Scattering theory for graphs isomorphic to a homogeneous tree at infinity

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Summary

- The setup: graphs asymptotic to a homogeneous tree
- The case of a homogeneous tree $\mathbb{T}_q$
  - the spectral decomposition of the adjacency matrix
  - the Fourier-Helgason transform
- A scattering problem for a Schrödinger operator with a compactly supported non local potential
  - Existence and unicity of the generalised eigenfunctions
  - The deformed Fourier-Helgason transform
  - Correlation of scattered plane waves
  - The S-matrix and the asymptotics of the sc. pl. waves
  - Computation of the transmission coefficients
- The spectral theory for a graph asymptotic to $\mathbb{T}_q$

Joint work with Y. Colin de Verdière, Grenoble

The setup

- $\Gamma = (V_\Gamma, E_\Gamma)$: a connected graph
  - V_Γ : the set of vertices, E_Γ : the set of edges
  - We write $x \sim y$ for $\{x, y\} \in E_\Gamma$.
- $q \geq 2$:fixed integer. Γ is asymptotic to a hom. tree of degree $q + 1$ $\Leftrightarrow \exists$ a finite sub-graph Γ_0 of Γ s.t.
  - $\Gamma' := \Gamma \setminus \Gamma_0$ is a disjoint union of a finite number of trees $T_l,\ l = 1, \cdots, L$, rooted at a vertex x_l linked to Γ_0
  - all vertices of T_l different from x_l are of degree $q + 1$.
  - The trees $T_l,\ l = 1, \cdots, L$, are called the ends of Γ.

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  - The trees $T_l, \ l = 1, \cdots, L$, are called the ends of Γ.
- $\partial\Gamma_0 =$ the boundary of Γ_0 : the set of edges of Γ connecting a vertex of Γ_0 to a vertex of Γ', (one of the x_l's).
  $|x|_{\Gamma_0}$: the distance of $x \in V_{\Gamma'}$ to Γ_0.

The adjacency operator

- $C(\Gamma) = \{f : V_\Gamma \longrightarrow \mathbb{C}\}$
- $C_0(\Gamma)$: the subspace of functions with finite support.
- $l^2(\Gamma) = \{f \in C(\Gamma);\ \sum_{x \in V_\Gamma} |f|^2(x) < \infty\}$. $\langle f, g \rangle = \sum_{x \in V_\Gamma} \overline{f(x)}.g(x)$.

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- On $C_0(\Gamma)$, we define the adjacency operator A_Γ by $(A_\Gamma f)\,(x) = \sum_{y \sim x} f\,(y)$
  - A_Γ is bounded on $l^2(\Gamma) \Leftrightarrow$ the degree of the vertices of Γ is bounded. (which is the case here.)
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  - A_Γ is bounded on $l^2(\Gamma) \Leftrightarrow$ the degree of the vertices of Γ is bounded. (which is the case here.)
  - In that case, the operator A_Γ is self-adjoint.
- Our goal : get an explicit spectral decomposition of the adjacency operator A_Γ.
  - get a S. D. for a Schrödinger operator with a compactly supported potential on a hom. tree
  - get a similar S. D. for the adjacency operator A_Γ via a combinatorial result

The points at infinity on the tree $\mathbb{T}_q$

- $\mathbb{T}_q = (V_q, E_q)$: homogeneous tree of degree $q + 1$
  - choose an origin O (a root)
  - $|x|$: the combinatorial distance of the vertex x to O .
- Ω_O : the set of infinite simple paths starting from O.
  - a sequence $y_n \in V_q$ tends to $\omega \in \Omega_O$ iff for n large enough, y_n belongs to the path ω and is going to infinity along that path.

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- $d\sigma_O$: canonical probability measure on Ω_O
- Busemann function $x \to b_\omega(x) := |x_\omega| - d(x, x_\omega)$.
  (x_ω the last point lying on ω in the geodesic path joining O to x)
- level sets of b_ω : horocycles associated to ω.

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Theorem A_0 : the adjacency operator on $\mathbb{T}_q$. The spectrum of A_0 is the interval $I_q = [-2\sqrt{q}, +2\sqrt{q}]$. Set $e_0(x,\omega,s) := q^{(1/2-is)b_\omega(x)}$, and $\lambda_s = q^{\frac{1}{2}+is} + q^{\frac{1}{2}-is}$. Then $\forall s \in S^0$, $\quad A_0 e_0(\omega,s) = \lambda_s e_0(\omega,s)$.

The spectral Riemann surface

- $S := /\tau\mathbb{Z} \times i\mathbb{R}$, $\tau = 2\pi/\log q$
  - $s \to \lambda_s$ holomorphic function defined on S by $\lambda_s = q^{\frac{1}{2}+is} + q^{\frac{1}{2}-is}$.
  - $S^+ := \{s \in S \mid \Im s > 0\}$ is mapped bijectively onto $\mathbb{C} \setminus I_q$.
  - $S^0 := \mathbb{R}/\tau\mathbb{Z}$: the circle $\Im s = 0$.

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G_0 : the Green's function on $\mathbb{T}_q$.

- Theorem
  - The Green's function of the tree $\mathbb{T}_q$ is given, for $s \in S^+$ by $G_0(\lambda_s, x, y) = \frac{q^{(-\frac{1}{2}+is)d(x,y)}}{q^{\frac{1}{2}-is} - q^{-\frac{1}{2}+is}}$.
  - G_0 extends merom. to S with two poles $-i/2$ and $-i/2 + \tau/2$.
  - for any $x \in V_q$ and any y belonging to the path ω,

$$G_0(\lambda_s, x, y) = C(s) q^{(-\frac{1}{2}+is)|y|} q^{(\frac{1}{2}-is)b_\omega(x)},$$

The density of states

- $\phi : \mathbb{R} \to \mathbb{R}$ given continuous function
- $\phi(A_\Gamma)$: operator on $l^2(\Gamma)$, (associated matrix $[\phi(A_\Gamma)](x, x')$)
- Consider for any $x \in V_\Gamma$, the linear form on $C(\mathbb{R}, \mathbb{R})$

$$L_x(\phi) = [\phi(A_\Gamma)](x, x) \ .$$

L_x is positive and verifies $L_x(1) = 1$, so we have $L_x(\phi) = \int_{\mathbb{R}} \phi de_x$ where de_x is a probability measure on $\mathbb{R}$, supported by the spectrum of A_Γ.

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- Theorem The spectral measure de_x of $\mathbb{T}_q$ is independent of the vertex x and is given by

$$de_x(\lambda) := de(\lambda) = \frac{(q+1)\sqrt{4q - \lambda^2}}{2\pi\left((q+1)^2 - \lambda^2\right)} d\lambda$$

The Fourier-Helgason transform

- Definition The Fourier-Helgason transform $\mathcal{FH} : f \to \hat{f}(\omega, s)$ of $f \in C_0(\mathbb{T}_q)$, where $\omega \in \Omega_O$ and $s \in S$, is given by

  $$\hat{f}(\omega, s) = \sum_{x \in V_q} f(x) q^{(1/2+is)b_\omega(x)} .$$

- Remark If $s \in S^0$, then

  $\hat{f}(\omega, s) = \langle e_0(\omega, s), f \rangle = \sum_{x \in V_\Gamma} f\,(x)\, \overline{e_0(x, \omega, s)}$.

  Completeness of the set $\{e_0(\omega, s),\ s \in S^0,\ \omega \in \Omega\}$:

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  $\hat{f}(\omega, s) = \sum_{x \in V_q} f(x) q^{(1/2+is)b_\omega(x)}$.
- **Remark** If $s \in S^0$, then
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  Completeness of the set $\{e_0(\omega, s),\ s \in S^0,\ \omega \in \Omega\}$:
- **Theorem (inversion formula)**
  - For any $f \in C_0(\mathbb{T}_q)$, we have
    $f(x) = \int_{S^0} \int_\Omega e_0(x, \omega, s) \hat{f}(\omega, s) d\sigma_O(\omega) d\mu(s)$
    where $d\mu(s) = \frac{(q+1)\log q}{\pi} \frac{\sin^2(s \log q)}{q+q^{-1}-2\cos(2s\log q)} |ds|$.
  - $\mathcal{FH}$ extends to a u. map from $l^2(\mathbb{T}_q)$ into $L^2(\Omega \times S^0, d\sigma_O \otimes d\mu)$.
  - its range is the subsp. of the f. F of $L^2(\Omega \times S^0, d\sigma_O \otimes d\mu)$ s.t.
    $\int_\Omega e_0(x, \omega, s) F(\omega, s) d\sigma_O(\omega) = \int_\Omega e_0(x, \omega, -s) F(\omega, -s) d\sigma_O(\omega)$.
  - Spectral resolution of A_0: if $\phi : \mathbb{R} \to \mathbb{R}$ is continuous,
    $\phi(A_0) = (\mathcal{FH})^{-1} \phi(\lambda_s) \mathcal{FH}$.

Scattering on $\mathbb{T}_q$ between A_0 and the Schrödinger operator $A = A_0 + W$

- the Hermitian matrix (also denoted W) assoc. to this potential is supported by $K \times K$ (K : a finite part of V_q)
- K is chosen minimal, so that: $K = \{x \in V_q \mid \exists y \in V_q \text{ with } W_{x,y} \neq 0\}$.
- A is a finite rank perturbation of A_0 .

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- A is a finite rank perturbation of A_0 .
- Proposition $\qquad l^2(\mathbb{T}_q) = \mathcal{H}_{\rm ac} \oplus \mathcal{H}_{\rm pp}$
  - $\mathcal{H}_{\rm ac}$ is the isometric image of $l^2(\mathbb{T}_q)$ by the wave operator $\Omega^+ = s - \lim_{t \to -\infty} e^{itA} e^{-itA_0}$. We have $A_{|\mathcal{H}_{\rm ac}} = \Omega^+ A_0 (\Omega^+)^\star \implies$ the corresponding part of the S.D. is isomorphic to that of A_0 which is an a. c. spectrum on I_q.
  - The space $\mathcal{H}_{\rm pp}$ is finite dimensional, admits an o.b. of l^2 eigenf. associated to a finite set of eigenv. (*Some of them can be embedded in the continuous spectrum* I_q.)

Formal derivation of the Lippmann-Schwinger equation

- We look for generalised eigenfunctions of A.
- they are particular solutions of
  $(\lambda_s - A)e(.,\omega,s) = 0$,
  (meaning not l^2 solutions, but only point-wise solutions.)

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If $e(\omega,s)$ is the image of $e_0(\omega,s)$ by Ω^+ in some sense (they are not in l^2!), then we should have formally $e_0(\omega,s) = \lim_{t\to-\infty} e^{itA_0}e^{-itA}e(\omega,s)$

$$= \lim_{t\to-\infty}[e(\omega,s) - i\int_0^t e^{iuA_0}We^{-iuA}e(\omega,s)du]$$

$$= e(\omega,s) - i\lim_{\varepsilon\to0}\int_0^{-\infty} e^{iuA_0}We^{-iu\lambda_s}e^{\varepsilon u}e(\omega,s)du$$

$$= e(\omega,s) + \lim_{\varepsilon\to0}[(A_0 - (\lambda_s + i\varepsilon))^{-1}We](\omega,s) \ .$$

So $e(\omega,s)$ should obey the following "Lippmann-Schwinger-type" equation

$$e(\omega,s) = e_0(\omega,s) + G_0(\lambda_s)We(\omega,s) \ .$$

- $\chi \in C_0(\mathbb{T}_q)$ be a compactly supported real-valued function s. t. $W\chi = \chi W = W$
- If $e(\omega, s)$ obeys (LSE) and $a(\omega, s) = \chi e(\omega, s)$, then a obeys (MLSE): $a(\omega, s) = \chi e_0(\omega, s) + \chi G_0(\lambda_s) W a(\omega, s)$.
  - K_s : the finite rank op. on $l^2(\mathbb{T}_q)$ defined by $K_s = \chi G_0(\lambda_s) W$. The map $s \to K_s$ extends holom. to $\Im s > -\frac{1}{2}$
  - analytic Fredholm theorem $\Longrightarrow$ $\exists$ a finite subset $\hat{\mathcal{E}}$ of S^0, defined by $\hat{\mathcal{E}} =: \{s \in S^0; \ker(\mathrm{Id} - K_s) \neq 0\}$, so that (MLSE) has a unique solution $a(\omega, s) \in C_0(\mathbb{T}_q)$ whenever $s \notin \hat{\mathcal{E}}$.

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For $s \notin \hat{\mathcal{E}}$, the function $e(\omega, s) = e_0(\omega, s) + G_0(\lambda_s) W a(\omega, s)$ is the unique solution of (LSE).

The set $\hat{\mathcal{E}}$ and the pure point spectrum

Propositions

- The set $\hat{\mathcal{E}}$ is independent of the choice of χ with $W\chi = \chi W = W$.
- If $(A - \lambda)f = 0$ with $\lambda \in I_q$ and $f \in l^2(\mathbb{T}_q)$, then $\mathrm{Supp}(f) \subset \hat{K}$
  $\hat{K}$: the smallest subset of V_q s. t. $\mathrm{Supp}(W) \subset \hat{K} \times \hat{K}$ and all connected components of $\mathbb{T}_q \setminus \hat{K}$ are infinite.
- Consequence $\#\{\sigma_{\mathrm{pp}}(A) \cap I_q\} \leq \#\hat{K}$.
- If $s \in S^0$, $(A - \lambda_s)f = 0$ and $f \in l^2(\mathbb{T}_q) \setminus 0$, then $s \in \hat{\mathcal{E}}$.
- Conversely, if $s \in \hat{\mathcal{E}} \subset S^0$, $\exists f \neq 0$ s. t. $(A - \lambda_s)f = 0$ and $f(x) = O\left(q^{-|x|/2}\right)$.

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Theorem The pure point spectrum $\sigma_{\mathrm{pp}}(A)$ of A splits into 3 parts

$\sigma_{\mathrm{pp}}(A) = \sigma_{\mathrm{pp}}^{-}(A) \cup \sigma_{\mathrm{pp}}^{+}(A) \cup \sigma_{\mathrm{pp}}^{0}(A)$

where $\sigma_{\mathrm{pp}}^{-}(A) = \sigma_{\mathrm{pp}}(A) \cap]-\infty, -2\sqrt{q}[$, $\sigma_{\mathrm{pp}}^{+}(A) = \sigma_{\mathrm{pp}}(A) \cap]2\sqrt{q}, +\infty[$, *and* $\sigma_{\mathrm{pp}}^{0}(A) = \sigma_{\mathrm{pp}}(A) \cap I_q$.

We have $\#\sigma_{\mathrm{pp}}^{\pm}(A) \leq \#\mathrm{Supp}(W)$ and $\#\sigma_{\mathrm{pp}}^{0}(A) \leq \#\hat{K}$.

The deformed Fourier-Helgason transform

- **Definition** The deformed Fourier-Helgason transform $\mathcal{FH}_{\rm sc}$ of $f \in C_0(\mathbb{T}_q)$ is the function $\hat{f}_{\rm sc}$ on $\Omega \times (S^0 \setminus \hat{\mathcal{E}})$ defined by

$$\hat{f}_{\rm sc}(\omega, s) = \langle e(\omega, s), f \rangle = \sum_{x \in V_\Gamma} f\,(x)\, \overline{e(x, \omega, s)}\,.$$

- **Remark** Since $K_s = K_{-s}$, the subset $\hat{\mathcal{E}}$ is invariant by $s \to -s$ and consequently is the inverse image by $s \to \lambda_s$ of a subset of I_q which we denote by $\mathcal{E}$.

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- **Theorem (inversion formula)**
  - $f \in C_0(\mathbb{T}_q)$, $J \subset I_q \setminus \mathcal{E}$ any closed interval
  - denote by $\hat{J}$ the inverse image of J by $s \to \lambda_s$,
  - then the following inverse transform holds
    $P_J f(x) = \int_{\hat{J}} \int_\Omega e(x, \omega, s) \hat{f}_{\rm sc}(\omega, s) d\sigma_O(\omega) d\mu(s)\;.$
  - Moreover $f \to \hat{f}_{\rm sc}$ extends to an isometry from $\mathcal{H}_{\rm ac}$ onto $L^2_{\rm even}(\Omega \times S^0, d\sigma_O \otimes d\mu)$.

Correlation of scattered plane waves

- Motivation: passive imaging in seismology (M. Campillo's seismology group in Grenoble).
  For a scattering problem in $\mathbb{R}^d$ the point-to-point correlations of the plane waves can be computed in terms of the Green's function (Y. C.d.V, '09): for a fixed spectral parameter, plane waves are viewed as random waves parametrised by the direction of their incoming part.

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- Consider the plane wave $e(x,\omega,s(\lambda))$ as a random wave
- Define the point-to-point correlation $C_\lambda^{sc}(x,y)$ of such a random wave in the usual way:

$$C_\lambda^{sc}(x,y) = \int_\Omega \overline{e(x,\omega,s(\lambda))} e(y,\omega,s(\lambda))\, d\sigma(\omega) \ .$$

- Theorem For any $\lambda \in I_q$ and any vertices x, y

$$C_\lambda^{sc}(x,y) = -\frac{2(q^2+2q+1-\lambda^2)}{(q+1)\sqrt{4q-\lambda^2}} \Im G(\lambda+i0,x,y) \ .$$

The S-matrix and the asymptotics of the deformed plane waves

The Lippmann-Schwinger eigenfunctions $e(x, \omega, s)$ are especially useful to describe the so-called $S-$matrix ($S = (\Omega^{-})^{*}\Omega^{+}$).

- For any f and $g \in C_0(\mathbb{T}_q)$

  $(f, (S - I)g) = -2\pi i \int_{S^0 \times S^0} \int_{\Omega \times \Omega} T(\omega, s; \omega', s') \overline{\hat{f}(\omega, s)} \delta(\lambda_s - \lambda_{s'}) \hat{g}(\omega', s') d\Sigma$

  - $d\Sigma = d\sigma_O(\omega) d\mu(s) d\sigma_O(\omega') d\mu(s')$
  - $T(\omega, s; \omega', s') = \langle e(\omega', s'), We_0(\omega, s)\rangle = \sum_{(x,y)} e(x, \omega', s') \overline{W(x, y) e_0(y, \omega, s)}$.

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  - $d\Sigma = d\sigma_O(\omega) d\mu(s) d\sigma_O(\omega') d\mu(s')$
  - $T(\omega, s; \omega', s') = \langle e(\omega', s'), We_0(\omega, s) \rangle = \sum_{(x,y)} e(x, \omega', s') \overline{W(x,y) e_0(y, \omega, s)}$.
- This can be written symbolically as

$$S(\omega, s; \omega', s') = \delta(s - s') - 2\pi i T(\omega, s; \omega', s') \delta(\lambda_s - \lambda_{s'}) \ .$$

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  $(f,(S-I)g) = -2\pi i \int_{S^0\times S^0}\int_{\Omega\times\Omega} T(\omega,s;\omega',s')\overline{\hat{f}(\omega,s)}\delta(\lambda_s-\lambda_{s'})\hat{g}(\omega',s')d\Sigma$
  - $d\Sigma = d\sigma_O(\omega)d\mu(s)d\sigma_O(\omega')d\mu(s')$
  - $T(\omega,s;\omega',s') = \langle e(\omega',s'),We_0(\omega,s)\rangle = \sum_{(x,y)} e(x,\omega',s')\overline{W(x,y)e_0(y,\omega,s)}$.
- This can be written symbolically as

$$S(\omega,s;\omega',s') = \delta(s-s') - 2\pi i T(\omega,s;\omega',s')\delta(\lambda_s-\lambda_{s'}) .$$

- There exist "transmission coefficients" $\tau(s,\omega,\omega')$ so that

$$e(x;\omega,s) = e_0(x;\omega,s) + \tau(s,\omega,\omega')q^{(-\frac{1}{2}+is)|x|}$$

for any x close enough to ω' , $\tau(s,\omega,\omega') = -\frac{C(s)}{2i\pi}S(\omega',-s;\omega,s)$
with $C(s)^{-1} = q^{\frac{1}{2}-is} - q^{-\frac{1}{2}+is}$.

Computation of the transmission coefficients in terms of the Dirichlet-to Neumann operator

- The functions $b_\omega(y)$ and $b_{\omega'}(y)$ are equal if ω and ω' belong to the same end of $\mathbb{T}_q \setminus K$.
- $\Longrightarrow$ the function $\omega' \to \tau(s,\omega,\omega')$ is constant in each end of $\mathbb{T}_q \setminus K$
- $\Longrightarrow$ the tr. coeff. $\tau(s,\omega,\omega')$ can be written as a function $\tau(s,\omega,l)$.

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- $\Longrightarrow$ the tr. coeff. $\tau(s, \omega, \omega')$ can be written as a function $\tau(s, \omega, l)$.
- Moreover the reduced Lippmann-Schwinger equation depends only on the restriction of e_0 to K
- $\Longrightarrow$ the function $\omega \to \tau(s, \omega, l)$ is also constant in each end of $\mathbb{T}_q \setminus K$.
- Finally, we get an $L \times L$ matrix depending on s, denoted by

$$\tilde{S}(s) = (S(l', -s, l, s))_{l,l'} = -\frac{2i\pi}{C(s)} (\tau(s, l, l'))_{l,l'}$$

.

Theorem

Consider n : the integer so that B_{n-2} is the smallest ball containing the finite graph K.

Set $\Gamma = B_n$, $\partial\Gamma = \{x_{l'}, 1 \leq l' \leq L\}$.

Set $\widehat{A_n}$: the restriction of A to B_n ($\widehat{A_n} = (A_{x,y})_{(x,y)\in B_n})$

define I_n in the same way,

set $B = \widehat{A_n} - \lambda_s I_n$

Consider $\mathcal{DN}_s$: the corresponding Dirichlet-to Neumann operator .

Then $\mathcal{DN}_s$ and the transmission vector

$\overrightarrow{\tau(s,l)} := (\tau(s,l,1), \cdots, \tau(s,l,l'), \cdots, \tau(s,l,L))$ exist for any

$$s \notin \mathcal{E}_0 = \{s \in S^0 \, , \lambda_s \in \sigma(\widehat{A}_{n-1})\}$$

and

$$(\tau(s,l,l')) = -\alpha^{-2n}\left[\frac{1}{C(s)}\left(\mathcal{DN}_s + q^{1/2+is} I\right)^{-1} + \mathcal{A}\right] ,$$

with $\widehat{A}_{n-1} = (A_{x,y})_{(x,y)\in B_{n-1}}$, $\mathcal{A} = (\mathcal{A}_{l,l'}) = (\alpha^{d(x_l,x_{l'})}), \quad \alpha = q^{-1/2+is}$.

The Dirichlet-to Neumann operator $\mathcal{DN}$ on a finite graph

$\Gamma = (V, E)$: a connected finite graph
$\partial\Gamma$: a subset of V called the "boundary of Γ".
$B = (b_{i,j}) : \mathbb{R}^V \to \mathbb{R}^V$: a sym. matrix assoc. to Γ, namely

$$b_{i,j} = 0 \quad \text{if } i \neq j \ \text{ and } \quad \{i, j\} \notin E.$$

Set $V_0 = V \setminus \partial\Gamma$, define $B_0 : \mathbb{R}^{V_0} \to \mathbb{R}^{V_0}$ as the restriction of B to the functions which vanish on $\partial\Gamma$.

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Lemma

Assume B_0 invertible. Then, $\forall f \in C(\partial\Gamma)$, $\exists$ a unique solution $F \in C(\Gamma)$ of the Dirichlet problem

$$(D_f) \ : \ F_{|\partial\Gamma} = f \text{ and } BF(l) = 0 \text{ if } l \in V_0 \ .$$

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The Dirichlet-to Neumann operator $\mathcal{DN}$ associated to B is the linear operator from $C(\partial\Gamma)$ to $C(\partial\Gamma)$ defined as follows:
if $l \in \partial\Gamma$,

$$\mathcal{DN}(f)(l) = \sum_{i=1}^{m} b_{l,i} F(i) (= BF(l)) \ .$$

The spectral theory for a graph asymptotic to an homogeneous tree

Some combinatorics

Theorem 1

If Γ is asymptotic to a homogeneous tree of degree $q+1$, then Γ is isomorphic to a connected component of a graph $\hat{\Gamma}$ which can be obtained from $\mathbb{T}_q$ by adding and removing a finite number of edges.

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Tools

- a combinatorial analogue of the reg. total curvature of a Riem. surface S

$$\nu(\Gamma) = \sum_{x \in V_\Gamma} (q + 1 - d(x)) + 2b_1 \ ,$$

$d(x)$: the degree of x, b_1 : the first Betti number of Γ

- Lemma 1 If, for $r \geq 2$, $B_r = \{x \in V_\Gamma \mid |x|_{\Gamma_0} \leq r\}$, then

$$\nu(\Gamma) = (q-1)m - M + 2 \ ,$$

(m : number of inner vertices of B_r, M :number of boundary vertices)

Lemma 2

F : a finite tree whose all vertices are of degree $q+1$ except the ends which are of degree 1.
M number of ends, m the number of inner vertices.
We have

$$M = 2 + (q-1)m \ . \quad (1)$$

Conversely, for each choice of (m, M) satisfying Equation (1), there exists such a tree F.

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- **Some modifications of Γ in order to get a new graph $\hat{\Gamma}$ with $\nu(\hat{\Gamma}) = 0$.**

  Lemma 3

  If $\Gamma' = M_1(\Gamma)$ is defined by adding to Γ a vertex and an edge connecting that vertex to a vertex of Γ_0, then

  $$\nu(\Gamma') = \nu(\Gamma) + q - 1\ .$$

  If $\Gamma' = M_2(\Gamma)$ is defined by adding to Γ a tree whose root x is of degree q and all other vertices of degree $q+1$ and connecting x by an edge to a vertex of Γ_0, Γ' is asymptotic to an homogeneous tree of degree $q+1$ and

  $$\nu(\Gamma') = \nu(\Gamma) - 1\ .$$

The spectral theory of Γ

- Theorem (1) $\Longrightarrow$ existence of a Hilbert space $\mathcal{H}$ so that
  - $l^2(\hat{\Gamma}) = l^2(\Gamma) \oplus \mathcal{H}$
  - this decomposition is invariant by $A_{\hat{\Gamma}}$.
- Moreover $A_{\hat{\Gamma}}$ is a finite rank perturbation of $A_0 = A_{\mathbb{T}_q}$. $\Longrightarrow$ this gives the spectral theory of A_Γ by using the results of the preceding section .

The spectral theory of Γ

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  - this decomposition is invariant by $A_{\hat{\Gamma}}$.
- Moreover $A_{\hat{\Gamma}}$ is a finite rank perturbation of $A_0 = A_{\mathbb{T}_q}$. $\Longrightarrow$ this gives the spectral theory of A_Γ by using the results of the preceding section .
- Lemma 4 Let $A_{\hat{\Gamma}} = A_{\mathbb{T}_q} + W$ with $\mathrm{Support}(W) \subset K \times K$ and K finite. Let Γ be an unbounded connected component of $\hat{\Gamma}$ and ω a point at infinity of Γ. Then, for any $s \notin \hat{\mathcal{E}}$, we have

  $$\mathrm{support}(e(.;s,\omega)) \subset V_\Gamma \ .$$

  Conversely, if ω' is a point at infinity of $\hat{\Gamma}$ which is not a point at infinity of Γ then

  $$\mathrm{support}(e(.;s,\omega')) \cap V_\Gamma = \emptyset \ .$$

- Theorem 2 The Hilbert space $l^2(\Gamma)$ splits into a finite dimensional part $\mathcal{H}_{\mathrm{pp}}$ and an absolutely continuous part $\mathcal{H}_{\mathrm{ac}}$. This decomposition is preserved by A_Γ. If $f \in C_0(\Gamma)$ and, for $\omega \in \Omega$, $\hat{f}_{sc}(s,\omega) = \langle f | e(.;s,\omega)\rangle$, then the map $f \to \hat{f}_{sc}$ extends to an isometry from $\mathcal{H}_{\mathrm{ac}}$ onto $L^2_{\mathrm{even}}(S_0 \times \Omega, d\sigma_0 \otimes d\mu)$ which intertwines the action of A_Γ with the multiplication by λ_s.