ depends only on the funnels .

The spectrum of constant magnetic Laplacians

- $$\mathcal{M} = \left(\bigcup_{j=0}^{J_1} M_j \right) \bigcup \left(\bigcup_{k=1}^{J_2} F_k \right) ;$$
- It is not always possible to have a constant magnetic field on $\mathcal{M}$,

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- but :
  $\forall (b, \beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2}$, $\exists A$, s. t. dA satisfies

$$dA = \widetilde{\mathbf{b}}(z) dm \quad \begin{cases} \widetilde{\mathbf{b}}(z) = b_j \, \forall \, z \in M_j \\ \widetilde{\mathbf{b}}(z) = \beta_k \, \forall \, z \in F_k \end{cases}$$

The spectrum of constant magnetic Laplacians

Theorem 1

- If $J_1 = 0$ and $J_2 > 0$, $\mathrm{sp}_{ess}(-\Delta_A) = [\frac{1}{4} + \inf_k \beta_k^2 \,,\, +\infty[$.
- If $J_1 > 0$, there exists a unique closed curve through z , $\mathcal{C}_{j,z}$ in $(M_j,\ g)$, not contractible and with zero $g-$curvature.

$$[\mathbf{A}]_{\mathbf{M_j}} = \lim_{d(z)\to+\infty} \int_{\mathcal{C}_{j,z}} A \,.$$

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$$[\mathbf{A}]_{\mathbf{M_j}} = \lim_{d(z)\to+\infty} \int_{\mathcal{C}_{j,z}} A \, .$$

- Define : $J_1^A = \{ j \in \mathbb{N} \, , \, 1 \leq j \leq J_1 \; s.t. \; [\mathbf{A}]_{\mathbf{M_j}} \in 2\pi\mathbb{Z} \, \}$, then
  - $\mathrm{sp}_{ess}(-\Delta_A) = [\frac{1}{4} + \min\{ \inf_{j \in J_1^A} b_j^2 \, , \, \inf_{1 \leq k \leq J_2} \beta_k^2 \, \} , +\infty[$.
  - If $J_2 = 0$ and $J_1^A = \emptyset$, then $\mathrm{sp}_{ess}(-\Delta_A) = \emptyset$: $-\Delta_A$ has purely discrete spectrum , (its resolvent is compact).

The case of finite area, with a non-integer class 1-form A

- Theorem 2

  Consider a geometrically finite hyperbolic surface $(\mathcal{M},\ g)$ of finite area, $(J_2 = 0)$, and assume that $J_1^A \ = \ \emptyset$.

  Then

$$N(\lambda, -\Delta_A) \ = \ \lambda \frac{|\mathcal{M}|}{4\pi} \ + \ \mathbf{O}(\frac{\lambda}{\ln \lambda}) \ .$$

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  Then

$$N(\lambda, -\Delta_A) = \lambda \frac{|\mathcal{M}|}{4\pi} + \mathbf{O}(\frac{\lambda}{\ln \lambda}) .$$

- - Consider $M = \mathbb{S} \times]\alpha^2, +\infty[$ equipped with the metric $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ for some $\alpha > 0$ and $L > 0$.
  - Denote by $-\Delta_A^M$ the Dirichlet operator on M , associated to $-\Delta_A$.
  - Proposition

$$N(\lambda, -\Delta_A^M) = \lambda \frac{|M|}{4\pi} + \mathbf{O}(\frac{\lambda}{\ln \lambda}) .$$

Proof of Theorem 1

- $$\mathrm{sp}_{ess}(-\Delta_A) = \left(\bigcup_{j=1}^{J_1} \mathrm{sp}_{ess}(-\Delta_A^{M_j})\right) \bigcup \left(\bigcup_{k=1}^{J_2} \mathrm{sp}_{ess}(-\Delta_A^{F_k})\right) ;$$

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- Lemma 1

$$\mathrm{sp}_{ess}(-\Delta_A^{F_k}) \;=\; [\frac{1}{4} + \beta_k^2 \;,\; +\infty[\;.$$

- Lemma 2

If $1 \leq j \leq J_1$ and $j \;\notin\; J_1^A$, then

$$\mathrm{sp}_{ess}(-\Delta_A^{M_j}) \;=\; \emptyset\;.$$

If $j \;\in\; J_1^A$, then

$$\mathrm{sp}_{ess}(-\Delta_A^{M_j}) \;=\; [\frac{1}{4} + b_j^2 \;,\; +\infty[\;.$$

Proof of Lemma 1

- $F_k \sim \mathbb{S} \times]\alpha_k^2, +\infty[$, (funnels) $ds_k^2 = \tau_k^2 \cosh^2 t \, d\theta^2 + dt^2$
- $\Longrightarrow - \Delta_A^{F_k} =$
$\tau_k^{-2} \cosh^{-2} t (D_\theta - A_1)^2 + \cosh^{-1} t (D_t - A_2) [\cosh t (D_t - A_2)]$.

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- $\widetilde{\mathbf{b}} = \beta_k = \tau_k^{-1} \cosh^{-1} t (\partial_\theta A_2 - \partial_t A_1)$, $\Longrightarrow \exists \varphi$ s.t.

$A - \widetilde{A} = d\varphi$ if $\widetilde{A} = (\xi - \beta_k \tau_k \sinh t) d\theta$, (for some constant ξ) .

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  $\Longrightarrow$ assume that $A = \widetilde{A}$.
- Define $Uf = (\tau_k \cosh t)^{1/2} f$
- $\Longrightarrow P = -U\Delta_A^{F_k}U^\star =$
  $\tau_k^{-2}\cosh^{-2} t(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}(1+\cosh^{-2} t)$.

  $(A_1 = (\xi - \beta_k\tau_k \sinh(t))$

- $\mathrm{sp}(-\Delta_A^{F_k}) = \mathrm{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \mathrm{sp}(P_\ell)$,

$$P_\ell = \tau_k^{-2} \cosh^{-2}(t)(\ell + \beta_k \tau_k \sinh(t) - \xi)^2 + D_t^2 + \frac{1}{4}(1 + \cosh^{-2}(t)) \ ,$$

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  for the Dirichlet condition on $L^2(I;dt)$; $I =]\alpha_k^2\ ,\ +\infty[$.

- Write

  $$P_\ell \;=\; \left(\frac{\ell-\xi}{\tau_k\cosh(t)}+\beta_k\tanh(t)\right)^2+\frac{1}{4}(1+\cosh^{-2}(t))+D_t^2$$

- Get the result

Proof of Lemma 2

- M_j isometric to $\mathbb{S}\times]a_j^2, +\infty[$, (cuspidal ends)
  $ds_j^2 \;=\; y^{-2}(\; L_j^2\, d\theta^2 \;+\; dy^2\;) \qquad (a_j > 0 \text{ and } L_j > 0\;)$
- Use the coordinate $t = \ln y$ instead of y , so
  - $M_j \;=\; \mathbb{S}\times]\alpha_j^2, +\infty[\quad \text{and} \quad ds_j^2 \;=\; L_j^2 e^{-2t} d\theta^2 \;+\; dt^2$; $(\alpha_j = e^{a_j})$.

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  - $-\Delta_A^{M_j} \;=\; L_j^{-2} e^{2t}(D_\theta - A_1)^2 + e^t(D_t - A_2)(e^{-t}(D_t - A_2))$,
  - $\widetilde{\mathbf{b}} = b_j = \; L_j^{-1} e^t(\partial_\theta A_2 - \partial_t A_1)$ and $dm = L_j e^{-t} d\theta dt$.

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- we have

$$A - \widetilde{A} \;=\; d\varphi \text{ if } \widetilde{A} = (\xi + L_j b_j e^{-t})d\theta \;,\; (\text{for some constant } \xi) \;.$$

- $\Longrightarrow$ we can assume that $A = \widetilde{A}$.

Proof continued

- Define $Uf = \sqrt{L_j} e^{-t/2} f$,
- $\Longrightarrow P = -U \Delta_A^{M_j} U^\star = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}$.

Proof continued

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- $\Longrightarrow P = -U \Delta_A^{M_j} U^\star = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}$.
- $\Longrightarrow \mathrm{sp}(-\Delta_A^{M_j}) = \mathrm{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \mathrm{sp}(P_\ell)$

  $$P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L_j} + b_j \right)^2 ,$$

  for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_j^2, +\infty[$.

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  for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_j^2, +\infty[$.
- When $\ell + \xi \neq 0$, the spectrum of P_ℓ is discrete.
- More precisely : $\mathrm{sp}(P_\ell) = \mathrm{sp}(P^\pm)$ $\quad P^\pm = D_t^2 + \dfrac{1}{4} + (\pm e^t + b_j)^2$

  for the Dirichlet condition on

  $L^2(I_{j,\ell}; dt)$; $I_{j,\ell} =]\alpha_j^2 + \ln(|\ell+\xi|/L_j), +\infty[$, and $\pm = \dfrac{\ell+\xi}{|\ell+\xi|}$.

Proof continued

- $\Longrightarrow$ if $\ell + \xi \neq 0 \, \forall \ell \in \mathbb{Z}$, $\lim_{|\ell| \to \infty} \inf \mathrm{sp}(P_\ell) \ = \ +\infty$,
- $\Longrightarrow$ the spectrum of $-\Delta_A^{M_j}$ is discrete.

Proof continued

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- $\Longrightarrow$ the spectrum of $-\Delta_A^{M_j}$ is discrete.

- What means this condition ? Recall
  - $A = (\xi + L_j b_j e^{-t}) d\theta$,
  - $[\mathbf{A}]_{\mathbf{M_j}} = \lim_{d(z) \to +\infty} \int_{\mathcal{C}_{j,z}} A$

    $\Longrightarrow [\mathbf{A}]_{\mathbf{M_j}} = \lim_{t \to +\infty} \int_0^{2\pi} (\xi + L_j b_j e^{-t}) d\theta = 2\pi\xi$, so
- $\ell + \xi \neq 0\ \forall \ell \in \mathbb{Z} \iff [\mathbf{A}]_{\mathbf{M_j}} \notin 2\pi\mathbb{Z} \iff J_1^A = \emptyset$.

Proof continued

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- $\ell + \xi \neq 0 \ \forall \ell \in \mathbb{Z} \iff [\mathbf{A}]_{\mathbf{M_j}} \notin 2\pi\mathbb{Z} \iff J_1^A = \emptyset$.
- If $\ell + \xi = 0$, the spectrum of P_ℓ is absolutely continuous :

$$\mathrm{sp}(P_{-\xi}) = \mathrm{sp}_{ess}(P_{-\xi}) = \mathrm{sp}_{ac}(P_{-\xi}) = [\frac{1}{4} + b_j^2 , +\infty[;$$

- $\Longrightarrow$ if $[A]_{M_j} \in 2\pi\mathbb{Z}$, $\mathrm{sp}_{ess}(-\Delta_A^{M_j}) = [\frac{1}{4} + b_j^2 , +\infty[$.

Proof of Proposition

- $M = \mathbb{S}\times]\alpha^2, +\infty[$ a cusp
  $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ the metric on M ($\alpha > 0$ and $L > 0$).
- $A = (\xi + Lbe^{-t})d\theta$, (for some constant ξ) .
- $N(\lambda, -\Delta_A^M) = \sum_{\ell\in\mathbb{Z}} N(\lambda, P_\ell)$
  with $P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell+\xi)}{L} \pm b\right)^2$,
  for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha^2, +\infty[$.

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- Define $Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell+\xi)^2}{L^2} e^{2t}$, for the D. C. on $L^2(I; dt)$.
- Then $Q_\ell - C\sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C\sqrt{Q_\ell}$.

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- Then $Q_\ell - C\sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C\sqrt{Q_\ell}$.
- $\Longrightarrow$ $\exists$ a constant $C(b)$, s. t. for any $\lambda >> 1$,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \leq N(\lambda, P_\ell) \leq N(\lambda + \sqrt{\lambda}C(b), Q_\ell) ;$$

Proof continued

- Weyl formula $\Longrightarrow$
  $\exists C_0 > 1$ s.t. for any $\lambda >> 1$ and any $\ell \in \mathbb{Z}$,

$$w_\ell(\lambda - C_0\sqrt{\lambda}) \ \leq \ \pi N(\lambda, P_\ell) \ \leq \ w_\ell(\lambda + C_0\sqrt{\lambda}) \ ,$$

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- with $w_\ell(\mu) \ = \ \displaystyle\int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dt$

Proof continued

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- We want to compute $\sum_{\ell \in \mathbb{Z}} w_\ell(\mu)$.
  - $\Longrightarrow$ it is enough to compute
    $\displaystyle\int_{\alpha^2}^{+\infty} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2} e^{2t}\right]_+^{1/2} dx dt$,
  - and this is equivalent to $\mu L e^{-\alpha^2} \displaystyle\int_{\mathbb{R}} \left[1 - x^2\right]_+^{1/2} dx$
  - use $|M| = 2\pi L e^{-\alpha^2}$ to conclude.