The magnetic Laplacian acting on discrete cusps

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Joint work with Sylvain Golénia (Bordeaux).

27/11/2017

Plan of the talk

Aim

- the spectral analysis of the Laplacian associated to a graph is strongly related to the geometry of the graph.
- graphs are discretized versions of manifolds.
- for a manifold with cusps, adding a magnetic field can drastically destroy the essential spectrum of the Laplacian.
- Our aim: go along this line in a discrete setting.

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Graph

- A *graph* is a triple $\mathcal{G} := (\mathcal{E},\mathcal{V},m)$, where $\mathcal{V}$ is a countable set (the *vertices*), $\mathcal{E} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_+$ is symmetric, and $m : \mathcal{V} \to (0,\infty)$ is a weight.
- $\mathcal{G}$ is *simple* $\iff m = 1$ and $\mathcal{E} : \mathcal{V} \times \mathcal{V} \to \{0,1\}$.
- Given $x,y \in \mathcal{V}$, (x,y) is an edge (or x and y are *neighbors*, or $x \sim y$) $\iff \mathcal{E}(x,y) > 0$.
- there is a *loop* at $x \in \mathcal{V} \iff \mathcal{E}(x,x) > 0$.
- A graph is *connected* $\iff$ for all $x,y \in \mathcal{V}$, there exists a path γ joining x and y.
- In the sequel, we assume that:
  All graphs are locally finite, connected with no loops.

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The magnetic Laplacian

- $C(\mathcal{V}) := \{f : \mathcal{V} \to \mathbb{C}\}$
- $C_c(\mathcal{V})$: functions with finite support.
- $$\ell^2(\mathcal{V},m) := \left\{ f \in C(\mathcal{V}), \sum_{x\in\mathcal{V}} m(x)|f(x)|^2 < \infty \right\}$$
- scalar product $\langle f,g\rangle := \sum_{x\in\mathcal{V}} m(x)\overline{f(x)}g(x)$.
- *magnetic potential* $\theta : \mathcal{V}\times\mathcal{V} \to \mathbb{R}/2\pi\mathbb{Z}$
  $\theta_{x,y} := \theta(x,y) = -\theta_{y,x}$ and $\theta(x,y) := 0$ if $\mathcal{E}(x,y) = 0$.

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The magnetic Laplacian (continued)

- Hermitian form :

$$Q_{\mathcal{G},\theta}(f) := \frac{1}{2} \sum_{x,y \in \mathcal{V}} \mathcal{E}(x,y) \left| f(x) - e^{i\theta_{x,y}} f(y) \right|^2 ,$$

for all $f \in \mathcal{C}_c(\mathcal{V})$.

- The magnetic Laplacian : the unique non-negative self-adjoint operator $\Delta_{\mathcal{G},\theta}$ satisfying $\langle f, \Delta_{\mathcal{G},\theta} f \rangle_{\ell^2(\mathcal{V},m)} = Q_{\mathcal{G},\theta}(f)$, for all $f \in \mathcal{C}_c(\mathcal{V})$.
- = *Friedrichs extension* of $\Delta_{\mathcal{G},\theta}|_{\mathcal{C}_c(\mathcal{V})}$
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$$(\Delta_{\mathcal{G},\theta} f)(x) = \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x,y) \left(f(x) - e^{i\theta_{x,y}} f(y) \right) ,$$

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$$0 \leq \langle f, \Delta_{\mathcal{G},\theta} f \rangle \leq \langle f, 2\deg_{\mathcal{G}}(\cdot) f \rangle, \text{ for all } f \in \mathcal{C}_c(\mathcal{V}). \qquad (1)$$

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Degree (continued)

$$\mathcal{D}\left(\deg_{\mathcal{G}}^{1/2}(\cdot)\right) = \mathcal{D}\left(\Delta_{\mathcal{G},\theta}^{1/2}\right) \quad (3)$$

is wrong in general. If $\theta = 0$, (3) is equivalent to a sparseness condition (for ex: planar simple graphs).

If (3) holds true, then

$$\sigma_{\text{ess}}(\Delta_{\mathcal{G},\theta}) = \emptyset \Leftrightarrow (\Delta_{\mathcal{G},\theta}+1)^{-1} \text{ is compact} \Leftrightarrow \lim_{|x|\to\infty} \deg_{\mathcal{G}}(x) = \infty,$$

where $|x| := \rho_{\mathcal{G}}(x_0,x)$ for a given $x_0 \in \mathcal{V}$.
If moreover the graph is sparse, then

$$\lim_{n\to\infty} \frac{\lambda_n\left(\Delta_{\mathcal{G},\theta}\right)}{\lambda_n\left(\deg_{\mathcal{G}}(\cdot)\right)} = 1$$

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The technique does not apply when the graph is a discrete cusp (thin at infinity). Our aim :

- establish new behaviors for the asymptotic of eigenvalues for the magnetic Laplacian in that case
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References

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Holonomy of a magnetic potential

- gauge transform U : unitary map on $\ell^2(\mathcal{V}, m)$ defined by

$$(Uf)(x) = u_x f(x), \quad u_x = e^{\mathrm{i}\sigma_x} .$$

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- U acts on the quadratic forms $Q_{\mathcal{G},\theta}$ by $U^*(Q_{\mathcal{G},\theta})(f) = Q_{\mathcal{G},\theta}(Uf)$, for all $f \in \mathcal{C}_c(\mathcal{V})$.
- The magnetic potential $U^\star(\theta)$ is defined by:

$$U^\star(Q_{\mathcal{G},\theta}) = Q_{\mathcal{G},U^\star(\theta)}.$$

More explicitly, we get:

$$U^\star(\theta)_{xy} = \theta_{x,y} + \sigma_y - \sigma_x .$$

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Holonomy

- $Z_1(\mathcal{G})$:the space of cycles of $\mathcal{G}$
- It is is a free $\mathbb{Z}-$module with a basis of geometric cycles $\gamma = (x_0,x_1) + (x_1,x_2) + \ldots + (x_{N-1},x_N)$ with, for $i = 0, \cdots, N-1$, $\mathcal{E}(x_i,x_{i+1}) \neq 0$, and $x_N = x_0$.
- Holonomy map $\mathrm{Hol}_\theta : Z_1(\mathcal{G}) \to \mathbb{R}/2\pi\mathbb{Z}$

$$\mathrm{Hol}_\theta\left((x_0,x_1) + (x_1,x_2) + \cdots + (x_N,x_0)\right) := \theta_{x_0,x_1} + \cdots + \theta_{x_N,x_0}.$$

The map $\theta \mapsto \mathrm{Hol}_\theta$ is surjective onto $\mathrm{Hom}_{\mathbb{Z}}(Z_1(\mathcal{G}),\mathbb{R}/2\pi\mathbb{Z})$.
$\mathrm{Hol}_{\theta_1} = \mathrm{Hol}_{\theta_2}$ if and only if there exists a gauge transform U so that $U^\star(\theta_2) = \theta_1$.
In consequence $\mathrm{Hol}_{\theta_1} = \mathrm{Hol}_{\theta_2}$ if and only if the magnetic Laplacians $\Delta_{\mathcal{G},\theta_1}$ and $\Delta_{\mathcal{G},\theta_2}$ are unitarily equivalent.

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Holonomy (end)

Let $\mathcal{G} := (\mathcal{E},\mathcal{V},m)$ be a connected graph such that $1 \in \ker\Delta_{\mathcal{G},0}$. Let θ be a magnetic potential. Then $\ker\Delta_{\mathcal{G},\theta} \neq \{0\}$ if and only if $\mathrm{Hol}_\theta = 0$.

Remark

The hypothesis $1 \in \ker\Delta_{\mathcal{G},0}$ is trivially satisfied if $\mathcal{G}$ is a finite graph.

In general, it is satisfied if and only if:

$(*)$ 1 belongs to the closure of $\mathcal{C}_c(\mathcal{V})$ with respect to the norm $(\|\cdot\|^2 + Q_{\mathcal{G},0}(\cdot))^{1/2}$. A sufficient condition to guarantee $(*)$ is

- $\mathcal{G}$ is of finite volume, i.e., such that $\sum_{x\in\mathcal{V}} m(x) < \infty$,
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A coupling constant effect

Let $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ be a connected graph of finite volume, i.e., such that $\sum_{x \in \mathcal{V}} m(x) < \infty$ and let θ be a magnetic potential such that $\mathrm{Hol}_\theta \neq 0$. Assume that the function 1 is in $\ker \Delta_{\mathcal{G},\theta}$. Then there is $\nu \in \mathbb{R}$ such that

$$\ker \Delta_{\mathcal{G},\lambda\theta} \neq \{0\} \Leftrightarrow \lambda = 0 \text{ in } \mathbb{R}/\nu\mathbb{Z}.$$

Modified Cartesian product: motivation

- A hyperbolic manifold of finite volume is the union of a compact part and of a cusp. The cusp part can be seen as the product of $(1,\infty) \times M$, where (M,g_M) is a Riemannian manifold, endowed with the metric,

$$y^{-1}(dy^2 + g_M).$$

- On the cusp part, the infimum of the radius of injectivity is 0.
- To analyze the Laplacian on this product one separates the variables and obtain a decomposition which is not of the type of a Cartesian product.
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Modified Cartesian product: motivation

- A hyperbolic manifold of finite volume is the union of a compact part and of a cusp. The cusp part can be seen as the product of $(1,\infty) \times M$, where (M, g_M) is a Riemannian manifold, endowed with the metric,

$$y^{-1}(dy^2 + g_M).$$

- On the cusp part, the infimum of the radius of injectivity is 0.
- To analyze the Laplacian on this product one separates the variables and obtain a decomposition which is not of the type of a Cartesian product.
- $\Longrightarrow$ we define a modified Cartesian product.

Modified Cartesian product

definition

Given $\mathcal{G}_1 := (\mathcal{E}_1,\mathcal{V}_1,m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2,\mathcal{V}_2,m_2)$ and $\mathcal{I} \subset \mathcal{V}_2$, we define the product of $\mathcal{G}_1$ by $\mathcal{G}_2$ through $\mathcal{I}$ by $\mathcal{G} := (\mathcal{E},\mathcal{V},m)$, where

- $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2$
- $m(x,y) := m_1(x) \times m_2(y)$,
- $\mathcal{E}\left((x,y),(x',y')\right) :=$
  $\mathcal{E}_1(x,x') \times \delta_{y,y'} \left(\sum_{z\in\mathcal{I}} \delta_{y,z}\right) + \delta_{x,x'} \times \mathcal{E}_2(y,y')$,
- $\theta\left((x,y),(x',y')\right) := \theta_1(x,x') \times \delta_{y,y'} + \delta_{x,x'} \times \theta_2(y,y')$, for all $x,x' \in \mathcal{V}_1$ and $y,y' \in \mathcal{V}_2$.

We denote $\mathcal{G}$ by $\mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$.

- If $\mathcal{I}$ is empty, the graph is disconnected.
- If $|\mathcal{I}| = 1$, $\mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ is the graph $\mathcal{G}_1$ decorated by $\mathcal{G}_2$.
- If $\mathcal{I} = \mathcal{V}_2$ and $m = 1$, we notice that $\mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2 = \mathcal{G}_1 \times \mathcal{G}_2$.

Modified Cartesian product

definition

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Definition

Given $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, the (weighted) Cartesian product $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ is $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$, where $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2$, and

$$\begin{cases} m(x,y) := & m_1(x) \times m_2(y), \\ \mathcal{E}\left((x,y),(x',y')\right) := & \mathcal{E}_1(x,x') \times \delta_{y,y'} m_2(y) + m_1(x) \delta_{x,x'} \times \mathcal{E}_2(y,y'), \\ \theta\left((x,y),(x',y')\right) := & \theta_1(x,x') \times \delta_{y,y'} + \delta_{x,x'} \times \theta_2(y,y'), \end{cases}$$

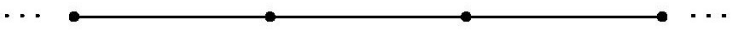
The terminology is motivated by the following decomposition:

$$\Delta_{\mathcal{G},\theta} = \Delta_{\mathcal{G}_1,\theta_1} \otimes 1 + 1 \otimes \Delta_{\mathcal{G}_2,\theta_2},$$

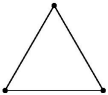
where $\ell^2(\mathcal{V}, m) \simeq \ell^2(\mathcal{V}_1, m_1) \otimes \ell^2(\mathcal{V}_2, m_2)$. The spectral theory of $\Delta_{\mathcal{G},\theta}$ is well-understood since

$$e^{it\Delta_{\mathcal{G},\theta}} = e^{it\Delta_{\mathcal{G}_1,\theta_1}} \otimes e^{it\Delta_{\mathcal{G}_2,\theta_2}}, \text{ for } t \in \mathbb{R}.$$

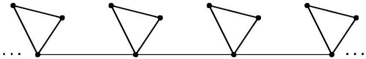
Modified Cartesian product:example



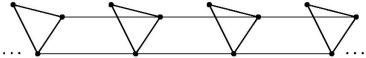
The graph of $\mathbb{Z}$



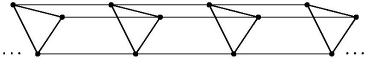
The graph of $\mathbb{Z}/3\mathbb{Z}$



The graph of $\mathbb{Z} \times_{\mathcal{I}} \mathbb{Z}/3\mathbb{Z}$, with $|\mathcal{I}| = 1$



The graph of $\mathbb{Z} \times_{\mathcal{I}} \mathbb{Z}/3\mathbb{Z}$, with $|\mathcal{I}| = 2$



Discrete cusps

If $\mathcal{G} = \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ then

- $\deg_{\mathcal{G}}(\cdot) = \deg_{\mathcal{G}_1}(\cdot) \otimes \frac{1_{\mathcal{I}}(\cdot)}{m_2(\cdot)} + \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot)$
- $\Delta_{\mathcal{G},\theta} = \Delta_{\mathcal{G}_1,\theta_1} \otimes \frac{1_{\mathcal{I}}(\cdot)}{m_2(\cdot)} + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}$.
- If m is non-trivial, $\Delta_{\mathcal{G},\theta}$ is usually not unitarily equivalent to the Laplacian obtained with the Cartesian product.

Definition

Set $\mathcal{G}_1 := (\mathcal{E}_1,\mathcal{V}_1,m_1)$, $\mathcal{G}_2 := (\mathcal{E}_2,\mathcal{V}_2,m_2)$, and $\mathcal{I} \subset \mathcal{V}_2$.
$\mathcal{G} = \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ is a discrete cusp if

(H1) $m_1(x)$ tend to 0 as $|x| \to \infty$,

(H2) $\mathcal{G}_2$ is finite,

(H3) $\Delta_{\mathcal{G}_1,\theta_1}$ is bounded (or equivalently $\sup_{x\in\mathcal{V}_1} \deg_{\mathcal{G}_1}(x) < \infty$).

Degree of $x \in \mathcal{V}$: $\deg_{\mathcal{G}}(x) := \frac{1}{m(x)} \sum_{y\in\mathcal{V}} \mathcal{E}(x,y)$

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Radius of injectivity

Definition

Given $\mathcal{G} := (\mathcal{E},\mathcal{V},m)$, the weighted length of an edge $(x,y) \in \mathcal{E}$ is defined by:

$$L_{\mathcal{G}}\big((x,y)\big) := \sqrt{\frac{\min\big(m(x),m(y)\big)}{\mathcal{E}(x,y)}}.$$

Given $x,y \in \mathcal{V}$, the weighted distance from x to y is defined by:

$$\rho_{L_{\mathcal{G}}}(x,y) := \inf_{\gamma} \sum_{i=0}^{|\gamma|-1} L_{\mathcal{G}}\big(\gamma(i),\gamma(i+1)\big),$$

where γ is a path joining x to y and with the convention that $\rho_{L_{\mathcal{G}}}(x,x) := 0$ for all $x \in \mathcal{V}$.

Radius of injectivity(continued)

Definition

Given $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$,

- the girth at $x \in \mathcal{V}$ of $\mathcal{G}$ w.r.t. the weighted length $L_{\mathcal{G}}$ is
$$\mathrm{girth}(x) := \inf\{L_{\mathcal{G}}(\gamma), \gamma \text{ simple cycle containing } x\},$$
- convention: the girth is $+\infty$ if there is no such cycle.
- $$\mathrm{girth}(\mathcal{G}) := \inf_{x \in \mathcal{V}} \mathrm{girth}(x).$$
- The radius of injectivity of $\mathcal{G}$ with respect to $L_{\mathcal{G}}$ ($\mathrm{rad}(\mathcal{G})$) is half the girth.

Radius of injectivity(end)

Proposition 1

Consider $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$ and $\mathcal{I} \subset \mathcal{V}_2$ such that $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ is a discrete cusp. We have:

1) $\mathrm{rad}(\mathcal{G}_1) > 0$.
2) If $\mathrm{rad}(\mathcal{G}_2) < \infty$, then $\mathrm{rad}(\mathcal{G}) = 0$.

Proposition 2

Consider $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$ and $\mathcal{I} \subset \mathcal{V}_2$ such that (H1), (H2), and (H3) are satisfied. Then $\mathrm{rad}(\mathcal{G}_1 \times \mathcal{G}_2) > 0$.

Radius of injectivity(end)

Proposition 1

Consider $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$ and $\mathcal{I} \subset \mathcal{V}_2$ such that $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ is a discrete cusp. We have:

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Consider $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$ and $\mathcal{I} \subset \mathcal{V}_2$ such that (H1), (H2), and (H3) are satisfied. Then $\mathrm{rad}(\mathcal{G}_1 \times \mathcal{G}_2) > 0$.

Absence of essential spectrum

Proposition

Set $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, and $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$, with $|\mathcal{I}| > 0$. Assume that (H1), (H2), and $\mathrm{Hol}_{\theta_2} \neq 0$ hold true. Then $\Delta_{\mathcal{G},\theta}$ has a compact resolvent, and

$$\mathcal{N}_\lambda \left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2} \right) \geq \mathcal{N}_\lambda(\Delta_{\mathcal{G},\theta}), \text{ for all } \lambda \geq 0.$$

Proof:

- $\Delta_{\mathcal{G},\theta} \geq \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}$ in the form sense on $\mathcal{C}_c(\mathcal{V})$.
- (H2) $+\mathrm{Hol}_{\theta_2} \neq 0+$ key Lemma $\Longrightarrow$ 0 is not in the spectrum of $(\Delta_{\mathcal{G}_2,\theta_2})$.
- Hence the spectrum of the r.h.s. is purely discrete.
- min-max Principle $\Longrightarrow$ $\Delta_{\mathcal{G},\theta}$ has a compact resolvent.

The asymptotic of the eigenvalues

Proposition(key-stone)

Set $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, and $\mathcal{I} \subset \mathcal{V}_2$ non-empty. Assume that $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ is a discrete cusp . We set

$$M := \sup_{x \in \mathcal{V}_1} \deg_{\mathcal{G}_1}(x) \times \max_{y \in \mathcal{V}_2}(1/m_2(y)) < \infty. \qquad (4)$$

We have:

$$\frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot) \le \deg_{\mathcal{G}}(\cdot) \le \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot) + M, \qquad (5)$$

$$\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2} \le \Delta_{\mathcal{G},\theta} \le 2M + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}, \qquad (6)$$

in the form sense on $\mathcal{C}_c(\mathcal{V})$.

The asymptotic of the eigenvalues(continued)

Theorem

Set $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, and $\mathcal{I} \subset \mathcal{V}_2$ non-empty. Assume that $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ is a discrete cusp. We have

- $\mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2}) = \mathcal{D}\left(m_1^{-1/2}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}^{1/2}\right)$.
- $\Delta_{\mathcal{G},\theta}$ has a compact resolvent if and only if $\mathrm{Hol}_{\theta_2} \neq 0$.
- If $\mathrm{Hol}_{\theta_2} \neq 0$, then $\mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2}) = \mathcal{D}\left(\deg_{\mathcal{G}}^{1/2}(\cdot)\right)$,

$$\lim_{n\to\infty} \frac{\lambda_n(\Delta_{\mathcal{G},\theta})}{\lambda_n\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}\right)} = 1, \quad \text{and} \tag{7}$$

$$\mathcal{N}_{\lambda-2M}\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}\right) \leq \mathcal{N}_\lambda(\Delta_{\mathcal{G},\theta}) \leq \mathcal{N}_\lambda\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}\right),$$

The asymptotic of the eigenvalues(corollary)

Aim : comparing the asymptotic with that of the degree.
New phenomenon: we can obtain a constant different from 1 in the asymptotic.

Corollary

Consider a discrete cusp $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$. Suppose that $\deg_{\mathcal{G}_2}$ is constant on $\mathcal{V}_2$ and take θ_2 such that $\mathrm{Hol}_{\theta_2} \neq 0$. Then, for all $a \in [1, +\infty[$, there exists $\widetilde{\mathcal{G}}_1 := (\widetilde{\mathcal{E}}_1, \mathcal{V}_1, \tilde{m}_1)$ such that

- $\widetilde{\mathcal{G}} := \widetilde{\mathcal{G}}_1 \times_{\mathcal{I}} \mathcal{G}_2$ is a discrete cusp.
- $\mathcal{E}_1$ and $\widetilde{\mathcal{E}}_1$ have the same zero set.
- $\deg_{\widetilde{\mathcal{G}}_1}(x) \leq \deg_{\mathcal{G}_1}(x)$ for all $x \in \mathcal{V}_1$.
- $\Delta_{\widetilde{\mathcal{G}},\theta}$ is with compact resolvent, and

$$\lim_{\lambda \to \infty} \frac{\mathcal{N}_\lambda\left(\Delta_{\widetilde{\mathcal{G}},\theta}\right)}{\mathcal{N}_\lambda\left(\deg_{\widetilde{\mathcal{G}}}(\cdot)\right)} = a.$$

The asymptotic of the eigenvalues(a specific example)

- $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$,
  $\mathcal{V}_1 := \mathbb{N}, \quad m_1(n) := e^{-n}, \mathcal{E}_1(n, n+1) := e^{-(2n+1)/2}$,
- $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, 1)$: a s.c. finite graph s. t. $|\mathcal{V}_2| = N$ $(N \geq 3)$.
- Set $\theta_1 := 0$, θ_2 s. t. $\mathrm{Hol}_{\theta_2} \neq 0$, and
  $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m) = \mathcal{G}_1 \times_{\mathcal{V}_2} \mathcal{G}_2$.

Then $\exists \nu > 0$ s. t. $\forall \kappa \in \mathbb{R}/\nu\mathbb{Z}$
$\sigma_{\mathrm{ess}}(\Delta_{\mathcal{G},\kappa\theta}) = \emptyset \Leftrightarrow \mathcal{D}\left(\Delta^{1/2}_{\mathcal{G},\kappa\theta}\right) = \mathcal{D}\left(\deg^{1/2}_{\mathcal{G}}(\cdot)\right) \Leftrightarrow \kappa \neq 0$ in $\mathbb{R}/\nu\mathbb{Z}$
Moreover:

- When $\kappa \neq 0$ in $\mathbb{R}/\nu\mathbb{Z}$, we have:

$$\lim_{\lambda \to \infty} \frac{\mathcal{N}_\lambda\left(\Delta_{\mathcal{G},\kappa\theta}\right)}{\mathcal{N}_\lambda\left(\deg_{\mathcal{G}}(\cdot)\right)} = 1,$$

The asymptotic of the eigenvalues(a specific example)

- When $\kappa = 0$ in $\mathbb{R}/\nu\mathbb{Z}$,

$$\sigma_{\mathrm{ac}}\left(\Delta_{\mathcal{G},\kappa\theta}\right) = \left[e^{1/2} + e^{-1/2} - 2, e^{1/2} + e^{-1/2} + 2\right],$$

with multiplicity 1 and

$$\lim_{\lambda\to\infty} \frac{\mathcal{N}_\lambda\left(\Delta_{\mathcal{G},\kappa\theta} P^{\perp}_{\mathrm{ac},\kappa}\right)}{\mathcal{N}_\lambda\left(\deg_{\mathcal{G}}(\cdot)\right)} = \frac{n-1}{n},$$

where $P_{\mathrm{ac},\kappa}$ denotes the projection onto the a.c. part of $\Delta_{\mathcal{G},\kappa\theta}$.

The asymptotic of the eigenvalues(a specific example)

Heuristic:

- switching on the magnetic field is not a gentle perturbation (the form domain of the operator is changed).
- second case: the constant $(n-1)/n$ encodes the fact that a part of the wave packet diffuses. the particle, which is localized in the a.c. part of the operator, escapes from every compact set.
- first case: (active magnetic potential) the spectrum of $\Delta_{\mathcal{G},\kappa\theta}$ is purely discrete. The particle cannot diffuse anymore. The particle is trapped by the magnetic field.

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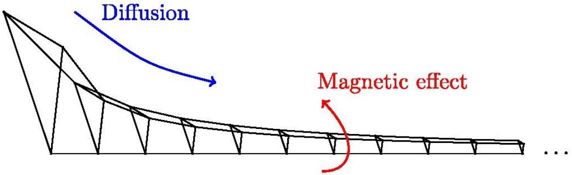
A specific example



Representation of a discrete cusp:
The magnetic field traps the particle by spinning it, whereas its absence lets the particle diffuse.