MAGNETIC BOTTLES ON POINCARE HALF-PLANE

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Summary

- The hyperbolic context
- The constant magnetic Laplacian
- Magnetic bottles: compacity of the resolvent
- Magnetic bottles: spectral asymptotics
- Related results
- Sketch of the proof

Joint work with Abderemane Morame, University of Nantes

Introduction

- Let (M, g) be a connected and oriented Riemannian manifold of dimension n.
- For any real one-form A on M , define

$$-\Delta_A \;=\; (i\,d{+}A)^\star(i\,d{+}A)\ ,\ ((i\;d+A)u = i\;du + uA\ ,\ \forall\; u\;\in\;C_0^\infty(M$$

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- The magnetic field is the two-form dA .
- To dA is associated the linear operator B defined on the tangent space by

$$dA(X,Y) \;=\; g(B.X,Y)\;; \quad \forall\, X\;,\;Y\;\in TM\times TM\;.$$

- The magnetic intensity $\mathbf{b}$ is given by

$$\mathbf{b} \;=\; \frac{1}{2} tr\left((B^{\star}B)^{1/2}\right)\;.$$

The Poincaré half-plane

- If $dim(M) = 2$, then $dA = \widetilde{\mathbf{b}}\, dv$, with $|\widetilde{\mathbf{b}}| = \mathbf{b}$, dv the Riemannian measure on M.
- The magnetic field is constant iff $\widetilde{\mathbf{b}}$ is constant.

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- The magnetic field is constant iff $\widetilde{\mathbf{b}}$ is constant.
- Let $M = \mathbb{H}$ be the hyperbolic half-plane :
  $\mathbb{H} = \mathbb{R}\times]0,+\infty[$, $g = \frac{dx^2+dy^2}{y^2}$. Then
  - $-\Delta_A = y^2(D_x - A_1)^2 + y^2(D_y - A_2)^2$,
    with $A = A_1(x,y)\, dx + A_2(x,y)\, dy$,
    and $A_j(x,y) \in C^\infty(\mathbb{H};\mathbb{R})$,
  - $\widetilde{\mathbf{b}} = y^2\,(\partial_x A_2 - \partial_y A_1)$
  - $\mathbf{b} = |\widetilde{\mathbf{b}}|$,
  - $dv = y^{-2} dx dy$.

Properties

- $-\Delta_A$ is essentially self-adjoint on $L^2(\mathbb{H})$.
- We are interested on its spectrum : $\mathrm{sp}(-\Delta_A)$.
  We will use that it is gauge invariant:

$$\mathrm{sp}(-\Delta_A) \;=\; \mathrm{sp}(-\Delta_{A+d\varphi})\;; \quad \forall\, \varphi \,\in\, C^\infty(\mathbb{H};\mathbb{R})\;.$$

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- If $\widetilde{\mathbf{b}} = y^2(\partial_x A_2(x,y) - \partial_y A_1(x,y))$ is constant, we choose a gauge such that $A_2 = 0$, so $A_1(x,y) = \pm \mathbf{b} y^{-1}$. We can assume that $A_1(x,y) = \mathbf{b} y^{-1}$, (the change $x \to -x$ is a unitary operator on $L^2(\mathbb{H})$. Then

$$-\Delta_{A^{\mathbf{b}}} = y^2(D_x - \mathbf{b} y^{-1})^2 + y^2 D_y^2\ , \quad \text{with} \quad \mathbf{b} \geq 0 \quad \text{constant.}$$

The spectrum for constant magnetic field

- The spectrum of $-\Delta_{A^{\mathbf{b}}}$ is essential: $\mathrm{sp}(-\Delta_{A^{\mathbf{b}}}) = \mathrm{sp}_{es}(-\Delta_{A^{\mathbf{b}}})$.
- Its absolutely continuous part is given by $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \;=\; [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.
- The remaining part of its spectrum is empty if $0 \leq \mathbf{b} \leq 1/2$.
- Otherwise it is formed by a finite number of eigenvalues of infinite multiplicity given by

$$\mathrm{sp}_p(-\Delta_{A^{\mathbf{b}}}) \;=\; \{(2j+1)\mathbf{b} - j(j+1)\,;\; j \;\in\; \mathbb{N}\;,\; j < \mathbf{b} - \frac{1}{2}\}\;,$$

$$(\text{if} \quad \frac{1}{2} < \mathbf{b}\;.)$$

Magnetic bottle : compact resolvent

$$-\Delta_A \;=\; y^2(D_x - A_1)^2 \;+\; y^2(D_y - A_2)^2$$

Magnetic bottle-type assumptions (MB)

- $\mathbf{b}(x,y) \;\to\; +\infty \quad \text{as} \quad d(x,y) \;\to\; +\infty$, $d(x,y)$: the hyperbolic distance of (x,y) to $(0,1)$.
- $\exists\, C_0 > 0$ such that, for any vector field X on $\mathbb{H}$,

$$|X\widetilde{\mathbf{b}}| \leq C_0(|\widetilde{\mathbf{b}}| + 1)\sqrt{g(X,X)} \;;$$

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Theorem

Under the assumptions (MB) $P(A) = -\Delta_A$ has a compact resolvent.

Proof

Lemma For any $\epsilon \in]0,1[$, there exists $C_\epsilon > 0$ s.t.

$$\forall\ f\ \in\ C_0^\infty(\mathbb{H})\ ,\quad \int_{\mathbb{H}} \mathbf{b}|f|^2 dv\ \leq\ (1+\frac{\epsilon}{2})\langle -\Delta_A f\,|\,f\rangle_{L^2(\mathbb{H})} + C_\epsilon \|f\|_{L^2(\mathbb{H})}\ .$$

Define the unitary operator

$$U\ :\ L^2(\mathbb{H})\ \to\ L^2(\mathbb{R}\times\mathbb{R}_+^*)\ ,\quad Uf(x,y) = y^{-1}f(x,y)\ ,$$

- $UP(A)U^\star = y^2(D_x - A_1)^2 + y(D_y - A_2)^2 y$.
- $UP(A)U^\star\ =\ K^\star K + \widetilde{\mathbf{b}} = \widetilde{K}^\star \widetilde{K} - \widetilde{\mathbf{b}}$
  with $K\ =\ y(D_x - A_1)\ -\ i(D_y - A_2)y$ and
  $\widetilde{K}\ =\ y(D_x - A_1)\ +\ i(D_y - A_2)y$. So

$$\pm\widetilde{\mathbf{b}}\ \leq\ UP(A)U^\star\ .$$

Magnetic bottle: spectral asymptotics

For any real $\lambda \leq \inf \mathrm{sp}_{es}(P)$, we denote by $N(\lambda; P)$ the number of eigenvalues of P , which are in $]-\infty, \lambda[$.

Theorem

Under the assumptions (MB) and for any $\delta \in]\frac{1}{3}, \frac{2}{5}[$, there exists a constant $C > 0$ such that

$$\frac{1}{2\pi} \int_{\mathbb{H}} (1 - \frac{C}{a_\delta(m)}) \mathbf{b}(m) \sum_{k=0}^{+\infty} [\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 \, dv$$

$$\leq N(\lambda, -\Delta_A) \leq$$

$$\frac{1}{2\pi} \int_{\mathbb{H}} (1 + \frac{C}{(a_\delta(m)}) \mathbf{b}(m) \sum_{k=0}^{+\infty} [\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 \, dv$$

$$[\rho]_+^0 = 1 \,, \text{ if } \rho > 0 \text{ and } 0 \text{ otherwise} \,,$$

$$a_\delta(m) := (\mathbf{b}(m) + 1)^{(2-5\delta)/2}.$$

Corollary

If moreover $\omega(\mu) := \int_{\mathbb{H}} [\mu - \mathbf{b}(m)]_+^0 dv$ satisfies (*) :

$$\exists\, C_1 > 0 \,\text{s.t.}\, \forall\, \mu > C_1\ ,\ \forall\, \tau\ \in\,]0,1[\ ,\quad \omega\,((1{+}\tau)\,\mu){-}\omega(\mu) \leq C_1\,\tau\,\omega(\mu)\ ,$$

then

$$N(\lambda; -\Delta_A)\ \sim\ \frac{1}{2\pi}\int_{\mathbb{H}} \mathbf{b}(m) \sum_{k\in\mathbb{N}} [\lambda - \tfrac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0\ dv\ .$$

Corollary

If moreover $\omega(\mu) := \int_{\mathbb{H}} [\mu - \mathbf{b}(m)]_+^0 \, dv$ satisfies (*) :

$$\exists\, C_1 > 0 \text{ s.t. } \forall\, \mu > C_1\,,\ \forall\, \tau \ \in]0,1[\,, \quad \omega\left((1+\tau)\,\mu\right) - \omega(\mu) \leq C_1\, \tau\, \omega(\mu)\,,$$

then

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- (*) is satisfied when $\omega(\lambda) \ \sim\ \alpha \lambda^k \ln^j(\lambda)$ with $k > 0$, or $k = 0$ and $j > 0$.
- Ex : $\mathbf{b}(x,y) \ = \ \left(\frac{x}{y}\right)^{2j} \ + \ g(y)$, with $j \ \in \ \mathbb{N}^\star$ and $g(y) \ = \ p_1(y) \ + \ p_2(1/y)$, $p_1(s)$, $p_2(s)$ polyn.f. of order ≥ 1 for large s . then $\omega(\lambda) \ \sim \ \alpha \lambda^{\frac{1}{2j}} \ln(\lambda)$ when $\lambda \to +\infty$.

Related results

- If (M,g) is the euclidean space $\mathbb{R}^d$,
$-\Delta_A = \sum_{j=1}^{d}(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j)^2$. $\forall x \in \mathbb{R}^d$, there exists $(e_j(x))$ s.t.
$B(x) = \sum_{j=1}^{r(x)} b_j(x) dx_j \wedge dy_j, \quad b_1(x) ... \geq b_r(x) > 0$.
The magnetic intensity is the norm of $B(x) := (b_j(x))_j$.

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- (Y. Colin de Verdière)
Under (EMB) conditions $-\Delta_A$ has compact resolvent and

$$N_B^{as}[\lambda(1-o(1))] \leq N(\lambda, -\Delta_A) \leq N_B^{as}[\lambda(1+o(1))] \quad (\lambda \to +\infty) \ .$$

  - $N_B^{as}(\lambda) = \int_{\mathbb{R}^d} \nu_{B(x)}(\lambda) dx$
  - $\nu_{B(x)}(\lambda) = C_{k,r} \ \sum^* \left(\lambda - \sum_{i=1}^{r}((2n_i+1) b_i(x)\right)_+^{k/2} \prod_{i=1}^{r} b_i(x)$
  - $\sum^* = \sum_{(n_1, \dots n_r) \in \mathbb{N}^r}$, $d = 2r + k$, $C_{k,r} = \frac{\gamma_k}{(2\pi)^{k+r}}$
  - $\gamma_k =$ volume of the unit ball in R^k .

Related results (2)

- (EMB) conditions
  - (B_1) $\lim_{\|x\|\to\infty} \|B(x)\| = \infty$
  - (B_2) $\|x - x'\| \leq 1$, $\|B(x)\| \leq C\|B(x')\|$
  - (B_3) $M(x) = o(\|B(x)\|^{\frac{3}{2}})$ when $\|x\| \to \infty)$
    ($M(x) = \max_{|\beta|=2} \left(\sup_{\|x-x'\|\leq 1} \|D^\beta A(x')\|\right)$.)

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    ($M(x) = \max_{|\beta|=2} \left(\sup_{\|x-x'\|\leq 1} \|D^\beta A(x')\|\right)$.)
- Results on the hyperbolic space
  - Maass Laplacian: Elstrodt(73), Grosche(88), Comtet(87), Ikeda-Matsumoto (99), Kim-Lee(02)
  - Asympt. constant magnetic fields: Inahama-Shirai (03)
  - Pauli operators: Inahama-Shirai (03)
  - Asymptotic distribution for Schrödinger operators: Inahama-Shirai (04)

Ingredients of the proof

- Technical transformations
- Technical lemmas
- Minimax technique on quadratic forms

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Main ideas :

- Get a problem in $\mathbb{R}^2$,
- find a good gauge,
- localize in a "good" rectangle $\Longrightarrow$ replace the initial problem by a problem with a constant magnetic field,
- use a partition of unity.

Change of variables

- diffeomorphism: $\phi \ : \ \mathbb{R}^2 \ \longrightarrow \ \mathbb{H}\ , \quad (x,y) = \phi(x,t) := (x, e^t)$
- unitary operator: $\widehat{U} \ : \ L^2(\mathbb{H}; dv) \ \longrightarrow \ L^2(\mathbb{R}^2; dxdt)$
  $(\widehat{U}u)(x,t) := e^{-t/2} u(x, e^t)$ (for any $u \in L^2(\mathbb{H})$.)

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- quadratic form related to $P(A) = -\Delta_A$:

$$q(u) := \int_{\mathbb{H}} \left[|y(D_x - A_1)u|^2 + |y(D_y - A_2)u|^2 \right] \frac{dxdy}{y^2} \, , (u \in L^2(\mathbb{H}))$$

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$$= \int_{\mathbb{R}^2} \left[|e^t (D_x - \tilde{A}_1)w|^2 + |(D_t - e^t \tilde{A}_2)w|^2 + 1/4 |w|^2 \right] dxdt \, .$$

with

$$\tilde{A}_i(x, t) := A_i(x, e^t) \quad , \ i = \ 1, 2 \ \ .$$

Gauge

We want to work with a gauge such that $A_2 = 0$. Since

$$\widetilde{\mathbf{b}} \; = \; y^2 \left(\partial_x A_2 - \partial_y A_1\right)$$

we take

$$A_1(x,y) = -\int_1^y \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds$$

which gives

$$\tilde{A}_1(x,t) \; := -\int_1^{e^t} \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds$$

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The associated quadratic form is

$$\widehat{q}^{\tilde{A}}(w) = \int_{\mathbb{R}^2} \left[|e^t(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 + 1/4|w|^2\right] dxdt\ ,$$

with $\tilde{A}(x,t) = (\tilde{A}_1, 0)$.

Localization in a suitable rectangle in $\mathbb{R}^2$

- Let $X_0 = (x_0, t_0) \in \mathbb{R}^2$ such that $\mathbf{b}(z_0) > 1$; $(z_0 = (x_0, e^{t_0})\ )$; $|X_0|$ can be very large.
- Let $a_0 > 1$ and $\varepsilon_0 \in]0, 1[$ be given.
- Take $a \in]1/a_0, a_0[$ and define $K := X_0 + K_0$, with $K_0 =] - \varepsilon_0 a \frac{e^{t_0}}{2},\ \varepsilon_0 a \frac{e^{t_0}}{2}[\times] - \frac{\varepsilon_0}{2},\ \frac{\varepsilon_0}{2} [$.

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- Let $a_0 > 1$ and $\varepsilon_0 \in]0,1[$ be given.
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Compare :

$$\widehat{q}_K^{\tilde{A}}(w) = \int_K \left[|e^t(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 + 1/4|w|^2 \right] dxdt$$

with the "frozen" form ($w \in W_0^1(K)$) :

$$\widehat{q}_K^{A^0,0}(w) = \int_K \left[|e^{t_0}(D_x - A_1^0)w|^2 + |D_t w|^2 + 1/4|w|^2 \right] dxdt \ .$$

Localization (2)

- $A^0(x,t) = (A_1^0, 0)$, $\quad A_1^0 := -(t - t_0)\, e^{-t_0}\, \widetilde{\mathbf{b}}(x_0, e^{t_0})$.
  - $\tilde{A}_1(x,t) \; := -\int_{e^{t_0}}^{e^t} \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds$ (gauge invariance .)

First step: Key Lemma

$$|\tilde{A}_1(x,t) - A_1^0(x,t)| \;\leq\; C\, \varepsilon_0^2\, e^{-t_0}\, \mathbf{b}(x_0, e^{t_0}) \; .$$

($\forall (x,t) \in K$, $\;C > 0$ depends only on a_0.)

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($\forall (x,t) \in K$, $\quad C > 0$ depends only on a_0.)

This comes from the assumption : For any vector field X on $\mathbb{H}$,

$|X\widetilde{\mathbf{b}}| \le C_0(|\widetilde{\mathbf{b}}| + 1)\sqrt{g(X,X)}$.

Proof of key lemma

- **Lemma**

  For any $C_0 > 0$ there exists $C_1 > 1$ s. t. $\forall (x_0, y_0) \in \mathbb{H}$ satisfying $\mathbf{b}(x_0, y_0) > 1$

  $$\frac{1}{C_1}\ \mathbf{b}(x_0, y_0) \le \mathbf{b}(x, y) \le C_1\ \mathbf{b}(x_0, y_0)\ ; \quad \forall\ (x, y) \in \Omega$$

  where $\Omega := \{(x, y)/|x - x_0| \le C_0\ y_0, |y - y_0| \le y_0/2\}$.

Proof of key lemma

- **Lemma**

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- **Proof** Perform Taylor expansion $\Longrightarrow$

  $$|\widetilde{\mathbf{b}}(x, y) - \widetilde{\mathbf{b}}(x_0, y_0)| \leq (|x - x_0| + |y - y_0|) \sup_{z \in \Omega}(|\partial_x \widetilde{\mathbf{b}}(z)| + |\partial_y \widetilde{\mathbf{b}}(z)|)$$

  so $|\widetilde{\mathbf{b}}(x, y) - \widetilde{\mathbf{b}}(x_0, y_0)| \leq\ (C_0 + 1)\ y_0\ \sup_{z \in \Omega} \dfrac{\mathbf{b}(z) + 1}{y}$

- **Lemma** $\Longrightarrow$ key lemma

Second step

- Define

$$\widehat{q}_K^{\tilde{A},0}(w) := \int_K \left[|e^{t_0}(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 + 1/4|w|^2\right] dxdt \ , \ \forall w \in W_0$$

- Write

$$e^{t_0}(D_x - \tilde{A}_1)w = e^{t_0}(D_x - A_1^0)w - e^{t_0}(\tilde{A}_1 - A_1^0)w \ ,$$

- and use the first step to get :

For any $\tau \ \in \,]0,1[\ ;$

$$(1-\tau^2)\ \widehat{q}_K^{A^0,0}(w) + (1 - \frac{1}{\tau^2})\ C\ \varepsilon_0^4\ \mathbf{b}^2(z_0)\|w\|^2 \leq \widehat{q}_K^{\tilde{A},0}(w)$$

$$\leq\ (1+\tau^2)\widehat{q}_K^{A^0,0}(w) + (1 + \frac{1}{\tau^2})\ C\ \varepsilon_0^4\ \mathbf{b}^2(z_0)\|w\|^2\ .$$

Third step

Proposition

$$(1-\varepsilon_0 C)\,\widehat{q}_K^{\tilde{A},0}(w) \leq \widehat{q}_K^{\tilde{A}}(w) \leq (1+\varepsilon_0 C)\widehat{q}_K^{\tilde{A},0}(w)\ .$$

Proof : Write

$$\widehat{q}_K^{\tilde{A}}(w) = \int_K \left[e^{2(t-t_0)}|e^{t_0}(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 dxdt| + 1/4|w|^2\right]$$

and use that $|t-t_0| \leq 1$ in K.

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Spectral asymptotics for a rectangle : upper bound

- **Proposition** For any real λ ,

$$N(\lambda, P_K^0(A^0)) \;\leq\; \frac{|K|\mathbf{b}(x_0, e^{t_0})}{2\pi e^{t_0}} \sum_{k=0}^{+\infty} [\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(x_0, e^{t_0})]_+^0 \; .$$

Spectral asymptotics for a rectangle : upper bound

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$$N(\lambda, P_K^0(A^0)) \le \frac{|K|\mathbf{b}(x_0, e^{t_0})}{2\pi e^{t_0}} \sum_{k=0}^{+\infty} [\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(x_0, e^{t_0})]_+^0 .$$

- This is a consequence of **Colin de Verdière's upper bound** : if $\lambda > 1$:

$$N(\lambda, H_{D,\mathbf{b}}^{\Omega_R}) \le \frac{\mathbf{b}|\Omega_R|}{2\pi} \sum_{k=0}^{+\infty} [\lambda - (2k+1)\mathbf{b}]_+^0 .$$

$H_{D,\mathbf{b}}^{\Omega_R}$: Dir. pb. associated to $H_0 = (D_x - \mathbf{b}\frac{y}{2})^2 + (D_y + \mathbf{b}\frac{x}{2})^2$ in $\Omega_R =]-\frac{R_1}{2}, \frac{R_1}{2}[\times]-\frac{R_2}{2}, \frac{R_2}{2}[$.
$R = (R_1, R_2) \in (\mathbb{R}_+^\star)^2$, $\mathbf{b} > 0$ is a constant.

Spectral asymptotics for a rectangle : lower bound

- **Proposition** $\exists\, C_0$ depending only on a_0 , s.t. , if $\varepsilon_0^{-2}/C_0 \leq \mathbf{b}(x_0, e^{t_0}) \leq \lambda$, then $\forall\, \tau\ \in]0,1[$,

$$(1-\tau)^2 \frac{|K|\mathbf{b}(x_0, e^{t_0})}{2\pi e^{t_0}} \sum_{k=0}^{+\infty} [\lambda - \frac{1}{4} - \frac{C_0}{(\tau\varepsilon_0)^2} - (2k+1)\mathbf{b}(x_0, e^{t_0})]_+^0$$

$$\leq\ N(\lambda, P_K^0(A^0))\ .$$

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$$\leq \; N(\lambda, P_K^0(A^0)) \; .$$

- This is a consequence of the **Theorem** : $\exists\, C_0 > 0$ s.t., if $0 < \mathbf{b} < \lambda$ and $1 \leq \sqrt{\mathbf{b}}\, \min R_j$, then $\forall\, \epsilon \;\in]0,1]$,

$$(1-\epsilon)^2 \frac{\mathbf{b}|\Omega_R|}{2\pi} \sum_{k=0}^{+\infty} [\lambda - \frac{C_0}{(\epsilon \min R_j)^2} - (2k+1)\mathbf{b}]_+^0 \leq \; N(\lambda, H_{D,\mathbf{b}}^{\Omega_R}) \; .$$

Partition of $\mathbb{R}^2$ of suitable rectangles

Let a_0 and δ_0 such that

$$1 < a_0 \quad \text{and} \quad \delta_0 \in \left]\frac{1}{3}, \frac{2}{5}\right[.$$

For any $\alpha \in \mathbb{Z}^2$, we denote the rectangle

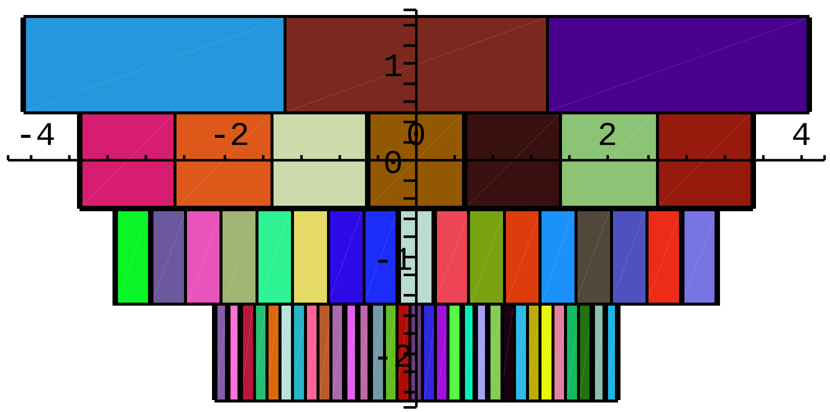
$$K(\alpha) = \left] -\frac{e^{\alpha_2}}{2} + e^{\alpha_2}\alpha_1 \,,\, e^{\alpha_2}\alpha_1 + \frac{e^{\alpha_2}}{2} \right[\times \left] -\frac{1}{2} + \alpha_2 \,,\, \alpha_2 + \frac{1}{2} \right[.$$

So

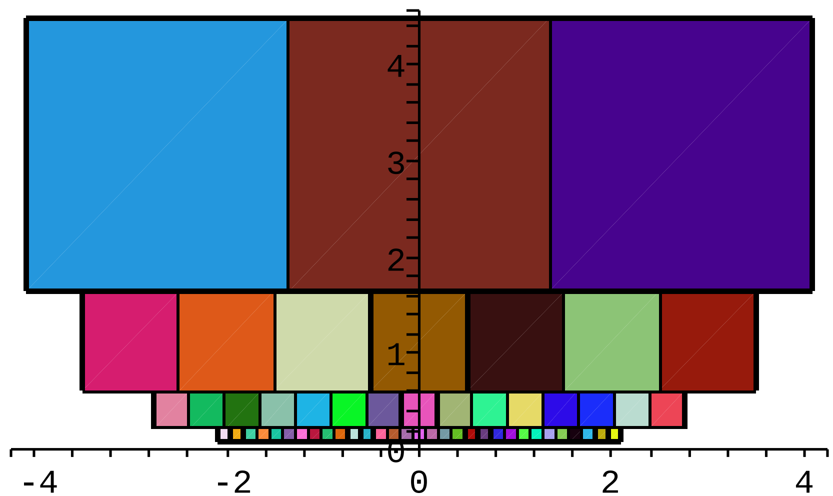
$$\mathbb{R}^2 = \cup_\alpha \overline{K}(\alpha)$$

and $K(\alpha) \cap K(\beta) = \emptyset$ if $\alpha \neq \beta$.

Covering of $\mathbb{R}^2$ with the rectangles $\overline{K}(\alpha)$



Covering of $\mathbb{H}$ with the cubes $\phi(\overline{K}(\alpha))$



$$\phi \ : \ \mathbb{R}^2 \to \mathbb{H} \ , \qquad \phi(x,t) = (x, e^t) \ .$$

A partition adapted to b

Each $K(\alpha)$ can be parted, (if necessary), into $M(\alpha)$ rectangles:

$$\overline{K}(\alpha) \;=\; \cup_{j=1}^{M(\alpha)} \overline{K_{\alpha,j}} \;,$$

$$K_{\alpha,j} \;=\,]-\frac{\epsilon_{\alpha,j} e^{t_{\alpha,j}}}{2} + x_{\alpha,j} \;,\; x_{\alpha,j} + \frac{\epsilon_{\alpha,j} e^{t_{\alpha,j}}}{2} [\times] - \frac{\epsilon_{\alpha,j}}{2} + t_{\alpha,j} \;,\; t_{\alpha,j} + \frac{\epsilon_{\alpha,j}}{2} [\;,$$

with

$$\frac{1}{a_0 (1 + \mathbf{b}^{\delta_0} (x_{\alpha,j}, e^{t_{\alpha,j}})\;)} \;\le\; \epsilon_{\alpha,j} \;\le\; \frac{a_0}{(1 + \mathbf{b}^{\delta_0} (x_{\alpha,j}, e^{t_{\alpha,j}})\;)} \;, \qquad (1)$$

and such that $K_{\alpha,k} \cap K_{\alpha,j} \;=\; \emptyset$ if $k \neq j$.

Conclusion

- **Remark** :

  Take $d = 2$ in Colin de Verdière's result. Then

  $N_B^{as}(\lambda) = \frac{1}{2\pi} \int_{R^2} \mathbf{b}(x) \sum_{k\in\mathbb{N}} \left[\lambda - (2k+1)\mathbf{b}(x)\right]_+^0 dx$

  $(b_1(x) = \|B(x)\| = \mathbf{b}(x))$

  to be compared with

  $N(\lambda; -\Delta_A) \sim \frac{1}{2\pi} \int_{\mathbb{H}} \mathbf{b}(m) \sum_{k\in\mathbb{N}} [\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 \, dv$.

  $[\rho]_+^0 = 1$ if $\rho > 0$

  $[\rho]_+^0 = 0$ if $\rho \leq 0$.

- **Possible developpements**
  - Role of the curvature
  - Application to cuspidal manifolds (with infinite volume)
  - 3-dimensional case?