

**COUNTING FUNCTION OF EMBEDDED
EIGENVALUES FOR SOME MANIFOLDS WITH
CUSPS, AND MAGNETIC LAPLACIAN**

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Outline

- The hyperbolic context
- The Poincaré half-plane
 - The constant magnetic Laplacian
 - Related results
- Geometrically finite hyperbolic surfaces
 - The essential spectrum of constant magnetic Laplacians
 - The Weyl formula (case of finite area)
- Higher dimensions: manifolds with cusps
 - The essential spectrum of constant magnetic Laplacians
 - The Weyl formula with remainder
 - Estimate on the embedded values of the Laplacian
 - Outline of proofs

Joint work with Abderemane Morame, University of Nantes

Framework

• Let (\mathcal{M}, g) be a connected and oriented Riemannian manifold of dimension n .

• For any real one-form A on \mathcal{M} , define

$$-\Delta_A = (i d + A)^*(i d + A), \quad ((i d + A)u = i du + uA, \quad \forall u \in C_0^\infty(\mathcal{M})) .$$

• The magnetic field is the two-form dA .

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• The magnetic field is the two-form dA .

• To dA is associated the linear operator B defined on the tangent space by $dA(X, Y) = g(B.X, Y)$; $\forall X, Y \in T\mathcal{M} \times T\mathcal{M}$.

• The magnetic intensity \mathbf{b} is given by $\mathbf{b} = \frac{1}{2} \text{tr} \left((B^* B)^{1/2} \right)$.

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● If $\dim(\mathcal{M}) = 2$, then

● $dA = \tilde{\mathbf{b}} dv$, with $|\tilde{\mathbf{b}}| = \mathbf{b}$,

dv the Riemannian measure on \mathcal{M} .

● The magnetic field is constant $\iff \tilde{\mathbf{b}}$ is constant.

The Poincaré half-plane

Let $\mathcal{M} = \mathbb{H}$ be the hyperbolic half-plane :

$$\mathbb{H} = \mathbb{R} \times]0, +\infty[, \quad g = \frac{dx^2 + dy^2}{y^2} .$$

● $-\Delta_A = y^2(D_x - A_1)^2 + y^2(D_y - A_2)^2 ,$
with $A = A_1(x, y) dx + A_2(x, y) dy$, and $A_j(x, y) \in C^\infty(\mathbb{H}; \mathbb{R})$,

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● $\tilde{\mathbf{b}} = y^2 (\partial_x A_2 - \partial_y A_1)$

● $\mathbf{b} = |\tilde{\mathbf{b}}| ,$

● $dv = y^{-2} dx dy .$

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- $dv = y^{-2} dx dy$.

Properties

- $-\Delta_A$ is essentially self-adjoint on $L^2(\mathbb{H})$.

- We are interested on its spectrum : $\text{sp}(-\Delta_A)$.

Gauge invariance:

$$\text{sp}(-\Delta_A) = \text{sp}(-\Delta_{A+d\varphi}) ; \quad \forall \varphi \in C^\infty(\mathbb{H}; \mathbb{R}) .$$

The spectrum for constant magnetic field

- The spectrum of $-\Delta_{A\mathbf{b}}$ is essential:

$$\text{sp}(-\Delta_{A\mathbf{b}}) = \text{sp}_{ac}(-\Delta_{A\mathbf{b}}) \cup S(\mathbf{b}) .$$

- Its absolutely continuous part is given by

$$\text{sp}_{ac}(-\Delta_{A\mathbf{b}}) = [\mathbf{b}^2 + \frac{1}{4}, +\infty[.$$

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 $\text{sp}_{ac}(-\Delta_{A\mathbf{b}}) = [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.
- If $0 \leq \mathbf{b} \leq 1/2$ the remaining part $S(\mathbf{b})$ is empty,
- if $\mathbf{b} > 1/2$ it is formed by a **finite** number of eigenvalues of **infinite** multiplicity given by

$$S(\mathbf{b}) = \left\{ (2j+1)\mathbf{b} - j(j+1) ; j \in \mathbb{N}, j < \mathbf{b} - \frac{1}{2} \right\}.$$

Related results : hyperbolic context

- Results on the hyperbolic space
 - Maass Laplacian: Elstrodt(73), Grosche(88), Comtet(87), Ikeda-Matsumoto (99), Kim-Lee(02)
 - Asympt. constant magnetic fields ,Pauli operators : Inahama-Shirai (03)
 - Asymptotic distribution for Schrödinger operators : Inahama-Shirai (04)
 - Asymptotic distribution for magnetic bottles : Morame-Truc (08)
- conformally cusp manifolds :
Asymptotic distribution for Schrödinger operators :
Golénia-Moroianu (08)
- Geometrically finite hyperbolic surfaces
Asymptotic distribution for magnetic bottles : Morame-Truc (09)

Geometrically finite hyperbolic surface of infinite area

● Definition

(\mathcal{M}, g) : a smooth connected Riemannian manifold of dimension

$$\text{two } \mathcal{M} = \left(\bigcup_{j=0}^{J_1} M_j \right) \cup \left(\bigcup_{k=1}^{J_2} F_k \right) ;$$

- M_j , F_k open sets of \mathcal{M} , M_0 compact closure,
- ($j \neq 0$: M_j isometric to $\mathbb{S} \times]a_j^2, +\infty[$, (cuspidal ends)

$$ds_j^2 = y^{-2} (L_j^2 d\theta^2 + dy^2)$$

(a_j and L_j are strictly positive constants)

- F_k isometric to $\mathbb{S} \times]\alpha_k^2, +\infty[$, (funnel ends)

$$ds_k^2 = \tau_k^2 \cosh^2 t d\theta^2 + dt^2$$

(α_k and τ_k are strictly positive constants) .

Constant magnetic field on \mathcal{M}

- $\mathcal{M} = \left(\bigcup_{j=0}^{J_1} M_j \right) \cup \left(\bigcup_{k=1}^{J_2} F_k \right) ;$

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- $\forall (b, \beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2} , \exists A ,$ s. t. dA satisfies

$$dA = \tilde{\mathbf{b}}(z) dm \quad \begin{cases} \tilde{\mathbf{b}}(z) = b_j \quad \forall z \in M_j \\ \tilde{\mathbf{b}}(z) = \beta_k \quad \forall z \in F_k \end{cases}$$

The essential spectrum of magnetic Laplacians (1)

Theorem 1

• If $J_1 = 0$ and $J_2 > 0$,

$$\text{sp}_{\text{ess}}(-\Delta_A) = \left[\frac{1}{4} + \inf_k \beta_k^2, +\infty \right] \cup \left(\bigcup_{k=1}^{J_2} S(\beta_k) \right).$$

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- if $\beta_k > 1/2$ it is formed by a **finite** number of eigenvalues of **infinite** multiplicity given by

$$S(\beta_k) = \left\{ (2j+1)\beta_k - j(j+1); j \in \mathbb{N}, j < \beta_k - \frac{1}{2} \right\}.$$

The essential spectrum of magnetic Laplacians (2)

If $J_1 > 0$, then

- $\forall j, 1 \leq j \leq J_1$ and $\forall z \in M_j$
 \exists a unique closed curve $C_{j,z}$ through z , in (M_j, g) , not contractible and with zero g -curvature.
- The following limit exists and is finite:

$$[\mathbf{A}]_{M_j} = \lim_{d(z) \rightarrow +\infty} \int_{C_{j,z}} A.$$

- Define : $J_1^A = \{j \in \mathbb{N}, 1 \leq j \leq J_1 \text{ s.t. } [\mathbf{A}]_{M_j} \in 2\pi\mathbb{Z}\}.$

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Theorem 2

Assume that $J_1 > 0$, $J_2 > 0$, and $J_1^A \neq \emptyset$. Then

- $\text{sp}_{\text{ess}}(-\Delta_A) =$
 $\left[\frac{1}{4} + \min \left\{ \inf_{j \in J_1^A} b_j^2, \inf_{1 \leq k \leq J_2} \beta_k^2 \right\}, +\infty \right] \cup \left(\bigcup_{k=1}^{J_2} S(\beta_k) \right).$

Weyl law for \mathcal{M} of finite area, with a non-integer class 1-form A

● Theorem 3

Consider a geometrically finite hyperbolic surface (\mathcal{M}, g) of finite area, ($J_2 = 0$), and assume that $J_1^A = \emptyset$. Then

- (i) $\text{sp}_{ess}(-\Delta_A) = \emptyset$:
 $-\Delta_A$ has **purely discrete spectrum**, (its resolvent is compact).
- (ii) $N(\lambda, -\Delta_A) = \lambda \frac{|\mathcal{M}|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda)$.

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● (Definitions) In the case when $\text{sp}(-\Delta_A)$ is **discrete**,

- denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue repeated according to its multiplicity)
- define $N(\lambda, -\Delta_A) = \sum_{\lambda_j < \lambda} 1$.
- **Weyl law** for the **Laplacian** on a **compact** n-dim Riemannian manifold M

$$N(\lambda, -\Delta_0^M) = \lambda^{n/2} \frac{|M|}{(4\pi)^{n/2} \Gamma(n/2 + 1)} + o(\lambda^{n/2}).$$

Remarks (1)

- Theorem 3 relies on the **Proposition (Weyl law for one cusp)**

- Consider $M = \mathbb{S} \times]\alpha^2, +\infty[$ equipped with the metric $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ for some $\alpha > 0$ and $L > 0$.

- Denote by $-\Delta_A^M$ the Dirichlet op. on M , ass. to $-\Delta_A$.

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$$N(\lambda, -\Delta_A^M) = \lambda \frac{|M|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda).$$

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- **Stability under perturbation of the metric**

Theorem 3 still holds if the metric of \mathcal{M} is modified in a compact set.

- One can see the case $J^A = \emptyset$ as an **Aharonov-Bohm phenomenon** :

- the magnetic field dA is not sufficient to describe $-\Delta_A$

- the use of the magnetic potential A is essential :

one can have magnetic bottle (magnetic Laplacian with compact resolvent) **with null intensity**.

Remarks (2)

- When $A = 0$, $-\Delta = -\Delta_0$ has **embedded** eigenvalues in its **essential** spectrum : $(\text{sp}_{\text{ess}}(-\Delta) = [\frac{1}{4}, +\infty[)$.
- Denote by $N_{\text{ess}}(\lambda, -\Delta)$ the number of these eigenvalues in $[\frac{1}{4}, \lambda[$, then
- $N_{\text{ess}}(\lambda, -\Delta) \leq \lambda \frac{|\mathcal{M}|}{4\pi}$; (Colin de Verdière, Hejhal)

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• Locally symmetric spaces and automorphic forms

Consider $\mathcal{M} = \Gamma(N) \backslash \mathbb{H}$, with

$$\Gamma(N) = \{\gamma \in SL(2, \mathbb{Z}) : \gamma = Id \text{ mod } N\}$$

then (Müller '07)

$$N_{\text{ess}}(\lambda, -\Delta) = \lambda \frac{|\mathcal{M}|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda).$$

Remarks (3)

- Definition

A **Maass automorphic form** is a smooth function $\mathbb{H} \rightarrow \mathbb{C}$ s.t.

- $f(\gamma z) = f(z) \quad \forall \gamma \in \Gamma(N)$
- $\exists \lambda$ s.t. $\Delta f = \lambda f$
- f is slowly increasing

- Exemples : Eisenstein series, cusp forms

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● **Selberg formula** (derived from the trace formula)

For any lattice $\Gamma \in SL(2, \mathbb{R})$:

$$N_{\Gamma}(\lambda, -\Delta) + M_{\Gamma}(\lambda, -\Delta) \sim \lambda \frac{|\Gamma \backslash \mathbb{H}|}{4\pi}.$$

- $M_{\Gamma}(\lambda, -\Delta)$: winding number of the determinant of the scattering matrix (given by the zeroth Fourier coefficients of the Eisenstein series)

Higher dimensions

(\mathcal{M}, g) : a smooth connected n -dim. Riemannian manifold

$$\mathcal{M} = M_0 \cup \left(\bigcup_{j=1}^J M_j \right) ;$$

- $M_j \cap M_k = \emptyset$,
- M_j open sets of \mathcal{M} , M_0 compact closure,
- $\forall j > 1, \exists$ a closed compact $(n - 1)$ -dim. Riem. manifold (X_j, \mathbf{h}_j) s.t. M_j is isometric to $X_j \times]a_j^2, +\infty[$, $(a_j > 0)$ (M_j $j \neq 0$ **cuspidal ends**)

$$ds_j^2 = y^{-2\delta_j} (\mathbf{h}_j + dy^2) ; \quad (1/n < \delta_j \leq 1)$$

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$$ds_j^2 = y^{-2\delta_j} (\mathbf{h}_j + dy^2) ; \quad (1/n < \delta_j \leq 1)$$

$\implies \forall j > 1, \exists$ a smooth real one-form $A_j \in T^*(X_j)$, s.t.

- $dA_j \neq 0$ or
- $dA_j = 0$ and $[A_j]$ is not integer

(\exists a smooth closed curve γ in X_j such that $\int_{\gamma} A_j \notin 2\pi\mathbb{Z}$.)

- One can find $A \in T^*(\mathcal{M})$ s.t.
 $\forall j, 1 \leq j \leq J, \quad A = A_j \quad \text{on } M_j .$
- Define the magnetic Laplacian,
$$-\Delta_A = (i d + A)^*(i d + A) ,$$
$$(i d + A)u = i du + uA , \quad \forall u \in C_0^\infty(\mathcal{M}; \mathbb{C})$$

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• \mathcal{M} complete metric space

\implies by Hopf-Rinow theorem \mathcal{M} is geodesically complete

$\implies -\Delta_A$ has a unique self-adjoint extension on $L^2(\mathcal{M})$.

The essential spectrum

● Theorem 4 (The case $A = 0$)

Under the above assumptions on \mathcal{M} ,

the **essential** spectrum of $-\Delta = -\Delta_0$ on \mathcal{M} is given by

● $\text{sp}_{\text{ess}}(-\Delta) = [0, +\infty[$ if $1/n < \delta < 1$

● $\text{sp}_{\text{ess}}(-\Delta) = \left[\frac{(n-1)^2}{4}, +\infty[$ if $\delta = 1$.

● $\delta = \min_{1 \leq j \leq J} \delta_j$.

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● Theorem 5 (The case $A \neq 0$)

Under the above assumptions on \mathcal{M} and A ,

- the magnetic Laplacian $-\Delta_A$ has a **compact resolvent**.
- $\text{sp}(-\Delta_A) = \text{sp}_d(-\Delta_A) = \{\lambda_j, j \in \mathbb{N}^*\}$ with
 - $\lambda_j \leq \lambda_{j+1}$
 - $\lim_{j \rightarrow +\infty} \lambda_j = +\infty$,
 - $\lambda_0 > 0$.
- the sequence of normalized eigenfunctions $(\varphi_j)_{j \in \mathbb{N}^*}$ is a Hilbert basis of $L^2(\mathcal{M})$.

Weyl formula

● Theorem 6 (The case $A \neq 0$)

Under the above assumptions on \mathcal{M} and on A ,
we have the Weyl formula with remainder as $\lambda \rightarrow +\infty$

$$N(\lambda, -\Delta_A) = |\mathcal{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} + \mathbf{O}(r(\lambda)),$$

$$\text{with } r(\lambda) = \begin{cases} \lambda^{(n-1)/2} \ln(\lambda), & \text{if } 1/(n-1) \leq \delta \\ \lambda^{1/(2\delta)}, & \text{if } 1/n < \delta < 1/(n-1) \end{cases}$$

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$$\bullet \delta = \min_{1 \leq j \leq J} \delta_j,$$

• $|\mathcal{M}|$ is the Riemannian measure of \mathcal{M}

• ω_d the euclidian volume of the unit ball of \mathbb{R}^d : $\omega_d = \frac{\pi^{d/2}}{\Gamma(1 + \frac{d}{2})}$.

● Remarks

- The asymptotic formula is given without remainder by Golenia-Moroianu (in a larger context),
- **Theorem 3** is a particular case of **Theorem 6** (for $n = 2$ and $\delta_j = 1$ for any $1 \leq j \leq J$).

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● Consequences

- The Laplace-Beltrami operator $-\Delta = -\Delta_0$ may have **embedded** eigenvalues in its **essential** spectrum $\text{sp}_{\text{ess}}(-\Delta)$.
- Denote by $N_{\text{ess}}(\lambda, -\Delta)$ the number of eigenvalues of $-\Delta$, (counted according to their multiplicity), less than λ .
- **Theorem 6** can be mimicked to get an upperbound for $N_{\text{ess}}(\lambda, -\Delta)$.

● Theorem 7

There exists a constant $C_{\mathcal{M}}$ such that, for any $\lambda \gg 1$,

$$N_{\text{ess}}(\lambda, -\Delta) \leq |\mathcal{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} + C_{\mathcal{M}} r_0(\lambda) \quad (*),$$

with $r_0(\lambda)$ defined by $r_0(\lambda) = \begin{cases} \lambda^{\frac{n-1}{2}} \ln(\lambda), & \text{if } 2/n \leq \delta \leq 1 \\ \lambda^{\frac{n-(n\delta-1)}{2}}, & \text{if } 1/n < \delta < 2/n \end{cases}$;

δ defined as in Theorem 6 .

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● Remarks

- (*) \implies any eigenvalue of $-\Delta$ has finite multiplicity.
- (*) is sharp when $n = 2$. (Müller '07: $\mathcal{M} = \Gamma(N) \backslash \mathbb{H}$, with $\Gamma(N) = \{\gamma \in SL(2, \mathbb{Z}) : \gamma = Id \pmod N\}$)
- when $n = 2$, a compact perturbation of the metric can destroy all embedded eigenvalues (Colin de Verdière '83).

Proof of Theorems 4-5

- $\text{sp}_{\text{ess}}(-\Delta_A) = \bigcup_{j=1}^J \text{sp}_{\text{ess}}(-\Delta_A^{M_j, D})$ (The essential spectrum of an elliptic operator on a manifold is invariant by compact perturbation)
- $-\Delta_A^{M_j, D}$: the self-adjoint operator on $L^2(M_j)$ associated to $-\Delta_A$ with Dirichlet boundary conditions on the boundary ∂M_j of M_j .

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- Consider a cusp $M_j = X_j \times]a_j^2, +\infty[$ equipped with the metric $ds_j^2 = y^{-2\delta_j} (\mathbf{h}_j + dy^2)$; $(1/n < \delta_j \leq 1)$.

- Then for any $u \in C^2(M_j)$,

$$-\Delta_A^{M_j, D} u = -y^{2\delta_j} \Delta_{A_j}^{X_j} u - y^{n\delta_j} \partial_y (y^{(2-n)\delta_j} \partial_y u) ,$$

- $\Delta_{A_j}^{X_j}$ is the magnetic Laplacian on X_j :

- if for local coordinates $\mathbf{h}_j = \sum_{k,\ell} G_{k\ell} dx_k dx_\ell$ and

$$A_j = \sum_{k=1}^{n-1} a_{j,k} dx_k, \text{ then}$$

$$-\Delta_{A_j}^{X_j} = \frac{1}{\sqrt{\det(G)}} \sum_{k,\ell} (i\partial_{x_k} + a_{j,k}) \left(\sqrt{\det(G)} G^{k\ell} (i\partial_{x_\ell} + a_{j,\ell}) \right) .$$

- Perform the change of variables $y = e^t$,
- define the unitary operator $U : L^2(X_j \times]2 \ln(a_j), +\infty[) \rightarrow L^2(M_j)$
 $U(f) := y^{(n\delta_j - 1)/2} f.$
- $-U^* \Delta_{A_j}^{M_j, D} U f =$
 $-e^{2\delta_j t} \Delta_{A_j}^{X_j} f + \frac{(n\delta_j - 1)[3 + \delta_j(n - 4)]}{4} e^{2t(\delta_j - 1)} f - \partial_t(e^{2t(\delta_j - 1)} \partial_t f) .$

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- $(\mu_\ell(j))_{\ell \in \mathbb{N}}$: the increasing sequence of eigenvalues of $-\Delta_{A_j}^{X_j}$
- $\text{sp}(-\Delta_{A_j}^{M_j, D}) = \text{sp}(\bigoplus_{\ell=0}^{+\infty} L_{j, \ell}^D) ,$
- $L_{j, \ell}^D$: the Dirichlet operator on $L^2(]2 \ln(a_j), +\infty[)$ associated to
 $L_{j, \ell} = e^{2\delta_j t} \mu_\ell(j) + \frac{(n\delta_j - 1)}{4} [3 + \delta_j(n - 4)] e^{2t(\delta_j - 1)} - \partial_t(e^{2t(\delta_j - 1)} \partial_t) .$

- If $\mu_\ell(j) = 0$ then $\text{sp}(L_{j,\ell}^D) = \text{sp}_{\text{ess}}(L_{j,\ell}^D) = [\alpha_n, +\infty[$,
 - $\alpha_n = 0$ if $\delta_j < 1$,
 - $\alpha_n = (n - 1)^2/4$ if $\delta_j = 1$.
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- If $\mu_\ell(j) > 0$ then $\text{sp}(L_{j,\ell}^D) = \text{sp}_d(L_{j,\ell}^D) = \{\mu_{\ell,k}(j); k \in \mathbb{N}\}$,
 - $(\mu_{\ell,k}(j))_{k \in \mathbb{N}}$ the increasing sequence of eigenvalues of $L_{j,\ell}^D$, $\lim_{k \rightarrow +\infty} \mu_{\ell,k}(j) = +\infty$.
- If A satisfies the previous assumptions, then $0 < \mu_0(j) \leq \mu_\ell(j)$ for all j and ℓ, \implies
 - $\text{sp}(-\Delta_A^{M_j, D}) = \{\mu_{\ell,k}(j); (\ell, k) \in \mathbb{N}^2\}$.
 - $\lim_{\ell \rightarrow +\infty} \mu_{\ell,k}(j) = +\infty, \implies$ each $\mu_{\ell,k}(j)$ is an eigenvalue of $-\Delta_A^{M_j, D}$ of finite multiplicity,
 $\implies \text{sp}(-\Delta_A^{M_j, D}) = \text{sp}_d(-\Delta_A^{M_j, D}) \implies \text{sp}_{\text{ess}}(-\Delta_A) = \emptyset \square$

Proof of Theorem 6

\implies wanted: Weyl formula for M_j , with $-\Delta_A^{M_j, D}$ instead of $-\Delta_A$.

• $\delta_j = 1 \implies$ same change of variables and functions as before

• $1/n < \delta_j < 1, \implies$ set $y = [(1 - \delta_j)t]^{1/(1-\delta_j)}$, and define the unitary operator

$$U : L^2(X_j \times]\alpha_j, +\infty[) \rightarrow L^2(M_j), \quad \left(\alpha_j = \frac{a_j^{2(1-\delta_j)}}{1-\delta_j} \right)$$

$$U(f) = y^{(n-1)\delta_j/2} f.$$

• $\text{sp}(-\Delta_A^{M_j, D}) = \text{sp}\left(\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D\right),$

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$$L_{j,\ell} = V_{j,\ell} - \partial_t^2.$$

$$V_{j,\ell}(t) = \mu_\ell(j) [(1 - \delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} + \frac{(n-1)\delta_j [(n-3)\delta_j + 2]}{4(1 - \delta_j)^2 t^2}$$

• Titchmarsh's method \implies there exists $C > 1$ so that for any $\lambda \gg 1$ and any $\ell \in K_\lambda = \{l \in \mathbb{N}; \mu_\ell(j) \in [0, \lambda/a_j^2[\}$,

$$|N(\lambda, L_{j,\ell}^D) - \frac{1}{\pi} w_{j,\ell}(\lambda)| \leq C \ln(\lambda), \text{ with}$$

$$w_{j,\ell}(\mu) = \int_{\alpha_j}^{+\infty} [\mu - V_{j,\ell}(t)]_+^{1/2} dt$$

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- Use

- the sharp asymptotic Weyl formula of L. Hörmander (SAWFH)

- + the fact that $N(\lambda, L_{j,\ell}^D) = 0$ when $\ell \notin K_\lambda$,

- + the formula $N(\lambda, -\Delta_A^{M_j, D}) = \sum_{\ell=0}^{+\infty} N(\lambda, L_{j,\ell}^D)$

- to get $|N(\lambda, -\Delta_A^{M_j, D}) - \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda)| \leq C \lambda^{(n-1)/2} \ln(\lambda)$.

- (SAWFH) There exists $C > 0$ so that for any $\mu \gg 1$

$$|N(\mu, -\Delta_A^{\mathbf{X}_j}) - \frac{\omega_{n-1}}{(2\pi)^{n-1}} |\mathbf{X}_j| \mu^{(n-1)/2}| \leq C \mu^{(n-2)/2}.$$

$$\text{Estimate on } \Theta_j(\lambda) = \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda)$$

• Write $V_{j,\ell}(t) = \mu_\ell(j)V_j(t) + W_j(t)$

• $\Rightarrow \Theta_j(\lambda) = \frac{1}{\pi} \int_{\alpha_j}^{T_j(\lambda)} V_j^{1/2}(t) R_j\left(\frac{\lambda - W_j(t)}{V_j(t)}\right) dt$. with

• $R_j(\mu) = \sum_{\ell=0}^{+\infty} [\mu - \mu_\ell(j)]_+^{1/2}$.

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• $R_j(\mu) = \sum_{\ell=0}^{+\infty} [\mu - \mu_\ell(j)]_+^{1/2}$.

• Use then

• (SAWFH) and

$$R_j(\mu) = \frac{1}{2} \int_0^{+\infty} [\mu - s]_+^{-1/2} N(s, -\Delta_{A_j}^{X_j}) ds,$$

to get that

• There exists a constant $C > 0$ such that, for any $\mu \gg 1$,

$$\left| R_j(\mu) - \frac{\omega_{n-1}}{2(2\pi)^{n-1}} |X_j| \int_0^{+\infty} [\mu - s]_+^{-1/2} s^{(n-1)/2} ds \right| \leq C \mu^{(n-1)/2} .$$

• and use the definitions of V_j and W_j to conclude.

Proof of Theorem 7

- **Proposition 1** When A satisfies the previous assumptions (A) and if A_j verifies property (P) for any $j \in \{1, \dots, J\}$, , then as $\lambda \rightarrow +\infty$,

$$N(\lambda, -\Delta_{(\lambda^{-\rho} A)}) = |\mathcal{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} + \mathbf{O}(r_0(\lambda)),$$

$$\text{with } \rho = \begin{cases} 1/2, & \text{if } 2/n \leq \delta \leq 1 \\ (n\delta - 1)/2, & \text{if } 1/n < \delta < 2/n \end{cases}$$

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- **Property (P)**

There exists $\tau_0 = \tau_0(A_j) > 0$ and $C = C(A_j) > 0$ such that, if $e(\tau, j)$ denotes the first eigenvalue of $-\Delta_{\tau A_j}^{X_j}$, then $e(\tau, j) \geq C\tau^2$; $\forall \tau \in]0, \tau_0]$.

- **Proposition 2** For any $j \in \{1, \dots, J\}$, there exists a one-form A_j verifying assumptions (A) and property (P).

Proof of Proposition 1

- A satisfies (P) $\implies C/\lambda^{2\rho} \leq \mu_0(j)$ and $C \leq \mu_1(j)$.
- $(\mu_\ell(j))_{\ell \in \mathbb{N}}$: the increasing sequence of eigenvalues of $-\Delta_{\lambda^{-\rho} A_j}^{X_j}$.
- mimick the proof of Theorem 6.
 - Titchmarsh's method holds for any $\ell \in K_\lambda, \ell \neq 0$
 - it remains to prove that $N(\lambda, L_{j,0}^D) = \mathbf{O}(r_0(\lambda))$.

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 - it remains to prove that $N(\lambda, L_{j,0}^D) = \mathbf{O}(r_0(\lambda))$.
- $\delta_j = 1$ ($\rho = 1/2$) $\implies N(\lambda, L_{j,0}^D) \leq N(\lambda + C, L^{D,\lambda}) \leq C\lambda^{1/2} \ln(\lambda)$,
 where $L^{D,\lambda}$ is the Dirichlet operator on $]0, +\infty[$ associated to

$$\frac{C}{\lambda} e^{2t} - \partial_t^2.$$
- $0 < \delta_j < 1 \implies$

$$N(\lambda, L_{j,0}^D) \leq N((\lambda + C)^{1+2\rho(1-\delta_j)}, L^D) \leq C\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)},$$
 where L^D is the Dirichlet operator on $]0, +\infty[$ associated to

$$\frac{1}{C^2} t^{\frac{2\delta_j}{1-\delta_j}} - \partial_t^2.$$

End of proof of Theorem 7

- Take a one-form A satisfying the assumptions (A) and (P).
- any eigenfunction u of $-\Delta$ on \mathcal{M} associated to an eigenvalue in $] \inf \text{sp}_{\text{ess}}(-\Delta), +\infty[$, verifies

$$\forall j = 1, \dots, J, \quad \int_{X_j} u(x_j, y) dx_j = 0, \quad \forall y \in]a_j^2, +\infty[,$$

- this implies that

- $\|u\|_{L^2(X_j)}^2 \leq \frac{1}{\mu_1(j)} \|idu\|_{L^2(X_j)}^2, \implies$

- $\|idu + \tau u A\|_{L^2(\mathcal{M})}^2 \leq (1 + \tau C_A) \|idu\|_{L^2(\mathcal{M})}^2 + C_A \|u\|_{L^2(\mathcal{M})}^2 .$

- Denote by H_λ is the subspace of $L^2(\mathcal{M})$ spanned by eigenfunctions of $-\Delta$ associated to eigenvalues in $]0, +\infty[$. Do $\tau = 1/\lambda^\rho$,

End of proof of Theorem 7

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- $\forall u \in H_\lambda, \quad \|idu + \frac{1}{\lambda^\rho} uA\|_{L^2(\mathcal{M})}^2 \leq (1 + \frac{C_A}{\lambda^\rho}) \|du\|_{L^2(\mathcal{M})}^2 + C_A \|u\|_{L^2(\mathcal{M})}^2$
 \implies

$$\dim(H_\lambda) \leq N((1 + \frac{C_A}{\lambda^\rho})\lambda + C_A, -\Delta_{(\lambda^{-\rho}A)}).$$

- finally notice that $\lambda^{n/2}/\lambda^\rho = \mathbf{O}(r_0(\lambda))$

Proof of Theorems 1,2, 3(i)

$$\bullet \operatorname{sp}_{ess}(-\Delta_A) = \left(\bigcup_{j=1}^{J_1} \operatorname{sp}_{ess}(-\Delta_A^{M_j}) \right) \cup \left(\bigcup_{k=1}^{J_2} \operatorname{sp}_{ess}(-\Delta_A^{F_k}) \right) ;$$

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• Lemma 1

$$\operatorname{sp}_{ess}(-\Delta_A^{F_k}) = \left[\frac{1}{4} + \beta_k^2, +\infty \right] \cup \left(\bigcup_{k=1}^{J_2} S(\beta_k) \right) .$$

• Lemma 2

If $1 \leq j \leq J_1$ and $j \notin J_1^A$, then

$$\operatorname{sp}_{ess}(-\Delta_A^{M_j}) = \emptyset .$$

If $j \in J_1^A$, then

$$\operatorname{sp}_{ess}(-\Delta_A^{M_j}) = \left[\frac{1}{4} + b_j^2, +\infty \right] .$$

Proof of Lemma 2 (1)

- M_j isometric to $\mathbb{S} \times]a_j^2, +\infty[$, (cuspidal ends)

$$ds_j^2 = y^{-2} (L_j^2 d\theta^2 + dy^2) \quad (a_j > 0 \text{ and } L_j > 0)$$

- Use the coordinate $t = \ln y$ instead of y , so

- $M_j = \mathbb{S} \times]\alpha_j^2, +\infty[$ and $ds_j^2 = L_j^2 e^{-2t} d\theta^2 + dt^2$;
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- - $-\Delta_A^{M_j} = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + e^t (D_t - A_2)(e^{-t} (D_t - A_2))$,
 - $\tilde{\mathbf{b}} = b_j = L_j^{-1} e^t (\partial_\theta A_2 - \partial_t A_1)$ and $dm = L_j e^{-t} d\theta dt$.

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 - $-\Delta_A^{M_j} = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + e^t (D_t - A_2) (e^{-t} (D_t - A_2))$,
 - $\tilde{\mathbf{b}} = b_j = L_j^{-1} e^t (\partial_\theta A_2 - \partial_t A_1)$ and $dm = L_j e^{-t} d\theta dt$.
- we have

$$A - \tilde{A} = d\varphi \text{ if } \tilde{A} = (\xi + L_j b_j e^{-t}) d\theta , \text{ (for some constant } \xi \text{) .}$$

- \implies we can assume that $A = \tilde{A}$.

Proof of Lemma 2 (2)

- Define $Uf = \sqrt{L_j} e^{-t/2} f$,
- $\implies P = -U \Delta_A^{M_j} U^* = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}$.

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- $\implies \text{sp}(-\Delta_A^{M_j}) = \text{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell)$
$$P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L_j} + b_j \right)^2,$$
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for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha_j^2, +\infty[$.

- When $\ell + \xi \neq 0$, the spectrum of P_ℓ is **discrete**.

- More precisely : $\text{sp}(P_\ell) = \text{sp}(P^\pm)$ $P^\pm = D_t^2 + \frac{1}{4} + (\pm e^t + b_j)^2$

for the Dirichlet condition on

$$L^2(I_{j,\ell}; dt) ; I_{j,\ell} =]\alpha_j^2 + \ln(|\ell + \xi|/L_j), +\infty[, \text{ and } \pm = \frac{\ell + \xi}{|\ell + \xi|}.$$

Proof of Lemma 2 (3)

- \implies if $l + \xi \neq 0 \forall l \in \mathbb{Z}$, $\lim_{|l| \rightarrow \infty} \inf \text{sp}(P_l) = +\infty$,
- \implies the spectrum of $-\Delta_A^{M_j}$ is **discrete**.

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• \implies the spectrum of $-\Delta_A^{M_j}$ is **discrete**.

• What means this condition ? Recall

• $A = (\xi + L_j b_j e^{-t}) d\theta$,

• $[\mathbf{A}]_{M_j} = \lim_{d(z) \rightarrow +\infty} \int_{C_{j,z}} A$

$$\implies [\mathbf{A}]_{M_j} = \lim_{t \rightarrow +\infty} \int_0^{2\pi} (\xi + L_j b_j e^{-t}) d\theta = 2\pi\xi, \text{ so}$$

• $l + \xi \neq 0 \forall l \in \mathbb{Z} \iff [\mathbf{A}]_{M_j} \notin 2\pi\mathbb{Z} \iff J_1^A = \emptyset.$

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• $\ell + \xi \neq 0 \forall \ell \in \mathbb{Z} \iff [\mathbf{A}]_{M_j} \notin 2\pi\mathbb{Z} \iff J_1^A = \emptyset$.

• If $\ell + \xi = 0$, the spectrum of P_ℓ is **absolutely continuous** :

$$\text{sp}(P_{-\xi}) = \text{sp}_{ess}(P_{-\xi}) = \text{sp}_{ac}(P_{-\xi}) = \left[\frac{1}{4} + b_j^2, +\infty[;$$

• \implies if $[A]_{M_j} \in 2\pi\mathbb{Z}$, $\text{sp}_{ess}(-\Delta_A^{M_j}) = \left[\frac{1}{4} + b_j^2, +\infty[$.

Proof of Proposition (1)

• $M = \mathbb{S} \times]\alpha^2, +\infty[$ a cusp

$ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ the metric on M ($\alpha > 0$ and $L > 0$).

• $A = (\xi + Lbe^{-t})d\theta$, (for some constant ξ).

• $N(\lambda, -\Delta_A^M) = \sum_{\ell \in \mathbb{Z}} N(\lambda, P_\ell)$

with $P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L} \pm b \right)^2$,

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for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha^2, +\infty[$.
- Define $Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell + \xi)^2}{L^2} e^{2t}$, for the D. C. on $L^2(I; dt)$.
- Then $Q_\ell - C\sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C\sqrt{Q_\ell}$.

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$ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ the metric on M ($\alpha > 0$ and $L > 0$).

• $A = (\xi + Lbe^{-t})d\theta$, (for some constant ξ).

• $N(\lambda, -\Delta_A^M) = \sum_{\ell \in \mathbb{Z}} N(\lambda, P_\ell)$

with $P_\ell = D_t^2 + \frac{1}{4} + \left(e^t \frac{(\ell + \xi)}{L} \pm b \right)^2$,

for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha^2, +\infty[$.

• Define $Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell + \xi)^2}{L^2} e^{2t}$, for the D. C. on $L^2(I; dt)$.

• Then $Q_\ell - C\sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C\sqrt{Q_\ell}$.

• $\implies \exists$ a constant $C(b)$, s. t. for any $\lambda \gg 1$,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \leq N(\lambda, P_\ell) \leq N(\lambda + \sqrt{\lambda}C(b), Q_\ell);$$

Proof of Proposition (2)

• Lemma

$\exists C > 1$ s. t. $\forall \mu \gg 1$ and $\forall \ell \in X_\mu$,

$$w_\ell(\mu) - \pi \leq \pi N\left(\mu - \frac{1}{4}, Q_\ell\right) \leq w_\ell(\mu) + \frac{1}{12} \ln \mu + C$$

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$$w_\ell(\mu) = \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dt$$

$$X_\mu = \left\{ \ell / e^{\alpha^2} \frac{|\ell + \xi|}{L} < \sqrt{\mu - 1/4} - b \right\}.$$

Proof of Proposition (2)

● Lemma

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● Proof of Lemma (Titchmarsh's method)

- define $V_\ell = \frac{(\ell + \xi)^2}{L^2} e^{2t}$, ϕ_μ^ℓ : a solution of $Q_\ell \phi = (\mu - \frac{1}{4})\phi$.
- Consider x_ℓ and y_ℓ s. t. $V_\ell(x_\ell) = \mu$ and $V_\ell(y_\ell) = \nu$, $0 < \nu < \mu$ to be determined later.
- n (resp. m): number of zeros of ϕ_μ^ℓ on $] \alpha^2, x_\ell [$ (resp. on $] \alpha^2, y_\ell [$)
- recall: $n = N(\mu - \frac{1}{4}, Q_\ell)$.

Proof of Proposition (3)

(Titchmarsh 's lemma)

$$m\pi = \int_{\alpha^2}^{y_\ell} [\mu - V_\ell]^{1/2} dt + R_\ell$$

with $R_\ell \leq \frac{1}{4} \ln(\mu - V_\ell(\alpha^2)) - \frac{1}{4} \ln(\mu - V_\ell(y_\ell)) + \pi,$

$$\implies \left| n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2} dt \right| \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2} + R_\ell + (n - m)\pi$$

Proof of Proposition (3)

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$$\text{Sturm comparison theorem } \implies (n - m)\pi \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2} + \pi$$

So **since** $x_\ell - y_\ell = (1/2) \ln(\frac{\mu}{\nu})$

$$\left| n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2} dt \right| \leq \ln\left(\frac{\mu}{\nu}\right)(\mu - \nu)^{1/2} + \frac{1}{4} \ln \mu - \frac{1}{4} \ln(\mu - \nu) + 2\pi$$

Now take $\nu = \mu - \mu^{2/3}$ to get the **Lemma**.

Proof of Proposition (4)

• We want to compute $\sum_{\ell \in \mathbb{Z}} w_\ell(\mu)$ with

$$w_\ell(\mu) = \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dt.$$

Proof of Proposition (4)

- We want to compute $\sum_{\ell \in \mathbb{Z}} w_\ell(\mu)$ with

$$w_\ell(\mu) = \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dt.$$

- \implies it is enough to compute $\mathcal{I} = \int_{\alpha^2}^{+\infty} \int_{\mathbb{R}} \left[\mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]_+^{1/2} dx dt$,
- \mathcal{I} is equivalent to $\mu L e^{-\alpha^2} \int_{\mathbb{R}} [1 - x^2]_+^{1/2} dx$
- Use $|M| = 2\pi L e^{-\alpha^2}$ to conclude.