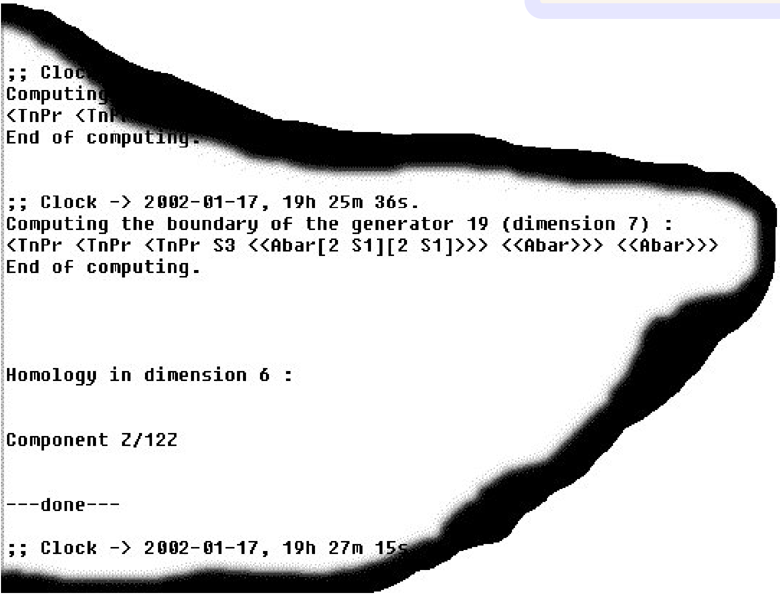
# The Computational Problem in Algebraic Topology

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*Quimper, October 2022*

**Semantics of colours:**

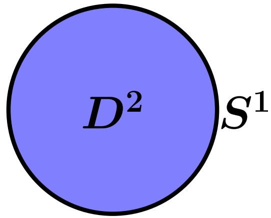
| | | |
|---|---|---|
| **Blue** | = | **“Standard” Mathematics** |
| **Red** | = | **Constructive, effective, algorithm, machine object, . . .** |
| **Violet** | = | **Problem, difficulty, obstacle, disadvantage, . . .** |
| **Green** | = | **Solution, essential point, mathematicians, . . .** |

**Plan.**

- **1. Typical example of Applied Algebraic Topology.**
- **2. The main groups of Algebraic Topology**
- **3. The computability problem.**
- **4. Constructive Algebraic Topology.**
- **5. Typical examples of Kenzo computation.**

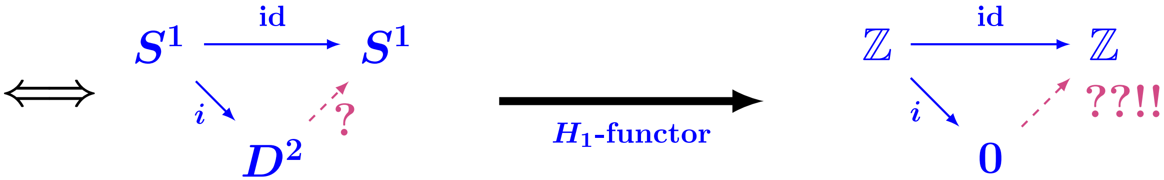
## 1. Simple example of Applied Algebraic Topology.

$D^2 = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2.$




$S^1 = \partial D^2 = \{x^2 + y^2 = 1\} \subset D^2 \subset \mathbb{R}^2.$

**Problem:** $\exists ?\, f : D^2 \to S^1$ continuous such that $f | S^1 = \mathrm{id}_{S^1}$ ?




$\Longrightarrow$ Impossible !!!

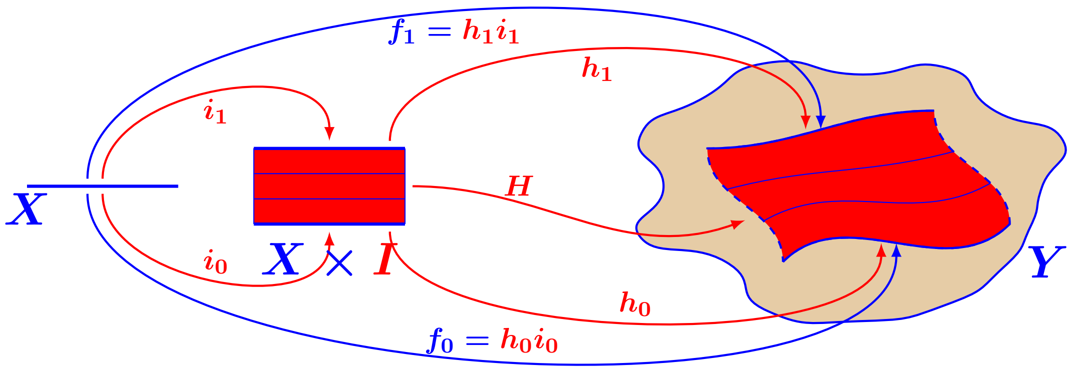
# 2. Main groups of Algebraic Topology.

**General notion of homotopy.**

**Given:** $f_0, f_1 : X \to Y$.

**Definition:** $f_0$ **homotope** to $f_1$ $\Leftrightarrow \exists H : X \times I \to Y$ satisfying:

$$f_0(x) = H(x, 0)$$
$$f_1(x) = H(x, 1)$$



**Notion of Homotopy Type.**

**<u>Definition</u>: Two spaces $X$ and $Y$ are homotopy equivalent if**

$$\exists \quad X \underset{f}{\overset{g}{\rightleftarrows}} Y \quad \text{with } gf \sim \mathrm{id}_X \text{ and } fg \sim \mathrm{id}_Y.$$

$$\{\text{Homotopy types}\} := \frac{\{\text{Topological Spaces}\}}{\text{Homotopy equivalence}}$$

**Examples:** $* \sim D^n \sim \mathbb{R}^n$

$$S^{n-1} \underset{f}{\overset{g}{\rightleftarrows}} \mathbb{R}^n - \{0\}$$

**with $f =$ canonical inclusion**

**and $g =$ radial projection.**

$(X, x_0)$ = topological space with base point $x_0 \in X$.

Loop of $X$ based at $x_0$ =

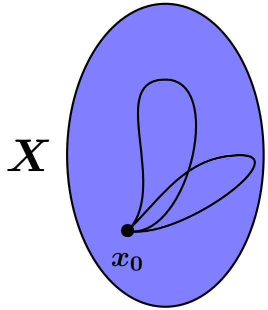
map $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

$\Omega(X, x_0) := \{$loops of $X$ based at $x_0\}$

$\pi_1(X, x_0) := \Omega(X, x_0)/$homotopy of based loops

$\pi_1(X, x_0)$ carries a natural group structure.

$\pi_1(X, x_0)$ is a homotopy invariant.




Obvious generalization to $\pi_n(X, x_0)$.

Computations of $\pi_n(X, x_0)$ ?

**Simple examples:**

$\pi_1(\mathbb{R}^n) = 0$

$X$ contractible $\Leftrightarrow$ $X$ has the homotopy type of $*$.

$X$ contractible $\overset{\sim}{\Leftrightarrow}$ $\pi_n(X) = 0 \quad \forall n$.

$\pi_*(S^n) = ?$

**Easy:**

$\pi_k(S^n) = 0$ for $k < n$.

$\pi_n(S^n) = \mathbb{Z}$

What about $\pi_k(S^n)$ for $k > n$?

# Table of $\pi_k(S^n)$ for $k \leq 16$ and $n \leq 9$ :

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ | $\pi_8$ | $\pi_9$ | $\pi_{10}$ | $\pi_{11}$ | $\pi_{12}$ | $\pi_{13}$ | $\pi_{14}$ | $\pi_{15}$ | $\pi_{16}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S^1$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S^2$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{12}\times\mathbb{Z}_2$ | $\mathbb{Z}_{84}\times\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_6$ |
| $S^3$ | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | | | | | | | | | | | |
| $S^4$ | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}\times\mathbb{Z}_{12}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{24}\times\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$ | $\mathbb{Z}_{84}\times\mathbb{Z}_2^5$ | $\mathbb{Z}_2^6$ |
| $S^5$ | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{30}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_{72}\times\mathbb{Z}_2$ | $\mathbb{Z}_{504}\times\mathbb{Z}_2^2$ |
| $S^6$ | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{60}$ | $\mathbb{Z}_{24}\times\mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_{72}\times\mathbb{Z}_2$ |
| $S^7$ | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_{120}$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^4$ |
| $S^8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}\times\mathbb{Z}_{120}$ | $\mathbb{Z}_2^4$ |
| $S^9$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_{240}$ |

**Other groups of Algebraic Topology:**

**Homology groups** **Cohomology groups**

**Cohomotopy groups** **K-theory groups**

**+ Rich extra algebra structures on these groups, examples :**

$\{\pi_n(X)\}$ = **Quasi-Lie algebra**

$\{H^n(X)\}$ = **Commutative algebra =**

**Module wrt the Steenrod Algebra**

$\{H_n(X)\}$ = **Module wrt the $E_\infty$-operad.**

## 3. The computability problem.

Computing a group = ???

Most groups $G$ of algebraic topology

are $\mathbb{Z}$-modules of finite type.

$$\Rightarrow G = \mathbb{Z}/d_1 + \cdots + \mathbb{Z}/d_n \quad \text{with } d_i \text{ divides } d_{i+1}.$$

$$\pi_7(S^4) = \mathbb{Z}/12 + \mathbb{Z} \quad \Longleftrightarrow \quad \pi_7(S^4) \text{ “=” } (12, 0)$$

**Computability of $\pi_n(X)$ ??**

**Meaningful only if $X$ is a "machine object",**

**must be "combinatorial".**

**$\Rightarrow$ $X$ simplicial complex, or much better $X$ simplicial set.**

**$\pi_1(X)$ in general non abelian + $\pi_n(X)$ abelian for $n \geq 2$**

**$\Rightarrow$ contexts totally different for $n = 1$ and $n \geq 2$.**

**Definition**: $X$ simply connected $\Leftrightarrow$

$$X \text{ connected} + \pi_1(X) = 0.$$

**Rabin's theorem**: $X$ finite simplicial set.

Then the decision problem

"$X$ simply connected ?" is undecidable.

More precisely the set of simply connected finite simplicial sets is enumerable but non-recursive.

$\exists$ program $n \mapsto X_n$ such that $\{X_n\}_{n \in \mathbb{N}}$ is the set of all the finite simplicial sets that are simply connected.

# ⇒ First case = $X$ where $X$ simply connected is known.

ANNALS OF MATHEMATICS
Vol. 65, No. 1, January, 1957
*Printed in U.S.A.*

**FINITE COMPUTABILITY OF POSTNIKOV COMPLEXES**[1]

BY EDGAR H. BROWN, JR.

(Received March 3, 1956)

In [4] Postnikov associates with each arcwise connected space $X$ a sequence

simply connected simplicial complex. From these results we are then able to prove:

(i) If $X$ is a simply connected simplicial complex, then $\pi_n(X)$ is finitely computable for each $n > 0$.

(ii) If $X$ and $Y$ are simply connected simplicial complexes with finite ho-

have a finite number of non-degenerate simplexes.)

It must be emphasized that although the procedures developed for solving these problems are finite, they are much too complicated to be considered practical.

In the first section of this paper we give some preliminary definitions, con-

**⇒ What about practical computability ?**

**practical computations ?**

1. **Rolf Schoen =**

   **Effective Algebraic Topology (Memoirs AMS 1990)**

   **= Elegant systematic organization of E. Brown's paper.**

2. **Justin Smith =**

   **Iterating the Cobar construction (Memoirs AMS 1994)**

   **= Use of operadic structures.**

3. **Julio Rubio + FS = Constructive Algebraic Topology.**

**4. HoTT = Homotopy Type Theory (Voevodsky).**

**HoTT = Organization of Constructive Mathematics**

**/ Martin-Löf type theory.**

**Ordinary Mathematics / ZF theory of sets**

**/ Various formalizations.**

**Constructive mathematics / Theory of groupoids**

**/ Various formalizations.**

**Theorem (?): HoTT $\ni$ Constructive versions of**

**Serre and Eilenberg-Moore spectral sequences.**

**$\Rightarrow$ The classical Theory of Classes due to Serre can be made constructive.**

**$\Rightarrow$**

**Every simply connected space with homology groups of finite type**

**have homotopy groups of finite type (Serre).**

**becomes:**

**$X$ = finite simply connected simplicial complex**

**$\Rightarrow$ Every $\pi_n(X)$ is computable.**

## 4. Constructive Algebraic Topology.

$X$ = simplicial complex.

$\exists$ ?? "Algebraic" model of $X \overset{???}{\Longleftrightarrow}$ Homotopy type of $X$ ?

Homology groups.

$(C_*(X), \partial)$ = chain complex canonically associated to $X$.

$C_n(X)$ = free $\mathbb{Z}$-module generated by the $n$-simplices of $X$.

Boundary operator $\partial : C_n(X) \to C_{n-1}(X)$.

$Z_n(X) = \{n\text{-cycles}\} = \ker \partial : C_n(X) \to C_{n-1}(X)$.

$B_n(X) = \{n\text{-boundaries}\} = \operatorname{im} \partial : C_{n+1}(X) \to C_n(X)$.

$B_n(X) \subset Z_n(X) \Rightarrow$

$H_n(X) = Z_n X / B_n(X) = n$-th Homology group.

Problem: $X$ often not of finite type

$\Rightarrow (C_*(X), \partial)$ cannot be a machine object.

Serre's theorem: $X$ "reasonable" $\Rightarrow H_n(X)$ of finite type.

But $\{H_n(X)\}$ does not determine the homotopy type.

**Notion of Strong Homology Equivalence**

**between chain complexes:**

$$C_* \overset{h}{\longleftrightarrow} C'_*$$

**with $h$ a (rich) collection of objects describing why and how the homological natures of $C_*$ and $C'_*$ are isomorphic.**

**Notion of locally effective object.**

**An object not of finite type is locally effective**

**when one or several algorithms**

**describe the nature of every component.**

**Typical example: the chain complex:**

$C_* := C_*(K(\mathbb{Z}, 1))$ **of** $K(\mathbb{Z}, 1)$.

$C_n =$ **the free** $\mathbb{Z}$**-module**

**generated by the sequences** $(a_1, \ldots, a_n) \in \mathbb{Z}^n$.

$C_n$ **not of finite type**

**Face operator** $\partial_k : K(\mathbb{Z},1)_n \to K(\mathbb{Z},1)_{n-1}$ **for** $0 \leq k \leq n$**:**

$$\partial_0(5,7,-8,21) = (7,-8,21)$$

$$\partial_1(5,7,-8,21) = (12,-8,21)$$

$$\partial_2(5,7,-8,21) = (5,-1,21)$$

$$\partial_3(5,7,-8,21) = (5,7,13)$$

$$\partial_4(5,7,-8,21) = (5,7,-8)$$

$d : C_n(K(\mathbb{Z},1)) \to C_{n-1}(K(\mathbb{Z},1))$ **:**

$$d(5,7,-8,21) = (7,-8,21) - (12,-8,21) + (5,-1,21) - (5,7,13) + (5,7,-8)$$

**Simplicial set $X$ with effective homology:**

$$(X, C_*(X) \overset{h}{\longleftrightarrow} EC_*^X)$$

**with:**

$$X = \text{Locally effective simplicial set}$$

$$C_*(X) = \text{Locally effective chain complex}$$

$$EC_*^X = \boxed{\text{Effective}} \text{ chain complex}$$

$$h = \text{Strong homology equivalence}$$

$EC_*^X$ = Chain complex of finite type

$\Rightarrow$ Computable homology.

# Fundamental Theorem of Effective Homology.

**Let $\phi : X \mapsto Y$ and $\psi : (X_1, X_2) \mapsto Y$ be "classical" constructors of Algebraic Topology.**

**Then there exist versions with effective homology of these constructors:**

$$\widetilde{\phi} : (X, C_*(X) \overset{h}{\Longleftrightarrow} EC_*) \mapsto (Y, C_*(Y) \overset{h}{\Longleftrightarrow} EC_*^Y)$$

$$\widetilde{\psi} : ((X_1, C_*(X_1) \overset{h}{\Longleftrightarrow} EC_*^{X_1}), (X_2, C_*(X_2) \overset{h}{\Longleftrightarrow} EC_*^{X_2})) \longmapsto (X, C_*(X) \overset{h}{\Longleftrightarrow} EC_*)$$

**Typical computability problem**

**pending in "classical" Algebraic Topology:**

**Definition** : $\Omega^n(X, x_0) := \Omega(\Omega^{n-1}(X, x_0), *)$

$$\simeq \mathcal{C}((S^n, *), (X, x_0))$$

**Given $X$, compute $H_*(\Omega^n(X, x_0)) = ?$ (Adam's problem).**

**$n = 1$ solved by Adams (1956, Cobar construction).**

**$n = 2$ solved by Baues (1980, Double Cobar construction).**

**$\forall n$ solved by Julio Rubio**

**(1990, Constructive Algebraic Topology).**

**Proof:**

$$
\begin{array}{rcl}
X, C_*(X) & \longleftrightarrow & C_*(X) \\
& \Big\downarrow & \\
(\Omega X, C_*(\Omega X)) & \longleftrightarrow & EC_*^{\Omega X}) \\
& \Big\downarrow & \\
(\Omega^2 X, C_*(\Omega^2 X)) & \longleftrightarrow & EC_*^{\Omega^2 X}) \\
& \Big\downarrow & \\
(\Omega^3 X, C_*(\Omega^3 X)) & \longleftrightarrow & EC_*^{\Omega^3 X}) \\
& \Big\downarrow & \\
(\Omega^4 X, C_*(\Omega^4 X)) & \longleftrightarrow & EC_*^{\Omega^4 X}) \\
& \Big\downarrow & \\
(\cdots, \cdots\cdots & \longleftrightarrow & \cdots\cdots)
\end{array}
$$

**QED**

**Example:** Let $P_4 = P^\infty \mathbb{R} / P^3 \mathbb{R}$.

$P_4$ **is a simplicial set of finite type**

$\Rightarrow$ **Trivially with effective homology:**

$$(P_4, C_*(P_4) \overset{=}{\longleftrightarrow} C_*(P_4))$$

Adams

$$(\Omega P_4, C_*(\Omega P_4) \overset{\neq}{\longleftrightarrow} EC_*^{\Omega P_4})$$

Baues

$$(\Omega^2 P_4, C_*(\Omega^2 P_4) \overset{\neq}{\longleftrightarrow} EC_*^{\Omega^2 P_4})$$

Rubio

$$(\Omega^3 P_4, C_*(\Omega^3 P_4) \overset{\neq}{\longleftrightarrow} EC_*^{\Omega^3 P_4})$$

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$$H^5(\Omega^3 P^4) = (\mathbb{Z}/2)^5 + \mathbb{Z}/3 + \mathbb{Z} = (\mathbb{Z}/2)^4 + \mathbb{Z}/6 + \mathbb{Z}$$

**The most difficult Kenzo computation:**

$$H_7(\Omega(\Omega(\Omega(P^\infty(\mathbb{R})/P^3(\mathbb{R})) \cup_4 D^4) \cup_2 D^3)) = \ ???$$

**Machine Puccini (2001) : 2 Months.**

**Machine IFNode2 (2018) : 1 Month.**

**Machine IFNode2 (2021) : 40 minutes**

$$H_7 = (\mathbb{Z}/2)^{113} + \mathbb{Z}/4 + (\mathbb{Z}/8)^3 + \mathbb{Z}/16 + \mathbb{Z}/32 + \mathbb{Z}$$

$$H_8 = (\mathbb{Z}/2)^{253} + (\mathbb{Z}/4)^9 + \mathbb{Z}/8 + \mathbb{Z}^5$$

**Based on many other subjects:**

- **Combinatorial topology:**

  **Loop spaces, Classifying spaces. . .**

- **Homological algebra,**

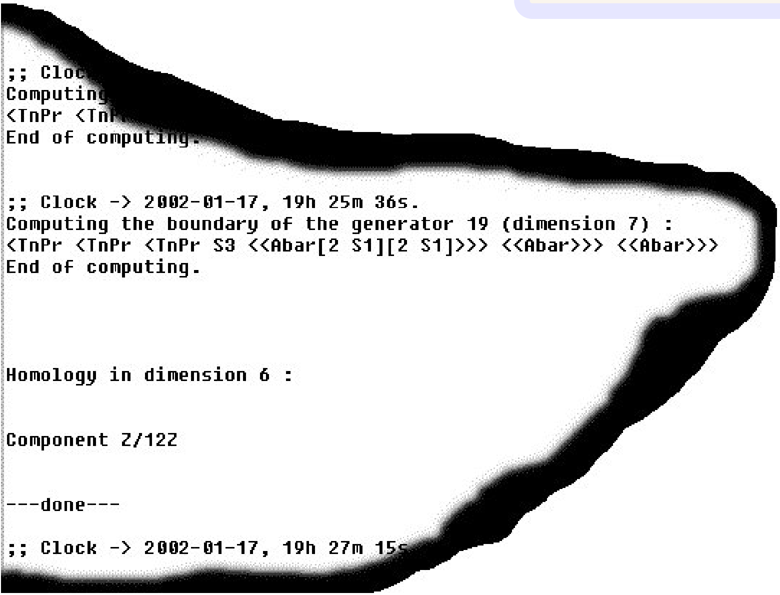
  **Exact sequences, Spectral sequences.**

- **Homological perturbation theory.**
- **Whitehead and Postnikov towers.**
- **Discrete vector fields.**
- **High level functional programming:**

  **computer closures, garbage collector design.**

- **Meta-object protocol.**

# The END

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