Full statistics of erasure processes: Isothermal adiabatic theory and a statistical Landauer principle

Joint work with Tristan Benoist (Toulouse)
Martin Fraas (LMU Munchen) & Vojkan Jakšić (McGill)

Claude-Alain Pillet (CPT – Université de Toulon)

Quantum Open & N-Body Systems
Grenoble — May 9 2016

Taming Maxwell's demon: a never ending story made short

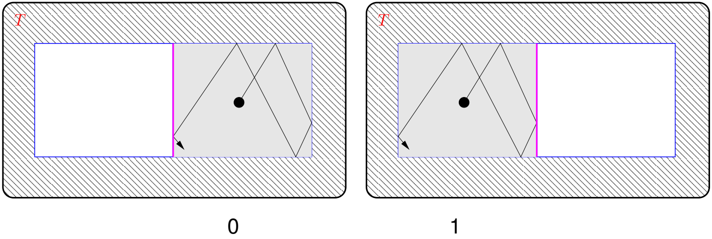
- 1871: Maxwell's demon violates the 2^{nd} Law
- 1929: Szilard's engine converts information into work
- 1956: Brillouin: irreversibility of quantum measurement processes
- 1961: Landauer: *logically irreversible* operations dissipate heat

$$\Delta Q = k_{\mathrm{B}} T \log 2 \text{ per bit}$$

- 1982: Bennett exorcises the demon
- 1999: Earman-Norton criticism...
- ... many attempts to "prove" Landauer's principle from "first principles" (stat. mech.) or conceive classical and quantum systems that violate it ...

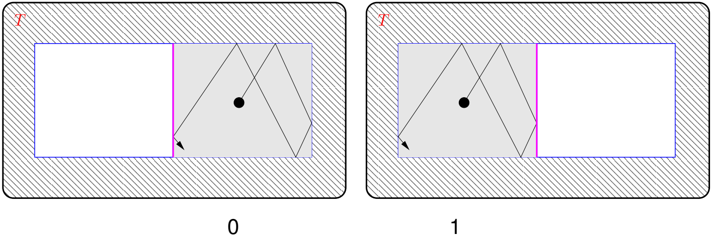
Thermodynamic “derivation” of Landauer’s Principle

The ideal gas 1-bit memory ($pV = k_B T$)



Thermodynamic "derivation" of Landauer's Principle

The ideal gas 1-bit memory ($pV = k_B T$)

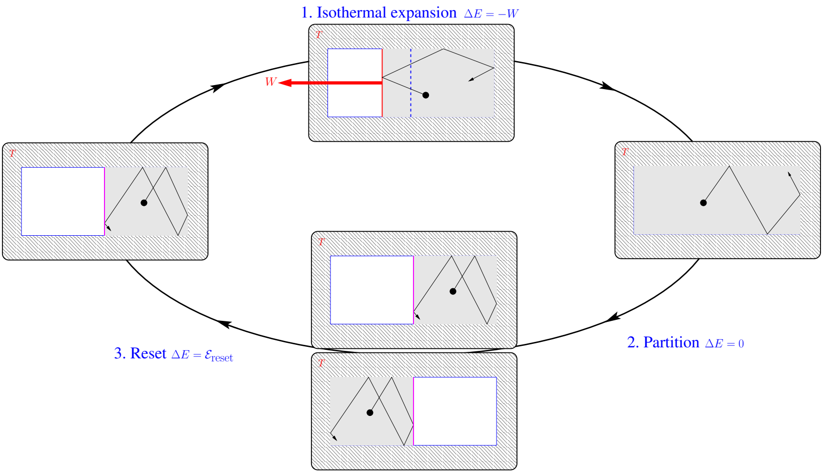


Assume there is a process which perform the reset operation **(0 or 1)** $\rightarrow 0$ with energy cost

$$\mathcal{E}_{\text{reset}}$$

Thermodynamic "derivation" of Landauer's Principle

Build a cyclic process – Szilard Engine



Thermodynamic "derivation" of Landauer's Principle

Work extracted during isothermal quasi-static expansion

$$W = \int_{V/2}^{V} p\,dV = \int_{V/2}^{V} \frac{k_{\mathrm{B}} T}{V}\,\mathrm{d}V = k_{\mathrm{B}} T \log 2$$

The second law imposes

$$\mathcal{E}_{\mathrm{reset}} \geq k_{\mathrm{B}} T \log 2$$

Thermodynamic “derivation” of Landauer’s Principle

Work extracted during isothermal quasi-static expansion

$$W = \int_{V/2}^{V} p\, dV = \int_{V/2}^{V} \frac{k_{\mathrm{B}} T}{V}\, \mathrm{d}V = k_{\mathrm{B}} T \log 2$$

The second law imposes

$$\mathcal{E}_{\mathrm{reset}} \geq k_{\mathrm{B}} T \log 2$$

[Landauer ’61] The energy injected in the reset process is released as heat in the reservoir. $k_{\mathrm{B}} T \log 2$ is the minimal energy dissipated by a reset operation. Moreover

$$k_{\mathrm{B}} T \log 2 = T \Delta S$$

ΔS being the decrease in entropy of the system in the reseting process (erasing entropy). Note that Landauer’s bound $\mathcal{E}_{\mathrm{reset}} \geq T \Delta S$ is saturated by the reverse process of quasi-static isothermal compression.

Landauer's Principle from statistical mechanics

[Earman-Norton 1999, Bennett 2003, Leff-Rex 2003, ...] All known derivations of Landauer's Principle assume the validity of one or another form of the 2^{nd} Law.

[Shizume 1995, Piechocinska 2000, ...] Landauer's Principle from classical and quantum microscopic dynamics of specific systems

[Reeb-Wolf 2014] *Much of the misunderstanding and controversy around Landauer's Principle appears to be due to the fact that its general statement has not been written down formally or proved in a rigorous way in the framework of quantum statistical physics*

Landauer's Principle from statistical mechanics

[Earman-Norton 1999, Bennett 2003, Leff-Rex 2003, ...] All known derivations of Landauer's Principle assume the validity of one or another form of the 2^{nd} Law.

[Shizume 1995, Piechocinska 2000, ...] Landauer's Principle from classical and quantum microscopic dynamics of specific systems

[Reeb-Wolf 2014] *Much of the misunderstanding and controversy around Landauer's Principle appears to be due to the fact that its general statement has not been written down formally or proved in a rigorous way in the framework of quantum statistical physics*

This formulation will definitively not close the philosophical discussions about Maxwell's demon and the relation between thermodynamics and information theory, but at least it provides a sound statement with well defined assumptions.

Landauer's Principle in statistical mechanics [Reeb-Wolf '14]

Finite quantum system $\mathcal{S}$ coupled to finite reservoir $\mathcal{R}$ at temperature $T > 0$

- Finite dimensional Hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R}$, reservoir Hamiltonian $H_\mathcal{R}$
- Product initial state + thermal reservoir $\omega_i = \rho_i \otimes \nu_i$

$$\nu_i = \mathrm{e}^{-(\beta H_\mathcal{R} + \log Z)}, \qquad \beta = \frac{1}{k_\mathrm{B} T}, \qquad Z = \mathrm{tr}\left(\mathrm{e}^{-\beta H_\mathcal{R}}\right)$$

- Unitary state transformation $U : \omega_i \mapsto \omega_f = U\omega_i U^*$
- Reduced final states

$$\rho_f = \mathrm{tr}_{\mathcal{H}_\mathcal{R}}(\omega_f), \qquad \nu_f = \mathrm{tr}_{\mathcal{H}_\mathcal{S}}(\omega_f)$$

- Energy dissipated in the reservoir $\mathcal{R}$:

$$\langle \Delta Q \rangle = \mathrm{tr}((\nu_f - \nu_i) H_\mathcal{R})$$

- Decrease in entropy of the system $\mathcal{S}$:

$$\Delta S = S(\rho_i) - S(\rho_f)$$

where $S(\rho) = -k_\mathrm{B}\,\mathrm{tr}\,(\rho \log \rho)$ is the von Neumann entropy of ρ

Landauer's Principle in statistical mechanics

Landauer's bound (statistical 2nd Law)[Reeb-Wolf '14, Tasaki '00]

$$\langle \Delta Q \rangle = T(\Delta S + \sigma), \qquad \sigma \geq 0 \tag{1}$$

$\sigma = 0$ iff $\langle \Delta Q \rangle = T\Delta S = 0$, in which case $\nu_f = \nu_i$, ρ_f and ρ_i being unitarily equivalent.

Landauer's Principle in statistical mechanics

Landauer's bound (statistical 2nd Law)[Reeb-Wolf '14, Tasaki '00]

$$\langle \Delta Q \rangle = T(\Delta S + \sigma), \qquad \sigma \geq 0 \tag{1}$$

$\sigma = 0$ iff $\langle \Delta Q \rangle = T\Delta S = 0$, in which case $\nu_f = \nu_i$, ρ_f and ρ_i being unitarily equivalent.

Remark 1. If $\mathcal{S}$ is a qubit,

$$\rho_i = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad \rho_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then the transformation $\rho_i \to \rho_f$ implements the state change **(0 or 1)** $\to 0$ and

$$T\Delta S = k_{\mathrm{B}} T \log 2$$

However, this transition can not be induced by a finite reservoir at positive temperature (more later).

Landauer's Principle in statistical mechanics

Landauer's bound (statistical 2nd Law)[Reeb-Wolf '14, Tasaki '00]

$$\langle \Delta Q \rangle = T(\Delta S + \sigma), \qquad \sigma \geq 0 \tag{1}$$

$\sigma = 0$ iff $\langle \Delta Q \rangle = T\Delta S = 0$, in which case $\nu_f = \nu_i$, ρ_f and ρ_i being unitarily equivalent.

Remark 1. If $\mathcal{S}$ is a qubit,

$$\rho_i = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad \rho_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then the transformation $\rho_i \to \rho_f$ implements the state change **(0 or 1)** $\to 0$ and

$$T\Delta S = k_{\mathrm{B}} T \log 2$$

However, this transition can not be induced by a finite reservoir at positive temperature (more later).

Remark 2. Von Neumann entropy is the quantum version of Shannon information theoretic entropy. It only coincides with thermodynamic (Clausius) entropy for thermal equilibrium states.

Landauer's Principle in statistical mechanics

Landauer's bound (statistical 2^{nd} Law)[Reeb-Wolf '14, Tasaki '00]

$$\langle \Delta Q \rangle = T(\Delta S + \sigma), \qquad \sigma \geq 0 \tag{1}$$

$\sigma = 0$ iff $\langle \Delta Q \rangle = T\Delta S = 0$, in which case $\nu_f = \nu_i$, ρ_f and ρ_i being unitarily equivalent.

Remark 1. If $\mathcal{S}$ is a qubit,

$$\rho_i = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad \rho_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then the transformation $\rho_i \to \rho_f$ implements the state change **(0 or 1)** $\to 0$ and

$$T\Delta S = k_{\mathrm{B}} T \log 2$$

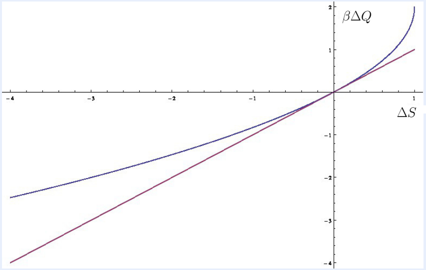
However, this transition can not be induced by a finite reservoir at positive temperature (more later).

Remark 2. Von Neumann entropy is the quantum version of Shannon information theoretic entropy. It only coincides with thermodynamic (Clausius) entropy for thermal equilibrium states.

Remark 3. Mathematically: simple, direct consequence of Klein's inequality.

Tightness of Landauer's Bound

[Reeb-Wolf '14] Most of their analysis consists in showing that Landauer's bound is not tight for reservoirs with finite dimensional Hilbert space and deriving tighter bounds in such cases.



Conjecture: *Landauer's Principle can probably be formulated within the general statistical mechanical framework of C^* and W^* dynamical systems and an equality version akin to (1) can possibly be proven.*

Tightness of Landauer's Bound

Macroscopic reservoir should be idealized as infinitely extended
$\rightsquigarrow$ Thermodynamic Limit

Familiar objects (Hamiltonians, density matrices,...) lose their meaning in the Thermodynamic Limit ...

...but other structures emerge (modular theory)

[Jakšić-P'15] Landauer's principle holds for infinitely extended reservoirs, under appropriate and physically reasonable ergodicity hypotheses. Moreover, the bound is saturated by isothermal quasi-static (i.e., adiabatic=infinitely slow) processes. Proof based on the gapless adiabatic theorem [Avron-Elgart'99, Teufel'01, Abou Salem-Fröhlich'05]

Tightness of Landauer's Bound

Macroscopic reservoir should be idealized as infinitely extended
$\rightsquigarrow$Thermodynamic Limit

Familiar objects (Hamiltonians, density matrices,...) lose their meaning in the Thermodynamic Limit ...

...but other structures emerge (modular theory)

[Jakšić-P'15] Landauer's principle holds for infinitely extended reservoirs, under appropriate and physically reasonable ergodicity hypotheses. Moreover, the bound is saturated by isothermal quasi-static (i.e., adiabatic=infinitely slow) processes. Proof based on the gapless adiabatic theorem [Avron-Elgart'99, Teufel'01, Abou Salem-Fröhlich'05]

Can we improve Landauer's Principle beyond the expected value $\langle \Delta Q \rangle$?

Improving Landauer's bound: Heat Full Statistics

Strategy: study the balance between ΔQ and ΔS in a quasi–static transition $\rho_i \to \rho_f$ induced by coupling $\mathcal{S}$ to a reservoir $\mathcal{R}$ of finite size L with a time-dependent Hamiltonian

$$H^{(L)}(t/T) = H_{\mathcal{R}}^{(L)} + H_{\mathcal{S}}(t/T) + \lambda(t/T)V$$

in the limits $L, T \to \infty$. Two time-scales:

$$\text{physical time } t \in [0, T], \qquad \text{epoch } s = t/T \in [0, 1]$$

Improving Landauer's bound: Heat Full Statistics

Strategy: study the balance between ΔQ and ΔS in a quasi–static transition $\rho_i \to \rho_f$ induced by coupling $\mathcal{S}$ to a reservoir $\mathcal{R}$ of finite size L with a time-dependent Hamiltonian

$$H^{(L)}(t/T) = H_{\mathcal{R}}^{(L)} + H_{\mathcal{S}}(t/T) + \lambda(t/T)V$$

in the limits $L, T \to \infty$. Two time-scales:

$$\text{physical time } t \in [0, T], \qquad \text{epoch } s = t/T \in [0, 1]$$

Practical implementation, assuming $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^d$, $\rho_i = d^{-1}I$ and $\rho_f > 0$:

- $\lambda(0) = 0$: at time $t = 0$, $\mathcal{S}$ and $\mathcal{R}$ being decoupled, in the product state $\omega_i^{(L)} = \rho_i \otimes \nu_i^{(L)}$, measure the total energy of $\mathcal{R}$ $\rightsquigarrow E_i \in \mathrm{sp}(H_{\mathcal{R}}^{(L)})$

Improving Landauer's bound: Heat Full Statistics

Strategy: study the balance between ΔQ and ΔS in a quasi–static transition $\rho_i \to \rho_f$ induced by coupling $\mathcal{S}$ to a reservoir $\mathcal{R}$ of finite size L with a time-dependent Hamiltonian

$$H^{(L)}(t/T) = H_{\mathcal{R}}^{(L)} + H_{\mathcal{S}}(t/T) + \lambda(t/T)V$$

in the limits $L, T \to \infty$. Two time-scales:

$$\text{physical time } t \in [0, T], \qquad \text{epoch } s = t/T \in [0, 1]$$

Practical implementation, assuming $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^d$, $\rho_i = d^{-1}I$ and $\rho_f > 0$:

- $\lambda(0) = 0$: at time $t = 0$, $\mathcal{S}$ and $\mathcal{R}$ being decoupled, in the product state $\omega_i^{(L)} = \rho_i \otimes \nu_i^{(L)}$, measure the total energy of $\mathcal{R} \rightsquigarrow E_i \in \mathrm{sp}(H_{\mathcal{R}}^{(L)})$
- $H_{\mathcal{S}}(0/1) = -\beta^{-1} \log \rho_{i/f} + F_{i/f}$: for $t \in]0, T[$ the joint system $\mathcal{S} + \mathcal{R}$ evolves according to the time-dependent Hamiltonian $H^{(L)}(t/T)$

$$T^{-1}\mathrm{i}\partial_s U_s^{(L,T)} = H^{(L)}(s)U_s^{(L,T)}$$

Improving Landauer's bound: Heat Full Statistics

Strategy: study the balance between ΔQ and ΔS in a quasi–static transition $\rho_i \to \rho_f$ induced by coupling $\mathcal{S}$ to a reservoir $\mathcal{R}$ of finite size L with a time-dependent Hamiltonian

$$H^{(L)}(t/T) = H_{\mathcal{R}}^{(L)} + H_{\mathcal{S}}(t/T) + \lambda(t/T)V$$

in the limits $L, T \to \infty$. Two time-scales:

$$\text{physical time } t \in [0, T], \qquad \text{epoch } s = t/T \in [0, 1]$$

Practical implementation, assuming $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^d$, $\rho_i = d^{-1}I$ and $\rho_f > 0$:

- $\lambda(0) = 0$: at time $t = 0$, $\mathcal{S}$ and $\mathcal{R}$ being decoupled, in the product state $\omega_i^{(L)} = \rho_i \otimes \nu_i^{(L)}$, measure the total energy of $\mathcal{R} \rightsquigarrow E_i \in \mathrm{sp}(H_{\mathcal{R}}^{(L)})$
- $H_{\mathcal{S}}(0/1) = -\beta^{-1} \log \rho_{i/f} + F_{i/f}$: for $t \in]0, T[$ the joint system $\mathcal{S} + \mathcal{R}$ evolves according to the time-dependent Hamiltonian $H^{(L)}(t/T)$

$$T^{-1}\mathrm{i}\partial_s U_s^{(L,T)} = H^{(L)}(s)U_s^{(L,T)}$$

- $\lambda(1) = 0$: at time $t = T$, $\mathcal{S}$ and $\mathcal{R}$ being again decoupled, measure the total energy of $\mathcal{R} \rightsquigarrow E_f \in \mathrm{sp}(H_{\mathcal{R}}^{(L)})$

Improving Landauer's bound: Heat Full Statistics

Strategy: study the balance between ΔQ and ΔS in a quasi–static transition $\rho_i \to \rho_f$ induced by coupling $\mathcal{S}$ to a reservoir $\mathcal{R}$ of finite size L with a time-dependent Hamiltonian

$$H^{(L)}(t/T) = H_{\mathcal{R}}^{(L)} + H_{\mathcal{S}}(t/T) + \lambda(t/T)V$$

in the limits $L, T \to \infty$. Two time-scales:

$$\text{physical time } t \in [0, T], \qquad \text{epoch } s = t/T \in [0, 1]$$

Practical implementation, assuming $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^d$, $\rho_i = d^{-1}I$ and $\rho_f > 0$:

- $\lambda(0) = 0$: at time $t = 0$, $\mathcal{S}$ and $\mathcal{R}$ being decoupled, in the product state $\omega_i^{(L)} = \rho_i \otimes \nu_i^{(L)}$, measure the total energy of $\mathcal{R} \rightsquigarrow E_i \in \mathrm{sp}(H_{\mathcal{R}}^{(L)})$
- $H_{\mathcal{S}}(0/1) = -\beta^{-1} \log \rho_{i/f} + F_{i/f}$: for $t \in]0, T[$ the joint system $\mathcal{S} + \mathcal{R}$ evolves according to the time-dependent Hamiltonian $H^{(L)}(t/T)$

$$T^{-1} \mathrm{i}\partial_s U_s^{(L,T)} = H^{(L)}(s) U_s^{(L,T)}$$

- $\lambda(1) = 0$: at time $t = T$, $\mathcal{S}$ and $\mathcal{R}$ being again decoupled, measure the total energy of $\mathcal{R} \rightsquigarrow E_f \in \mathrm{sp}(H_{\mathcal{R}}^{(L)})$
- Full Statistic of dissipated heat = Probability distribution of $\Delta Q = E_f - E_i$

Assumption I (Thermodynamic Limit)

For $s \in [0, 1]$ and $T > 0$, as $L \to \infty$

$$\mathrm{e}^{\mathrm{i}tH^{(L)}(s)}(\,\cdot\,)\mathrm{e}^{-\mathrm{i}tH^{(L)}(s)} \to \gamma^t_{(s)}(\,\cdot\,)$$

$$U_s^{(L,T)*}(\,\cdot\,)U_s^{(L,T)} \to \tau^s_{(T)}(\,\cdot\,)$$

$$\eta_s^{(L)}(\,\cdot\,) = \frac{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}(\,\cdot\,)}{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}} \to \eta_s(\,\cdot\,)$$

η_s being the unique equilibrium state at temperature β^{-1} for $\gamma^t_{(s)}$ (KMS-condition)

Improving Landauer's bound: Heat Full Statistics

Assumption I (Thermodynamic Limit)

For $s \in [0,1]$ and $T > 0$, as $L \to \infty$

$$\mathrm{e}^{\mathrm{i}tH^{(L)}(s)}(\,\cdot\,)\mathrm{e}^{-\mathrm{i}tH^{(L)}(s)} \to \gamma^t_{(s)}(\,\cdot\,)$$

$$U_s^{(L,T)*}(\,\cdot\,)U_s^{(L,T)} \to \tau^s_{(T)}(\,\cdot\,)$$

$$\eta_s^{(L)}(\,\cdot\,) = \frac{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}(\,\cdot\,)}{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}} \to \eta_s(\,\cdot\,)$$

η_s being the unique equilibrium state at temperature β^{-1} for $\gamma^t_{(s)}$ (KMS-condition)

Assumption II (Ergodicity)

For $s \in]0,1[$ the state η_s is ergodic for $\gamma^t_{(s)}$

Improving Landauer's bound: Heat Full Statistics

Assumption I (Thermodynamic Limit)

For $s \in [0,1]$ and $T > 0$, as $L \to \infty$

$$\mathrm{e}^{\mathrm{i}tH^{(L)}(s)}(\,\cdot\,)\mathrm{e}^{-\mathrm{i}tH^{(L)}(s)} \to \gamma^t_{(s)}(\,\cdot\,)$$

$$U_s^{(L,T)*}(\,\cdot\,)U_s^{(L,T)} \to \tau^s_{(T)}(\,\cdot\,)$$

$$\eta_s^{(L)}(\,\cdot\,) = \frac{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}(\,\cdot\,)}{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}} \to \eta_s(\,\cdot\,)$$

η_s being the unique equilibrium state at temperature β^{-1} for $\gamma^t_{(s)}$ (KMS-condition)

Assumption II (Ergodicity)

For $s \in]0,1[$ the state η_s is ergodic for $\gamma^t_{(s)}$

Remark 1. By (I) and the boundary conditions on $\lambda(0/1)$ and $H_{\mathcal{S}}(0/1)$,

$$\eta_0 = \rho_i \otimes \nu_{\mathrm{i}}, \qquad \eta_1 = \rho_f \otimes \nu_{\mathrm{i}}$$

where $\nu_i = \lim_{L\to\infty} \nu_i^{(L)}$ is the thermal equilibrium state of $\mathcal{R}$ at temperature β^{-1}

Improving Landauer's bound: Heat Full Statistics

Assumption I (Thermodynamic Limit)

For $s \in [0,1]$ and $T > 0$, as $L \to \infty$

$$\mathrm{e}^{\mathrm{i}tH^{(L)}(s)}(\,\cdot\,)\mathrm{e}^{-\mathrm{i}tH^{(L)}(s)} \to \gamma^t_{(s)}(\,\cdot\,)$$

$$U_s^{(L,T)*}(\,\cdot\,)U_s^{(L,T)} \to \tau^s_{(T)}(\,\cdot\,)$$

$$\eta_s^{(L)}(\,\cdot\,) = \frac{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}(\,\cdot\,)}{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}} \to \eta_s(\,\cdot\,)$$

η_s being the unique equilibrium state at temperature β^{-1} for $\gamma^t_{(s)}$ (KMS-condition)

Assumption II (Ergodicity)

For $s \in]0,1[$ the state η_s is ergodic for $\gamma^t_{(s)}$

Remark 1. By (I) and the boundary conditions on $\lambda(0/1)$ and $H_{\mathcal{S}}(0/1)$,

$$\eta_0 = \rho_i \otimes \nu_{\mathrm{i}}, \qquad \eta_1 = \rho_f \otimes \nu_{\mathrm{i}}$$

where $\nu_i = \lim_{L\to\infty} \nu_i^{(L)}$ is the thermal equilibrium state of $\mathcal{R}$ at temperature β^{-1}

Remark 2. (II) enforces interaction $\lambda(s)V$ to be non-trivial for $s \in]0,1[$

Improving Landauer's bound: Heat Full Statistics

Applying Avron-Elgart gapless adiabatic theorem in the GNS representation yields the

Isothermal Adiabatic Theorem [Abou Salem-Fröhlich'05, Jakšić-P'14]

If the functions $\lambda(s)$ and $H_{\mathcal{S}}(s)$ are $C^1([0,1])$ and Assumptions (I)-(II) are satisfied then

$$\lim_{T\to\infty}\sup_{s\in[0,1]}\|\eta_0\circ\tau_{(T)}^s-\eta_s\|=0$$

Improving Landauer's bound: Heat Full Statistics

Applying Avron-Elgart gapless adiabatic theorem in the GNS representation yields the

Isothermal Adiabatic Theorem [Abou Salem-Fröhlich'05, Jakšić-P'14]

If the functions $\lambda(s)$ and $H_{\mathcal{S}}(s)$ are $C^1([0,1])$ and Assumptions (I)-(II) are satisfied then

$$\lim_{T\to\infty}\sup_{s\in[0,1]}\|\eta_0\circ\tau_{(T)}^s-\eta_s\|=0$$

By the previous remark,

$$\lim_{T\to\infty}\eta_0\circ\tau_{(T)}^1=\eta_1$$

i.e., in the adiabatic limit our process induces the state transformation $\rho_i\to\rho_f$ on the system $\mathcal{S}$.

Improving Landauer's bound: Heat Full Statistics

Energetic Balance in average

Expected work done on the joint system $\mathcal{S} + \mathcal{R}$ (from Duhamel's formula):

$$\begin{aligned}\langle \Delta W \rangle &= \lim_{T\to\infty} \int_0^1 \eta_0 \circ \tau_{(T)}^s (\dot{H}_{\mathcal{S}}(s) + \dot{\lambda}(s)V)\mathrm{d}s \\ &= \int_0^1 \eta_s (\dot{H}_{\mathcal{S}}(s) + \dot{\lambda}(s)V)\mathrm{d}s \\ &= \lim_{L\to\infty} \int_0^1 \frac{\mathrm{tr}\, \mathrm{e}^{-\beta H^{(L)}(s)}(\dot{H}^{(L)}(s))}{\mathrm{tr}\, \mathrm{e}^{-\beta H^{(L)}(s)}} \mathrm{d}s = F_f - F_i = \Delta F\end{aligned}$$

Improving Landauer's bound: Heat Full Statistics

Energetic Balance in average

Expected work done on the joint system $\mathcal{S} + \mathcal{R}$ (from Duhamel's formula):

$$\begin{aligned}\langle \Delta W \rangle &= \lim_{T\to\infty} \int_0^1 \eta_0 \circ \tau_{(T)}^s (\dot{H}_{\mathcal{S}}(s) + \dot{\lambda}(s)V)\mathrm{d}s \\ &= \int_0^1 \eta_s(\dot{H}_{\mathcal{S}}(s) + \dot{\lambda}(s)V)\mathrm{d}s \\ &= \lim_{L\to\infty} \int_0^1 \frac{\mathrm{tr}\, \mathrm{e}^{-\beta H^{(L)}(s)}(\dot{H}^{(L)}(s))}{\mathrm{tr}\, \mathrm{e}^{-\beta H^{(L)}(s)}}\mathrm{d}s = F_f - F_i = \Delta F\end{aligned}$$

Expected change in energy of $\mathcal{S}$:

$$\langle \Delta U \rangle = \rho_f(H_{\mathcal{S}}(1)) - \rho_i(H_{\mathcal{S}}(0)) = -\beta^{-1}\Delta S + \Delta F$$

Improving Landauer's bound: Heat Full Statistics

Energetic Balance in average

Expected work done on the joint system $\mathcal{S}+\mathcal{R}$ (from Duhamel's formula):

$$\begin{aligned}\langle\Delta W\rangle &= \lim_{T\to\infty}\int_0^1 \eta_0\circ\tau^s_{(T)}(\dot H_{\mathcal{S}}(s)+\dot\lambda(s)V)\mathrm{d}s\\ &= \int_0^1 \eta_s(\dot H_{\mathcal{S}}(s)+\dot\lambda(s)V)\mathrm{d}s\\ &= \lim_{L\to\infty}\int_0^1 \frac{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}(\dot H^{(L)}(s))}{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}}\mathrm{d}s = F_f - F_i = \Delta F\end{aligned}$$

Expected change in energy of $\mathcal{S}$:

$$\langle\Delta U\rangle = \rho_f(H_{\mathcal{S}}(1)) - \rho_i(H_{\mathcal{S}}(0)) = -\beta^{-1}\Delta S + \Delta F$$

Expected heat released in $\mathcal{R}$ (from the First Law!):

$$\langle\Delta Q\rangle = \langle\Delta W\rangle - \langle\Delta U\rangle = \beta^{-1}\Delta S$$

saturates Landauer's bound

Improving Landauer's bound: Heat Full Statistics

Full Statistics of dissipated heat
[Shimizu-Sakaki'91, Levitov-Lesovik'92, Kurchan'00, Tasaki'00,. . .]

$$\mathbb{P}_{\rm heat}^{(L,T)}(\Delta Q) = \sum_{E_f - E_i = \Delta Q} \mathrm{tr}(P_{\{E_f\}}(H_{\mathcal{R}}^{(L)})U_1^{(L,T)}P_{\{E_i\}}(H_{\mathcal{R}}^{(L)})\eta_0^{(L)}P_{\{E_f\}}(H_{\mathcal{R}}^{(L)})U_1^{(L,T)*})$$

Improving Landauer's bound: Heat Full Statistics

Full Statistics of dissipated heat
[Shimizu-Sakaki'91, Levitov-Lesovik'92, Kurchan'00, Tasaki'00,...]

$$\mathbb{P}_{\rm heat}^{(L,T)}(\Delta Q) = \sum_{E_f - E_i = \Delta Q} \mathrm{tr}\big(P_{\{E_f\}}(H_{\mathcal{R}}^{(L)}) U_1^{(L,T)} P_{\{E_i\}}(H_{\mathcal{R}}^{(L)}) \eta_0^{(L)} P_{\{E_f\}}(H_{\mathcal{R}}^{(L)}) U_1^{(L,T)*}\big)$$

Cumulant generating function = Rényi relative entropy (use $H^{(L)}(0) = H_{\mathcal{R}}^{(L)} + \mathrm{const.}$)

$$\begin{aligned}
\chi_{\rm heat}^{(L,T)}(\alpha) &= \log \sum_{\Delta Q \in \mathrm{sp}(H_{\mathcal{R}}^{(L)}) - \mathrm{sp}(H_{\mathcal{R}}^{(L)})} \mathrm{e}^{-\alpha \Delta Q}\, \mathbb{P}_{\rm heat}^{(L,T)}(\Delta Q) \\
&= \log \mathrm{tr}\big(\mathrm{e}^{-\alpha H_{\mathcal{R}}^{(L)}} U_1^{(L,T)} \mathrm{e}^{\alpha H_{\mathcal{R}}^{(L)}} \eta_0^{(L)} U_1^{(L,T)*}\big) \\
&= \log \mathrm{tr}\big(\mathrm{e}^{-\alpha H^{(L)}(0)} U_1^{(L,T)} \mathrm{e}^{(\alpha-\beta) H^{(L)}(0)} U_1^{(L,T)*}\big) \\
&= \log \mathrm{tr}\big(\eta_0^{(L)\alpha/\beta} U_1^{(L,T)} \eta_0^{(L)(1-\alpha/\beta)} U_1^{(L,T)*}\big) \\
&= S_{\frac{\alpha}{\beta}}\big(\eta_0^{(L)} | U_1^{(L,T)} \eta_0^{(L)} U_1^{(L,T)*}\big)
\end{aligned}$$

Assumption III (Thermodynamiic Limit)

$$\chi_{\mathrm{heat}}^{(T)}(\alpha) = \lim_{L\to\infty} \chi_{\mathrm{heat}}^{(L,T)}(\alpha) = S_{\frac{\alpha}{\beta}}(\eta_0 | \eta_0 \circ \tau_{(T)}^1)$$

Improving Landauer's bound: Heat Full Statistics

Assumption III (Thermodynamiic Limit)

$$\chi_{\text{heat}}^{(T)}(\alpha) = \lim_{L\to\infty} \chi_{\text{heat}}^{(L,T)}(\alpha) = S_{\frac{\alpha}{\beta}}(\eta_0|\eta_0 \circ \tau_{(T)}^1)$$

Remark. $S_{\frac{\alpha}{\beta}}(\eta_0|\eta_0 \circ \tau_{(T)}^1)$ can be expressed in terms of the modular structure

Improving Landauer's bound: Heat Full Statistics

Assumption III (Thermodynamiic Limit)

$$\chi^{(T)}_{\mathrm{heat}}(\alpha) = \lim_{L\to\infty} \chi^{(L,T)}_{\mathrm{heat}}(\alpha) = S_{\frac{\alpha}{\beta}}(\eta_0|\eta_0 \circ \tau^1_{(T)})$$

Remark. $S_{\frac{\alpha}{\beta}}(\eta_0|\eta_0 \circ \tau^1_{(T)})$ can be expressed in terms of the modular structure

Large time limit follows from isothermal adiabatic theorem and previous remark:

$$\begin{aligned}\chi_{\mathrm{heat}}(\alpha) &= \lim_{T\to\infty} \chi^{(T)}_{\mathrm{heat}}(\alpha) = S_{\frac{\alpha}{\beta}}(\eta_0|\eta_1)\\ &= S_{\frac{\alpha}{\beta}}(\rho_i \otimes \nu_{\mathcal{R}}|\rho_f \otimes \nu_{\mathcal{R}}) = S_{\frac{\alpha}{\beta}}(\rho_i|\rho_f) = -\frac{\alpha}{\beta}\log d + \mathrm{tr}\,\rho_f^{1-\frac{\alpha}{\beta}}\end{aligned}$$

Improving Landauer's bound: Heat Full Statistics

Assumption III (Thermodynamiic Limit)

$$\chi_{\rm heat}^{(T)}(\alpha) = \lim_{L\to\infty} \chi_{\rm heat}^{(L,T)}(\alpha) = S_{\frac{\alpha}{\beta}}(\eta_0|\eta_0 \circ \tau_{(T)}^1)$$

Remark. $S_{\frac{\alpha}{\beta}}(\eta_0|\eta_0 \circ \tau_{(T)}^1)$ can be expressed in terms of the modular structure

Large time limit follows from isothermal adiabatic theorem and previous remark:

$$\chi_{\rm heat}(\alpha) = \lim_{T\to\infty} \chi_{\rm heat}^{(T)}(\alpha) = S_{\frac{\alpha}{\beta}}(\eta_0|\eta_1)$$

$$= S_{\frac{\alpha}{\beta}}(\rho_i \otimes \nu_{\mathcal{R}}|\rho_f \otimes \nu_{\mathcal{R}}) = S_{\frac{\alpha}{\beta}}(\rho_i|\rho_f) = -\frac{\alpha}{\beta}\log d + \mathrm{tr}\,\rho_f^{1-\frac{\alpha}{\beta}}$$

Convergence of Heat Full Statistics

$$\mathbb{P}_{\rm heat}^{(L,T)} \Rightarrow \mathbb{P}_{\rm heat}^{(T)}\ (L \to \infty), \qquad \mathbb{P}_{\rm heat}^{(T)} \Rightarrow \mathbb{P}_{\rm heat}\ (T \to \infty)$$

with

$$\chi_{\rm heat}(\alpha) = \log \int \mathrm{e}^{-\alpha \Delta Q} \mathrm{d}\mathbb{P}_{\rm heat}(\Delta Q)$$

Improving Landauer's bound: Heat Full Statistics

While ΔQ is a discrete random variable for finite L, in the thermodynamic limit it acquires a continuous range. It becomes atomic again in the adiabatic limit.

Improving Landauer's bound: Heat Full Statistics

While ΔQ is a discrete random variable for finite L, in the thermodynamic limit it acquires a continuous range. It becomes atomic again in the adiabatic limit.

Denote by p_k and m_k the distinct eigenvalues of ρ_f and their multiplicities and set

$$Q_k = \frac{\log d + \log p_k}{\beta}$$

then

$$\mathbb{P}_{\rm heat}(\Delta Q = Q_k) = p_k m_k = \frac{m_k}{d} \mathrm{e}^{\beta Q_k}$$

Improving Landauer's bound: Heat Full Statistics

While ΔQ is a discrete random variable for finite L, in the thermodynamic limit it acquires a continuous range. It becomes atomic again in the adiabatic limit.

Denote by p_k and m_k the distinct eigenvalues of ρ_f and their multiplicities and set

$$Q_k = \frac{\log d + \log p_k}{\beta}$$

then

$$\mathbb{P}_{\mathrm{heat}}(\Delta Q = Q_k) = p_k m_k = \frac{m_k}{d}\mathrm{e}^{\beta Q_k}$$

Under the same assumptions, a similar construction yields the Full Statistics of the work done on the joint system $\mathcal{S} + \mathcal{R}$

$$\chi_{\mathrm{work}}(\alpha) = -\alpha \Delta F$$

which shows that work does not fluctuate in the adiabatic limit

$$\mathrm{d}\mathbb{P}_{\mathrm{work}}(\Delta W) = \delta(\Delta W - \Delta F)\mathrm{d}\Delta W$$

it converges a.s. towards ΔF, a result which leads us to interpret ΔF a the change in free energy of the joint system $\mathcal{S} + \mathcal{R}$.

Improving Landauer's bound: The Perfect Erasure Limit

Until now $\rho_f > 0$. Perfect erasure aims at pure final state and can not be reached by coupling to a thermal reservoir at positive temperature. Consider the simplest case $d = 2$ with

$$\rho_f = (1 - \epsilon)|+\rangle\langle+| + \epsilon|-\rangle\langle-|, \qquad \epsilon \in]0, 1[$$

as an approximation of the aimed pure state $|+\rangle$.

Improving Landauer's bound: The Perfect Erasure Limit

Until now $\rho_f > 0$. Perfect erasure aims at pure final state and can not be reached by coupling to a thermal reservoir at positive temperature. Consider the simplest case $d = 2$ with

$$\rho_f = (1-\epsilon)|+\rangle\langle+| + \epsilon|-\rangle\langle-|, \qquad \epsilon \in]0,1[$$

as an approximation of the aimed pure state $|+\rangle$.

Differentiating $\chi_{\text{heat}}(\alpha)$ at $\alpha = 0$ yields

$$\langle\Delta Q\rangle = \beta^{-1}\log 2 + \mathcal{O}(\epsilon\log\epsilon)$$

and higher cumulants

$$\langle\langle\Delta Q^n\rangle\rangle = \mathcal{O}(\epsilon(\log\epsilon)^n)$$

In the limit $\epsilon \to 0$

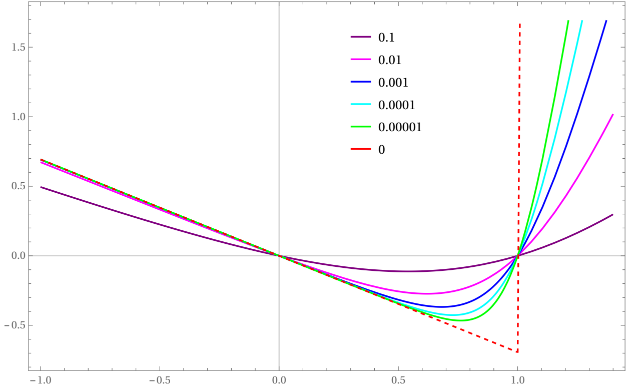
$$\mathbb{P}_{\text{heat}} \Rightarrow \delta_{\beta^{-1}\log 2}$$

but this weak convergence does not capture the singular nature of this limit: for small $\epsilon > 0$ there is a small $\mathcal{O}(\epsilon)$ but non-vanishing probability of violating Landauer's bound which is associated to failure of the erasing process

Improving Landauer's bound: The Perfect Erasure Limit

The limiting cumulant generating function is singular

$$\lim_{\epsilon\to 0}\chi_{\text{heat}}(\alpha)=\begin{cases}-\frac{\alpha}{\beta}\log d & \alpha<\beta\\ 0 & \alpha=\beta\\ +\infty & \alpha>\beta\end{cases}$$



Thank you !