Semi-classical asymptotics for magnetic bottles

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ABSTRACT . — In this paper we investigate the asymptotic behaviour of the counting function of the eigenvalues for a semi-classical Schrödinger operator with a magnetic field, for a fixed energy, when the small parameter h goes to zero. We require for the magnetic field assumptions of the type "magnetic bottles" and we use a method of subdivision of R^d in cubes, in order to apply Courant's minimax variational principle. This method was previously used by R.Courant in the case of the classical counting function for minus Laplacian .

1. Introduction

Let $B(x)$ be a magnetic field in R^d, which means a closed real 2-form on R^d. For every x there exists two integers $r(x)$ and $k(x)$ such that $d = 2r(x) + k(x)$, and we can find an orthonormal basis $(dx_1, \dots, dx_r, dy_1, \dots, dy_r, dx_{r+1}, \dots, dx_{r+k})$ on R^d for which B has the following expression :

$$B(x) = \sum_{i=1}^{r(x)} b_i(x) dx_i \wedge dy_i \text{ with } b_1 \geq b_2 \geq \cdots \geq b_r > 0 \ .$$

The $b_i(x)$ are the modules of non zero eigenvalues for the antisymmetric endomorphism associated to $B(x)$ and $2r(x)$ denotes its rank. In odd dimension in particular 0 is always an eigenvalue . We shall denote by $a = \sum_{i=1}^{d} a_i dx_i$ a magnetic potential for B, in other words a one-form related to B by : $da = B$.

It is moreover assumed that B satisfies the following conditions $(*)$:

(B_1) $\lim_{\|x\| \to \infty} \|B(x)\| = \infty$.

(B_2) there exists $C > 0$ such that, for every x and x' verifying : $\|x - x'\| \leq 1$, $\|B(x)\| \leq C \|B(x')\|$.

(B_3) Let $M(x)$ be: $M(x) = \max_{|\beta|=2}\Big(\sup_{\|x-x'\|\leq 1} \|D^\beta a(x')\|\Big)$.

$$M(x) = o(\|B(x)\|^{\frac{3}{2}}) \text{ when } \|x\| \to \infty.$$

We are interested in the Schrödinger operator $\widehat{H_h}$ associated to this magnetic field, which has the following expression :

$$\widehat{H_h} = \sum_{j=1}^{d}\Big(\frac{h}{i}\,\frac{\partial}{\partial x_j} - a_j\Big)^2 .$$

This operator is essentially selfadjoint with compact resolvant on $L^2(R^d)$ ([1]). Let us denote by $N_h(E)$ the counting function of its spectrum :

$$N_h(E) = \text{card}\{\lambda_i(h); \lambda_i(h) \quad \text{eigenvalue of} \quad \widehat{H_h} \text{ and } \lambda_i(h) \leq E\} .$$

We clearly have $\widehat{H_h} = h^2 H_h$, where $H_h = \sum_{j=1}^{d} (\frac{1}{i}\frac{\partial}{\partial x_j} - \frac{a_j}{h})^2$ is the Schrödinger operator associated to the magnetic field $\frac{B}{h}$. As a consequence, $N_h(E) = N(\frac{E}{h^2})$ for any fixed energy E , if we denote by N the counting function relative to H_h.

The behaviour of $N(\lambda)$, for a fixed h , and for the great values of λ is well known (cf. [3]):

$$N^{as}_{\frac{B}{h}}[\lambda(1 - o(1))] \leq N(\lambda) \leq N^{as}_{\frac{B}{h}}[\lambda(1 + o(1))] .$$

In the general case, the expression for N^{as}_B is the following :

$$N^{as}_B(\lambda) = \sum_{r=1}^{[d/2]} C_{k,r} \sum_{(n_1,\ldots,n_r)\in Z^+_r} \int_{A_r} \big(\lambda - \sum_{i=1}^{r}(2n_i+1)b_i(x)\big)_+^{k/2} \prod_{i=1}^{r} b_i(x)dx .$$

We used the following notations :

$$A_r = \{x \in R^d; r(x) = r\}$$

$$C_{k,r} = \frac{\gamma_k}{(2\pi)^{k+r}}$$

$$\gamma_k = \text{volume of the unit ball of } R^k .$$

Our aim in this paper is to determine an asymptotic formula of $N_h(E)$ with fixed energy E when h tends to zero. Using an adaptation of the method explained in [3], we prove the following result :

THEOREM. — *Under the conditions* $(*)$, *we have, for any energy* E :

$$\frac{1}{h^d} N^{as}_{hB}[E(1 - o(1))] \leq N_h(E) \leq \frac{1}{h^d} N^{as}_{hB}[E(1 + o(1))] \quad (h \to 0) .$$

Remark 1. — The expression for N_B^{as} becomes more explicit when $d = 2$. By the way if we set $T(\lambda) = \text{card}\{n \geq 0; 2n + 1 \leq \lambda\}$ et $z = (x, y)$, we get :

$$\frac{1}{h^d} N_{hB}^{as}(E) = \frac{1}{2\pi h} \int_{R^2} \|B(z)\| \, T(\frac{E}{hB(z)}) dx dy \; .$$

Remark 2. — It is possible to recover the conclusions of this theorem by studying the semi-group $\exp(-tH_h)$ (cf. [14]). The following asymptotic formula is obtained :

$$\text{Tr}(\exp(-tH_h)) \sim \frac{1}{h^d} Z_{hB}(t), \; h \to 0,$$

where $Z_{hB}(t) = (4\pi t)^{-d/2} \int_{R^d} \prod_{i=1}^{r(x)} \frac{htb_i(x)}{\sinh htb_i(x)} dx$ is precisely the Laplace transform of the function $N_{hB}^{as}(\lambda)$. To see this let us compute , for simplicity, in the case where the rank of B is constant and $k = 0$. (see [14] for the general case .)

We have then :

$$N_B^{as}(\lambda) = \frac{1}{(2\pi)^r} \sum_{(n_1,\ldots,n_r) \in Z_r^+} \int_{x \in R^d, N_i(x) < \lambda} \prod_{i=1}^{r} b_i(x) dx$$

where $N_i(x) = \sum_{i=1}^{r} (2n_i + 1) b_i(x)$.

So the Laplace transform can be computed rather simply :

$$\int_0^\infty e^{-t\lambda} dN_B^{as}(\lambda) = \frac{1}{(2\pi)^r} \sum_{(n_1,\ldots,n_r) \in Z_r^+} \int_{x \in R^d} e^{-tN_i(x)} \prod_{i=1}^{r} b_i(x) dx$$

$$= \frac{1}{(2\pi)^r} \int_{x \in R^d} \prod_{i=1}^{r} \sum_{n_i=0}^{\infty} e^{-t(2n_i+1)b_i(x)} \prod_{i=1}^{r} b_i(x) dx$$

and :

$$\int_0^\infty e^{-t\lambda} dN_B^{as}(\lambda) = \frac{1}{(2\pi)^r} \int_{x \in R^d} \prod_{i=1}^{r} \frac{b_i(x) e^{-tb_i(x)}}{1 - e^{-2tb_i(x)}} dx$$

and finally :

$$\int_0^\infty e^{-t\lambda} dN_B^{as}(\lambda) = \frac{1}{(4\pi t)^r} \int_{x \in R^d} \prod_{i=1}^{r} \frac{tb_i(x)}{\sinh tb_i(x)} dx \; .$$

However the method used in [14] requires stronger conditions for B, in order to make sure that $\exp(-tH_h)$ is a trace-class semi-group.

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2. Two necessary results for the proof

2.1. Schrödinger operator with a constant magnetic field in a cube and Dirichlet conditions . — Let us set

$$\nu_{B(x)}(\lambda) = C_{k,r} \sum_{(n_1,\dots,n_r)\in Z_r^+} \left(\lambda - \sum_{i=1}^{r}(2n_i+1)b_i(x)\right)_+^{k/2} \prod_{i=1}^{r} b_i(x) \ .$$

(In this definition, the numbers k et r depend on x.) The function $N_B^{as}(\lambda)$ has then the following expression :

$$N_B^{as}(\lambda) = \int_{R^d} \nu_{B(x)}(\lambda)dx \ .$$

In the case of a constant magnetic field, the function $\nu_B(\lambda)$ can be seen as a density of states, since it makes possible to estimate $N_{B,R}(\lambda)$, the counting function of the spectrum concerning Dirichlet problem for the Schrödinger operator with the constant field B in the cube $[0,R]^d$. We recall the precise estimate, given in [3] :

PROPOSITION . — *There exists a constant c depending only on d such that, for any A with $0 < A < R/2$, the following inequalities hold :*

$$N_{B,R}(\lambda) \leq R^d \nu_B(\lambda)$$
$$N_{B,R}(\lambda) \geq (R-A)^d \nu_B(\lambda - c/A^2) \ .$$

2.2. Existence of a subdivision of R^d in cubes "adaptated" to B and to h.

LEMMA. — *Under the assumptions $(*)$, and for a fixed $\varepsilon > 0$, there exists for any h a subdivision of R^d in cubes $(\Omega_i)_{i\geq 0}$ of sides r_i, and numbers $(a_i)_{i\geq 1}$ $(0 < a_i \leq r_i/2)$ such that, if we set $M_i = \max_{\|\beta\|=2} \sup_{x\in\Omega_i} \|D^\beta a(x)\|$, the following inequalities hold, for any x in Ω_i and for any integer $i \geq 1$:*

i) $r_i^2 M_i \leq \varepsilon h \|B(x)\|^{1/2}$

ii) $1/a_i^2 \leq \max\left(\dfrac{4\varepsilon\|B(x)\|}{h}, \dfrac{1}{\varepsilon}\right).$

Proof. — Let us consider first a subdivision of R^d in cubes (C_α) of sides 1. According with the assumption (B_3), it is possible to put together in a cube called Ω_0 the cubes (C_α) such that there exists x in (C_α) verifying : $M_\alpha \geq \varepsilon^3 \|B(x)\|^{\frac{3}{2}}$. This cube does not depend on h.

We subdivide the cubes (C_α) external to Ω_0 into small cubes of sides r_α with $1/r_\alpha$ an integer. We shall denote these small cubes by Ω_i. Two cases may occur concerning the cubes (C_α), each one determines a special choice of r_α :

1) $M_\alpha > \varepsilon h \min_{x \in C_\alpha} \|B(x)\|^{\frac{1}{2}}$

In this case it is possible to choose r_α such that $1/r_\alpha$ is a integer and the following inequalities hold :

$$\frac{\varepsilon h}{4} \min_{x \in C_\alpha} \|B(x)\|^{\frac{1}{2}} \leq r_\alpha^2 M_\alpha \leq \varepsilon h \min_{x \in C_\alpha} \|B(x)\|^{\frac{1}{2}} \ .$$

The inequality *i)* of the lemma is therefore satisfied, since $r_i = r_\alpha$ for the Ω_i contained in C_α, and since we have : $M_i \leq M_\alpha$. We choose now all the a_i equal in C_α to $a_\alpha = \sqrt{\varepsilon} r_\alpha$, which yields

$$1/a_i^2 = 1/(\varepsilon r_i^2) \leq \frac{4M_\alpha}{h\varepsilon^2 \min_{x \in C_\alpha} \|B(x)\|^{\frac{1}{2}}} \leq \frac{4\varepsilon \|B(x)\|}{h} \ .$$

This last inequality comes in fact as a result of the choice of Ω_0.

2) $M_\alpha \leq \varepsilon h \min_{x \in C_\alpha} \|B(x)\|^{\frac{1}{2}}$.

We set in this case $r_i = r_\alpha = 1$, so that inequality *i)* is satisfied and that $1/a_i^2 = 1/\varepsilon$.

Remark. — The assumption (B_2) ensures that the real number $\min_{x \in C_\alpha} \|B(x)\|^{\frac{1}{2}}$ is not zero.

In the following paragraph, ε is fixed. We shall be using the subdivision (Ω_i) associated to a fixed value of h for the moment.

3. Proof of the theorem

3.1. Minoration of $N_h(E)$.

We denote by ∇ the connexion $d + ia/h$, associated to the field B/h , and by ∇_i the connexion $d + ia_i/h$, associated to the field $B(x_i)/h$ for an arbitrarily fixed point x_i in Ω_i, and defined on the Sobolev spaces $H_0^1(\Omega_i)$.

We now use the isometric injection given by : $j : \bigoplus_{i\geq 1} H_0^1(\Omega_i) \hookrightarrow D(q_{B/h})$, $j(\bigoplus f_i) = \sum_i f_i$, so we can introduce the following quadratic forms :

$$q_1(\bigoplus_{i\geq 1} f_i) = \sum_{i\geq 1}\int_{\Omega_i} |\nabla_i f_i|^2$$

$$q_{B/h}(f) = \int_{R^d} |\nabla f|^2 \quad \text{with } f = \sum_{i\geq 1} f_i \ .$$

It is possible to choose for the potential of $B(x_i)$ the Taylor series up to order 1 of a in x_i, so that we can write :

$$|(\nabla - \nabla_i)f_i(x)| = |\frac{i}{h}[a(x) - a(x_i) - da(x_i)(x - x_i)]f_i(x)| \leq \frac{1}{h}\|x - x_i\|^2 M_i |f_i(x)| \ .$$

It results from that, and from the inequality *i)* of the lemma, the following inequality, for any η :

$$\int_{R^d} |\nabla f|^2 \leq (1+\eta^2)\ q_1(\bigoplus_{i\geq 1} f_i) + (1+1/\eta^2)\ \varepsilon^2 \sum_{i\geq 1}\|B(x_i)\|\int_{\Omega_i}|f_i|^2,$$

so that, by application of the minimax principle :

$$N(\lambda) \geq \sum_{i\geq 1} N_{B(x_i)/h, r_i}\left[\frac{\lambda}{1+\eta^2} - \frac{\varepsilon^2}{1+\eta^2}(1+\frac{1}{\eta^2})\|B(x_i)\|\right] \ .$$

Let us choose η so that : $\varepsilon = \frac{\eta^2}{\sqrt{1+\eta^2}}$. Then we have :

$$\frac{\varepsilon^2}{1+\eta^2}(1+\frac{1}{\eta^2}) = \frac{\eta^2}{1+\eta^2},$$

whence :

$$N(\lambda) \geq \sum_{i\geq 1} N_{B(x_i)/h, r_i}\left[\frac{\lambda}{1+\eta^2} - \frac{\eta^2}{1+\eta^2}\|B(x_i)\|\right] \ .$$

As a result of the statement in the case of a constant magnetic field in a cube (see Proposition 2.1), and of the inequality *ii)* of lemma 2.2 we can write consecutively :

$$N(\lambda) \geq \sum_{i\geq 1}(r_i - a_i)^d\ \nu_{B(x_i)/h}\left[\frac{1}{1+\eta^2}(\lambda - \eta^2\|B(x_i)\|) - \frac{c}{a_i^2}\right] ,$$

$$N(\lambda) \geq \sum_{i\geq 1}(r_i - a_i)^d\ \nu_{B(x_i)/h}\left[\frac{\lambda}{1+\eta^2} - \frac{\eta^2}{1+\eta^2}(1+\frac{4ce_\eta}{h})\|B(x_i)\|) - \frac{ce_\eta}{\eta^2}\right] \ .$$

(We have set : $e_\eta = \sqrt{1+\eta^2}$.)

Considering the expression of $\nu_{B(x_i)/h}$, and the equivalence between $\|B(x_i)\|$ and the greatest eigenvalue b_1 of B at the point x_i, we notice that the preceding sum takes in account only the indices i which satisfy :

$$\frac{\lambda}{1+\eta^2} - \frac{\eta^2}{1+\eta^2}(1+\frac{4ce_\eta}{h})\|B(x_i)\|) - \frac{ce_\eta}{\eta^2} \geq \frac{b_1}{h} \ .$$

We get therefore the following minoration :

$$N(\lambda) \geq \sum_{i\geq 1}(r_i - a_i)^d \ \nu_{B(x_i)/h}[\frac{F\lambda - K}{hG+1}]$$

with :

$$F = \frac{1}{1+\eta^2}$$
$$G = \frac{\eta^2}{1+\eta^2}(1+\frac{4ce_\eta}{h})$$
$$K = \frac{ce_\eta}{\eta^2}.$$

So we compute :

$$\frac{F}{hG+1} = \frac{1}{\eta^2(h+4ce_\eta)+1+\eta^2} \text{ whence : } \frac{F}{hG+1} = 1 - 0(h) - 0(\varepsilon) \ .$$

In the same way we have :

$$\frac{K}{hG+1} = c_{\varepsilon,h} = \frac{c}{\varepsilon}(1 - 0(h) - 0(\varepsilon)) \ .$$

Finally :

$$N(\lambda) \geq (1 - 0(\sqrt{\varepsilon}))\sum_{i\geq 1} r_i^d \ \nu_{B(x_i)/h}\big[\lambda(1 - 0(h) - 0(\varepsilon)) - c_{\varepsilon,h}\big] \ .$$

Now we use the fact that x_i is an arbitrary number in Ω_i and we write $\lambda = \frac{E}{h^2}$, so we get :

$$N(\frac{E}{h^2}) \geq (1 - 0(\sqrt{\varepsilon}))\int_{x\notin\Omega_0} \nu_{B(x)/h}\big[\frac{E}{h^2}(1 - 0(h) - 0(\varepsilon)) - c_{\varepsilon,h}\big] \ .$$

Ω_0 takes part in this inequality only as an element of the subdivision, element which does not depend on h. For that reason we can allow h to go to 0 . Finally, using the following fact :

$$\int_{\Omega_0} \nu_{B(x)/h}(\frac{E}{h^2}) = \ 0\Big(\int_{R^d} \nu_{B(x)/h}(\frac{E}{h^2})\Big) \text{ when } \ h \to 0$$

we get the final minoration :

$$N_h(E) \geq \int_{R^d} \nu_{B(x)/h}\big[\frac{E}{h^2}(1 - 0(h) - 0(\varepsilon))\big] \ .$$

3.2. Majoration of $N_h(E)$.

We are going to use now the covering of R^d by the $\widetilde{\Omega_i}$, which will be cubes centered on the Ω_i but of sides $r_i + a_i$.

We set : $\widetilde{\Omega_0} = \{x \in R^d; d(x, \Omega_0) \le \sqrt{\varepsilon}/2\}$. From that we define : $a_0 = \sqrt{\varepsilon}$. We construct then a family of functions $\phi_i \in C_o^\infty(\widetilde{\Omega_i})$ verifying : $\sum \phi_i^2 = 1$ et $\forall x \in \widetilde{\Omega_i}$, $\|d\phi_i(x)\| \le c/a_i$. ($c = \max_{x \in R^d} \# I(x)$, with $I(x) = \{i \ge 0; x \in \Omega_i\}$. c is a constant which depends only of d). The construction of such a family is explained in details in [3] or [9]. We define an injection from $D(q_{B/h})$ into $\bigoplus_{i\ge 0} H_0^1(\Omega_i)$ by the following formula : $f \to \bigoplus_{i \ge 0} (f\phi_i)$, and also a quadratic form $q_1(\bigoplus f_i)$ in the same way as before.

a) *Link between $\nabla f \phi_i$ and ∇f.* — We have :

$$\sum_{i\ge 0} |\nabla f \phi_i|^2 = \sum_{i \ge 0} |\phi_i \nabla f + f(d\phi_i)|^2 = \sum_{i\ge 0} |\phi_i \nabla f|^2 + |f|^2 \sum_{i \ge 0} |d\phi_i|^2 \ .$$

Notice that the double product is zero due to the choice of the ϕ_i : $0 = d(\sum_{i\ge 0} \phi_i^2) = 2 \sum_{i \ge 0} \phi_i d\phi_i$. By integration we get the following :

$$\int_{R^d} \sum_{i \ge 0} |\nabla f \phi_i|^2 \le \int_{R^d} |\nabla f|^2 + \int_{R^d} \sum_{i\ge 0} |d\phi_i|^2 |f|^2 \ .$$

Moreover, the inequality *ii)* of the lemma gives :

$$\sum_{i \ge 0} |d\phi_i(x)|^2 \le \sum_{i \in I(x)} |d\phi_i(x)|^2 \le c[c^2/\varepsilon + 4c^2 \varepsilon \|B(x)\|/h]$$

and then

$$\int_{R^d} \sum_{i\ge 0} |d\phi_i|^2 |f|^2 \le \frac{c^3}{\varepsilon} \int_{R^d} |f|^2 + 4c^3 \varepsilon \int_{R^d} \frac{\|B(x)\|}{h} |f|^2 \ .$$

We are going to use the following property :

PROPERTY. — If B satisfies the assumption (B_3), there exists a constant k which depends only on B and on d such that the following inequality holds :

$$\int_{R^d} \frac{\|B(x)\|}{h} |f|^2 \le k \int_{R^d} |\nabla f|^2 \ .$$

The proof of that result is given in the appendix; it uses a calculus explained in [10].

Finally we obtain :

$$\int_{R^d} \sum_{i\geq 0} |\nabla f\phi_i|^2 \leq \frac{c^3}{\varepsilon}\int_{R^d} |f|^2 + (1+4c^3k\varepsilon)\int_{R^d} |\nabla f|^2 \ .$$

b) Link between ∇ *and* ∇_i. — For any function f_i defined on $\widetilde{\Omega_i}$ we have :

$$|\nabla_i f_i|^2 \leq |(\nabla - \nabla_i)f_i|^2 + |\nabla f_i|^2 \ ,$$

so we get, after integration and the use of the inequality *i)* of the lemma, in the same way as precedingly :

$$\sum_{i\geq 0}\int_{\widetilde{\Omega}_i} |\nabla f_i|^2 \geq \sum_{i\geq 0}\int_{\widetilde{\Omega}_i} |\nabla_i f_i|^2 - \varepsilon^2 \sum_{i\geq 1} \|B(x_i)\| \int_{\widetilde{\Omega}_i} |f_i|^2 - \frac{M_0^2 r_0^4}{h^2}\int_{\widetilde{\Omega}_0} |f|^2 \ .$$

c) Using minimax principle for the new cubes. — Let us set $C = 4kc^3$. ¿From *a)* and *b)* we obtain the following minoration :

$$(1+C\varepsilon)\int_{R^d} |\nabla f|^2 \geq \sum_{i\geq 1}\Big(\int_{\widetilde{\Omega}_i} |\nabla_i f\phi_i|^2 - \varepsilon^2\|B(x_i)\|\int_{\widetilde{\Omega}_i} |f\phi_i|^2\Big) - \frac{c^3}{\varepsilon}\int_{R^d} |f|^2 + A_0 \ ,$$

with :

$$A_0 = \int_{\widetilde{\Omega}_0} |\nabla_0 f_0|^2 - \frac{M_0^2 r_0^4}{h^2}\int_{\widetilde{\Omega}_0} |f|^2 \ .$$

We get then, by using the equality : $\sum_{i\geq 1}\int_{\widetilde{\Omega}_i} |f\phi_i|^2 = \int_{R^d} |f|^2$ and the minimax principle :

$$N(\lambda) \leq \sum_{i\geq 1} N_{B(x_i)/h, r_i+a_i}\big[\lambda F + K + G\|B(x_i)\|\big] + N_0$$

with

$$F = 1 + C\varepsilon$$
$$G = \varepsilon^2(1+C\varepsilon)$$
$$K = \frac{c^3}{\varepsilon}(1+C\varepsilon)$$
$$N_0 = N_{B(x_0)/h, r_0+a_0}\big[\lambda F + K + F\frac{M_0^2 r_0^4}{h^2}\big] \ .$$

We obtain therefore the following majoration :

$$N(\lambda) \leq \sum_{i\geq 1}(r_i+a_i)^d \ \nu_{B(x_i)/h}\big[\frac{F\lambda + K}{1-hG}\big] + (r_0+a_0)^d \ \nu_{B(x_0)/h}\big[F(\lambda + \frac{M_0^2 r_0^4}{h^2}) + K\big] \ .$$

Making analogous computations as in the preceding section we can write now :

$$N(\frac{E}{h^2}) \leq (1+0(\sqrt{\varepsilon}))\Big(\sum_{i\geq 1} r_i^d \ \nu_{B(x_i)/h}\big[\frac{E}{h^2}(1+0(h)+0(\varepsilon)) + c_{\varepsilon,h}\big] + r_0^d\nu_0\Big) \ ,$$

with :

$$\nu_0 = \nu_{B(x_0)/h}\left[\frac{1}{h^2}(E+M_0^2r_0^4)(1+0(\varepsilon))+K\right]$$

and $c_{h,\varepsilon} = c^3(1+0(h)+0(\varepsilon))/\varepsilon$, so that :

$$N(\frac{E}{h^2}) \geq (1+0(\sqrt{\varepsilon}))\int_{x\notin\Omega_0}\nu_{B(x)/h}\left[\frac{E}{h^2}(1+0(h)+0(\varepsilon))+c_{\varepsilon,h}\right]$$

$$+\int_{\Omega_0}\nu_{B(x)/h}\left[\frac{1}{h^2}(E+M_0^2r_0^4)(1+0(\varepsilon))+K\right] .$$

As before, we can allow now h to go to 0 and conclude :

$$N_h(E) \leq \int_{R^d}\nu_{B(x)/h}\left[\frac{E}{h^2}(1+0(h)+0(\varepsilon))\right] .$$

3.3. Expression of $N^{as}_{B/h}$.

Let us set : $A_r = \{x\in R^d;\ r(x)=r\}$. We have :

$$N^{as}_{B/h}(\lambda) = \sum_{r=1}^{[d/2]} C_{k,r}\sum_{(n_1,\ldots,n_r)\in Z_r^+}\int_{A_r}(\lambda-\sum(2n_i+1)\frac{b_i(x)}{h})_+^{k/2}\prod_{i=1}^{r}\frac{b_i(x)}{h}dx,$$

so to say :

$$N^{as}_{B/h}(\lambda) = \frac{1}{h^d}\sum_{r=1}^{[d/2]} C_{k,r}\sum_{(n_1,\ldots,n_r)\in Z_r^+}\int_{A_r}(h^2\lambda-\sum(2n_i+1)hb_i(x))_+^{k/2}\prod_{i=1}^{r}hb_i(x)dx .$$

We have then :

$$N^{as}_{B/h}(\lambda) = \frac{1}{h^d}N^{as}_{hB}(h^2\lambda) ,$$

and finally :

$$\frac{1}{h^d}N^{as}_{hB}[E(1-o(1))] \leq N_h(E) \leq \frac{1}{h^d}N^{as}_{hB}[E(1+o(1))] .$$

Appendix

Proof of property 3.2.

Let δ be the application from $R^d\backslash B^{-1}(0)$ into the unit sphere S of $R^{d(d-1)/2}$ defined as follows : $\delta : x \to B/\|B\|$ (B can be seen as an application in $R^{d(d-1)/2}$). We first show that, if B satisfies the assumption (B_3), there exists a function ψ defined in R^d with limit zero at infinity and such that $\|d\delta\|^2 = \psi(x)\|B\|$. The proof consists on computing $\|d\delta\|$: for h in $R^{d(d-1)/2}$ we have :

$$(d\delta)(h) = -\frac{B(x)}{\|B(x)\|^3}\overrightarrow{dB_x(h)}\cdot\overrightarrow{B(x)}+\frac{dB_x(h)}{\|B(x)\|}$$

and we conclude by noticing that assumption (B_3) gives the following inequality : there exists a function ψ defined in R^d with limit zero at infinity and such that $\|dB\| = \psi(x)\|B\|^{3/2}$.

Now we can construct a covering of the sphere S in the following way. We set $m = d(d-1)/2 + 1$, we choose m points $A_1, \ldots, A_m$ of S composing a regular simplex and for $k = 1, \ldots, m$ we consider a smooth function χ_k on S such that :

i) $\chi_k(M) = 1$ si $d(M, A_k) \leq d(M, A_j)$ pour $j = 1, \ldots, m$

ii) $\mathrm{Supp}\chi_k \subset \{M \in S; \overrightarrow{OM} \cdot \overrightarrow{OA_k} > 0$

iii) $\|\chi_k\|_\infty = 1$

Let us set $\eta_k = \chi_k \circ \delta$, and $c_1 = \max\limits_{k=1,\ldots,m} \max\limits_{x\in S} \|\nabla\chi_k(x)\|$, where ∇ denotes the gradient in R^{m-1}. We have : $\|\nabla\eta_k(x)\| \leq c_1\|d\delta(x)\|$.

We denote by Π_j the operator $-i\frac{\partial}{\partial x_j} - \frac{a_j}{h}$ and set $B_k = \overrightarrow{B} \cdot \overrightarrow{OA_k}$. If we see B as a two-form, it means we chose 2 unit vectors e_k and u_k in R^d in order to have : $B_k = F(e_k, u_k) = \sum\limits_{1<i<j\leq d} (\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j})(e_k, u_k)$.

We notice now, according to a work of Simon, that if f is a function C^∞ with compact support in S, the formal relation :$[\Pi_i, \Pi_j] = \frac{i}{h}(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j})$ is justified to yield, by Schwarz inequality :

$$|\int_{R^d} \frac{1}{h}(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j})|f|^2 \leq \|\Pi_i f\|^2 + \|\Pi_j f\|^2$$

(see [1], p.852, th 2.9)

The preceding remark allows us to write the following inequality :

$$\int_{R^d} \sum_{k=1}^{m} \frac{B_k}{h}\eta_k^2|f|^2 \leq m \sum_{j=1}^{d} \int_{R^d} |\Pi_j(\eta_k f)|^2$$

and then, using the additional inequality : $\sum \eta_k^2 \leq m$

$$\int_{R^d} \sum_{k=1}^{m} \frac{B_k}{h}\eta_k^2|f|^2 \leq m \sum_{j=1}^{d} \int_{R^d} |\nabla_j(f)|^2 + c_1^2 \int_{R^d} \|d\delta\|^2|f|^2 \ .$$

Finally we notice that : $\sum\limits_{k=1}^{m} B_k\eta_k^2 \geq c_2\|B\|$, so we conclude :

$$(c_2 - hc_1^2 \max_{x\in R^d} \psi(x)) \int_{R^d} \frac{\|B\|}{h}|f|^2 \leq m \int_{R^d} |\nabla f|^2 \ .$$

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